ELLIPTIC PDE: PROBLEM SET 1

(1) Suppose $0 < \lambda < \Lambda < +\infty$ and $n > 2$. Show that there exists $\alpha > 0$ depending only on λ, Λ, n such that $v(x) := |x|^{-\alpha}$ satisfies

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2 v) > 0$$

on $B_1 \backslash \{0\}$.

(2) Suppose $u \in C^2(B) \cap C(\overline{B})$ satisfies

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2 u) \leqslant 0 \leqslant \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u).$$

Show that there exist measurable coefficients $a_{ij}(x)$ so that $a_{ij}u_{ij} = 0$.

(3) (*Extension of super solutions*) The following statement is used to glue, or extend, viscosity super solutions. It is extremely useful. Let Ω, Ω_1 be bounded domains such that $\overline{\Omega} \subset \Omega_1$. Suppose $u \in C(\Omega_1)$ is a viscosity supersolution of $F(D^2u, x) = f(x)$ in Ω_1, and $v \in C(\overline{\Omega})$ is a viscosity super solution in Ω of $F(D^2v, x) = g(x)$. Assume that $v \geqslant u$ on $\partial\Omega$. Define

$$w(x) = \begin{cases} u(x) & x \in \Omega_1 \backslash \Omega \\ \min\{u, v\}(x) & x \in \overline{\Omega} \end{cases} \qquad h(x) = \begin{cases} f(x) & x \in \Omega_1 \backslash \Omega \\ \max\{f(x), g(x)\} & x \in \overline{\Omega}. \end{cases}$$

Show that w is a supersolution of $F(D^2w, x) = h(x)$. Draw a picture in the one dimensional case for super solutions of $\Delta u = 0$. What happens if you take max instead of min?

(4) Recall the identification of $\mathrm{Sym}(2 \times 2)$ with $\mathbb{R}^3$.

$$\sqrt{2}e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sqrt{2}e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sqrt{2}e_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(i) Show that this map defines an isometry where $\mathrm{Sym}(2 \times 2)$ is equipped with the inner product $A \cdot B = \mathrm{Tr}(AB)$.
(ii) Describe the process of diagonalization geometrically.
(iii) Draw some level sets of the trace and determinant maps. Is the equation $\det(A) = 1$ uniformly elliptic on the space of positive definite symmetric matrices?

(5) (*Hyperplane separation lemma*) Suppose $K \subset \mathbb{R}^n$ is a convex open set, and suppose that $y \in \partial K$. Without appealing the the Hahn-Banach Theorem, show that there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ so that $L(y) = 0$ and $L \geqslant 0$ on K.

(i) Suppose first that $d(y, K) = r > 0$. Then there exists $x_0 \in \partial K$ so that $r = d(x_0, y)$. Consider the plane Σ with unit normal vector parallel to $x_0 - y$. Show that Σ is disjoint from K.

(ii) Now if $y \in \partial K$, take a sequence $y_i \in \overline{K}^c$, and apply the above argument to find a sequence of linear functions with L_i with $L_i > 0$ on K. Show that it is possible to extract a convergence subsequence $L_{i_j} \to L$. Show that L is the desired linear function.

(6) Prove the strong maximum principle for viscosity super solutions. Namely, suppose $u \in C(\Omega)$, and $u \in \overline{S}(0)$, with $u \geqslant 0$. Prove that if $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$. (**Hint**: The function from problem (1) should be useful here).

(7) Prove the Hopf lemma for viscosity super solutions. Namely, suppose $u \in C(\overline{B_r})$, and $u \in \overline{S}(0)$, with $u \geqslant 0$ and $u \not\equiv 0$. Prove that if $u(y) = 0$ for some $y \in \partial B_r$, then u grows at least linearly away from 0 near y.

(8) Extend the ABP estimate from balls to general domains, by using the extension of supersolutions. What happens to the contact set?