MATH 18.966: HOMEWORK 4

DUE TUESDAY, MAY 7, 2019

(1) In this exercise you will prove the following

Lemma 0.1. *Let (M, g) be a Riemannian manifold, and suppose that the sectional curvature satisfies $K(g) \leqslant \kappa$. Fix a point $p \in M$, then*

$$\text{inj}\,(p) \geqslant \min\{\frac{\pi}{\sqrt{\kappa}}, \frac{1}{2} \text{ length of shortest geodesic loop passing through } p\}$$

In particular,

$$\text{inj}\,(M, g) \geqslant \min\{\frac{\pi}{\sqrt{\kappa}}, \frac{1}{2} \text{ length of shortest geodesic loop in } (M, g)\}$$

As usual, we use the convention that if $\kappa \leqslant 0$, then we just ignore the symbol $\frac{\pi}{\sqrt{\kappa}}$. This estimate is called Klingenberg's estimate.

(a) First recall that we proved that the distance to the conjugate locus of p is no less that $\frac{\pi}{\sqrt{\kappa}}$ as a consequence of Rauch's theorem, (or our proof of Bishop-Gromov volume comparsion, for example). So we just need to show that if q is in the cut locus of p, inj(p) $= d(p, q)$ but q is not a conjugate point, then there is a geodesic loop containing p and q. To do this let $v_1, v_2 \in T_pM$ be unit length vectors so that $\gamma_1(t) := \exp_p tv_1$ and $\gamma_2(t) := \exp_p tv_2$ have $\gamma_1(1) = \gamma_2(1) = q$. It suffices to show that $\gamma_1'(1) = -\gamma_2'(1)$, since implies that γ_1, γ_2 can be combined to form a closed geodesic loop. Suppose not. Argue that there is $w \in T_qM$ such that $\langle w, \gamma_1'(1)\rangle < 0$ and $\langle w, \gamma_2'(1)\rangle < 0$.

(b) For $s \in (0, \varepsilon)$ consider the points $q(s) := \exp_q(sw)$. Show that $d(p, q(s)) < d(p, q)$ for $s > 0$.

(c) Now argue that there are distinct tangent vectors $v_1(s), v_2(s)$ such that $\exp_p(v_1(s)) = \exp_p(v_2(s))$. Explain why this contradicts the fact that $d(p, q) = \text{inj}(p)$.

(2) Next you will prove a lemma relating volume non-collapsing and the injectivity radius, under a bound for the sectional curvature. This result is originally due to Cheeger with a different argument.

Lemma 0.2. *Given $n \geqslant 2$, $\nu, \kappa > 0$, there is a constant $R = R(\nu, \kappa, n) > 0$ such that any compact manifold (M^n, g) with $|K(g)| \leqslant \kappa$, and $\mathrm{Vol}(B(p,1)) > \nu$ for all $p \in M$ has $\mathrm{inj}(M, g) \geqslant R$.*

The argument is by contradiction. Assume the result is false, so that there is a sequence (M_i, g_i) with n fixed, sectional curvature bounded by κ, and $\mathrm{Vol}(B_i(p,1)) > \nu$ for any $p \in M_i$.

(a) Rescale the manifolds to get $(M_i, \bar{g}_i)$ with $\mathrm{inj}(M, \bar{g}_i) = 1$. What happens to the sectional curvature?

(b) Choose points $p_i \in M_i$ achieving the injectivity radius. Show that the sequence $(M_i, \bar{g}_i, p_i)$ converges in the pointed $C^{1,\alpha}$ topology to a flat manifold $(M_\infty, g_\infty, p_\infty)$. By applying by Klingenberg's lemma to $(M_i, \bar{g}_i)$, argue that there must be a geodesic loop of length 2 in (M_∞, g_∞) passing through p_∞. In particular, $\mathrm{inj}(M_\infty, g_\infty) \leqslant 1$

(c) Using the volume comparison theorem, prove the following claim: **Claim**: There is a constant $\nu' > 0$ depending only on ν, K, n such that
$$\mathrm{Vol}(B_\infty(p_\infty, r)) \geqslant \nu' r^n$$
for all $r > 0$.

In order to obtain a contradiction, note that by (b) it suffices to prove that $(M_\infty, g_\infty) = (\mathbb{R}^n, g_{Euc})$. To do this, you will prove the following lemma

Lemma 0.3. *Suppose (M, g) is a flat manifold which is not simply connected. Then, for any $p \in M$ we have*
$$\lim_{r \to \infty} \frac{\mathrm{Vol}(B(p,r))}{r^{n-1}} < +\infty$$

As a first step toward proving this result recall that by the classification of manifolds with constant sectional curvature, (M, g) is the quotient of $(\mathbb{R}^n, g_{Euc})$ by a group of isometries Γ acting totally discontinuously (in particular without fixed points). The goal is to argue that if Γ acts non-trivially, and totally discontinuously and $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma = M$ is a covering map and a local isometry, then the volume of any ball $B(p, R)$ in M grows at most like R^{n-1} for $R \gg 0$. To build some intuition, consider the case when Γ acts by translation along a fixed vector!

(d) Begin by recalling that any isometry of $\mathbb{R}^n$ is given by $h(v) = Av + w$ for some $A \in O(n)$, and $w \in \mathbb{R}^n$. It follows that $0, w$ are identified under the quotient map π. Argue that the curve $\gamma(t) = tw$ descends to a closed geodesic $\bar{\gamma} \subset (M, g)$ with $\bar{\gamma}(0) = \bar{\gamma}(1)$ and $\bar{\gamma}'(0) = \bar{\gamma}'(1)$.

(e) Next we argue that $Aw = w$. If not, then the curve $h(tw) = w + tAw$ descends to $\bar{\gamma}(t)$ under the projection map. Argue that this implies that the covering map π cannot be a local diffeomorphism (**Hint**: Consider $d\pi|_w : \mathbb{R}^n \to T_{\pi(p)}M$).

(f) By a rotation we can assume $w = (a, 0, \ldots, 0)$ for some $a > 0$. Identify $(0, x_2, \ldots, x_n)$ with $\mathbb{R}^{n-1}$. Show that A has the block diagonal form

$$\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where $A' \in O(n-1)$. Conclude that $[0, a) \times \mathbb{R}^{n-1}$ covers (M, g). Argue that this implies the lemma. (**Note**: This is an ad-hoc argument to understand a part of the group of *deck transformations* acting on the universal cover).

(g) Find a counterexample to Lemma 0.1 if we drop the assumption $\mathrm{Vol}(B(p, 1)) > \nu$ for all $p \in M$.

(3) Use Lemma 0.1 together with results proved in class to prove Cheeger's finiteness theorem.

Theorem 0.4 (Cheeger's finiteness theorem)**.** *Let $n \geqslant 2$, and fix constants $\kappa, D, \nu > 0$. The class of Riemannian manifolds (M^n, g) satisfying the bounds*

$$|K(g)| \leqslant \kappa$$
$$\mathrm{diam}(M, g) \leqslant D$$
$$\mathrm{Vol}(M, g) \geqslant \nu$$

contains only finitely many diffeomorphism types.