MATH 18.966: HOMEWORK 2

DUE TUESDAY, MARCH 19, 2019

(1) Here is another application of the Riccati technique, which is fairly down to earth, and makes explicit our claim that "larger sectional curvature makes the metric smaller". You will prove the following

Proposition 0.1. *Suppose (M, g) is a complete, Riemannian manifold, $p \in M$. Let $r(p)$ denote the injectivity radius of (M, g) at p. Suppose that the sectional curvature satisfies $K(y) > -\kappa_0$ for all $y \in B_{r(p)}(p)$, and all two-planes in T_yM. Then on $B_{r(p)}(p)$, which we identify with $B_{r(p)}(0) \subset \mathbb{R}^n$ we have*

$$g \leqslant g_{-\kappa_0}$$

where $g_{-\kappa_0}$ is the metric of constant sectional curvature $-\kappa_0$ on $B_{r(p)}(0)$. Furthermore, if also $K(y) < \kappa_1$ for all $y \in B_{r(p)}(p)$, and all two-planes in T_yM then, with the same notation

$$g \geqslant g_{\kappa_1}.$$

In particular, if $-\kappa < K(y) < \kappa$ for all $y \in B_{r(p)}(p)$, then there is a universal constant $R > 0$ so that, if $r \leqslant \min\{r(p), R\kappa^{-\frac{1}{2}}\}$, then

$$\frac{1}{2} g_{Euc} \leqslant g \leqslant 2 g_{Euc}$$

on $B_r(p)$, where g_{Euc} denotes the Euclidean metric from $\mathbb{R}^n$.

For simplicity we will assume that $\kappa_0 = \kappa_1 = \kappa$ is positive, but this isn't necessary. It just saves you from some annoying symbology. If you want, just prove the result in full generality.

(a) Let (M, g) be a Riemannian manifold, $p \in M$, and let $(r, x_1, \dots, x_{n-1})$ be normal spherical coordinates on T_pM. In particular $(x_1, \dots, x_, n-1)$ are coordinates on S^{n-1}. For simplicity we will set $r = x_0$, and use the following notation. Greek indices $\{\alpha, \beta, \eta\}$ run over $\{0, \dots, n-1\}$, while Roman indices $\{i, j, k\}$ run over $\{1, \dots, n-1\}$. Show that the Christoffel symbols are given simply by

$$\Gamma^{\alpha}_{00} \equiv 0, \qquad \Gamma^{0}_{0p} \equiv 0, \qquad \Gamma^{\ell}_{0p} = \frac{1}{2} g^{\ell k} \partial_0 g_{pk}.$$

Show that the curvature is given by

$$R_{0\ell}{}^{p}{}_{0} = \partial_0 \Gamma^{p}_{0\ell} + \Gamma^{p}_{0s} \Gamma^{s}_{0\ell}. \tag{0.1}$$

In particular, if we view Γ_0 as an endomorphism of the the tangent bundle, the Γ_0 satisfies a Riccati equation.

(b) To expand on this last remark, recall that Γ_0 is only a locally well-defined endomorphism of the tangent bundle. Nevertheless, show that if $V(r)$ is a unit vector parallel along ∂_r, satisfying $\langle V, \partial_r \rangle = 0$, then we have

$$\frac{\partial}{\partial r}\langle \Gamma_0 V(r), V(r)\rangle \leqslant \kappa_0 - (\langle \Gamma_0 V(r), V(r)\rangle)^2$$

provided $K(y) \geqslant -\kappa_0$. Show that $\Gamma_{0\ell}^p$ approaches $\frac{1}{r}\delta_\ell^p$ as $r \to 0$, where δ_ℓ^p is the identity matrix tangent to S^{n-1}.

(c) Use the Riccati comparison argument to conclude that, for any $V(r)$ as above we have

$$\langle \Gamma_0 V(r), V(r)\rangle \leqslant \frac{\sqrt{\kappa_0}\cosh(\sqrt{\kappa_0}r)}{\sinh(\sqrt{\kappa}r)}.$$

By applying the formula for Γ_0, and showing that

$$\lim_{r\to 0}\frac{1}{r^2}g_{pk} = g_{S^{n-1}}$$

conclude that

$$g = (dr)^2 + g_{pk}dx_p \otimes dx_k \leqslant (dr)^2 + \frac{1}{\kappa_0}\sinh^2(\sqrt{\kappa_0}r)g_{S^{n-1}}.$$

which is our upper bound.

The lower bound is a little different, since the Cauchy-Schwarz inequality does not allow us to obtain the Riccati equation from (0.1). It turns out that the sectional curvature upper bound can be used by reducing to a 2-dimensional submanifold.

(d) Consider the (local) submanifold (S, g) of (M, g) defined by $(x_0, 0, \ldots, x_p, 0 \ldots, 0)$. Use the Gauss-Codazzi equation, together with $\nabla_{\partial r}\partial_r = 0$ to show that S has sectional curvature bounded above by κ_1.

(e) Write the induced metric on S as $dr^2 + G_p(r, x_p)dx_p^2$. Apply (0.1) on S, and the Riccati comparison argument to show that

$$G_p \geqslant \frac{1}{\kappa_1}\sin^2(\sqrt{\kappa_1}r)$$

(f) Conclude that

$$g \geqslant dr^2 + \frac{1}{\kappa_1}\sin^2(\sqrt{\kappa_1}r)g_{S^{n-1}}$$

(g) Finish the proof of the proposition by noting that

$$\frac{1}{2}r^2 \leqslant \frac{1}{\kappa}\sin^2(\sqrt{\kappa}r)g_{S^{n-1}} \leqslant \left(\frac{1}{\sqrt{\kappa}}\sinh(\sqrt{\kappa}r)\right)^2 g_{S^{n-1}} \leqslant 2r^2$$

for $r < R(\kappa)^{-\frac{1}{2}}$ for some universal R.

(2) Next we give a more streamlined derivation of the Laplacian comparison theorem for manifolds with Ricci curvature lower bounds. Let (M, g) be a Riemannian manifold, and $f, h : M \to \mathbb{R}$ a smooth functions (we will also consider the case when f, h are only locally defined).

(a) Recall that $\Delta f := \mathrm{Tr}_g \mathrm{Hess}(f)$. Show that, in local coordinates $(x_1, \ldots, x_n)$, we can write

$$\Delta f = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\, g^{ij}\partial_j f\right).$$

In particular, show that

$$\Delta f = \sum_{ij} g^{ij}\partial_i\partial_j f + \sum_j \partial_j f \Delta x_j.$$

(b) Show that

$$\Delta\frac{1}{2}\langle\nabla f, \nabla h\rangle = \langle \mathrm{Hess}(f), \mathrm{Hess}(h)\rangle + \langle\nabla f, \nabla\Delta h\rangle + \langle\nabla\Delta f, \nabla h\rangle + \mathrm{Ric}(\nabla f, \nabla h).$$

Apply this with $g = f$ to obtain Bochner's formula. Formulas of this type play a fundamental role in geometry by linking curvature with analysis.

(c) Fix a point $p \in M$, and let $r = d(p, \cdot)$, which is smooth away from $\mathrm{cut}(p)$. Apply the above formula to derive

$$\partial_r \Delta r = -\mathrm{Ric}(\partial_r, \partial_r) - |\mathrm{Hess}(r)|^2.$$

In particular, if $Ric > (n-1)\rho$, then

$$\partial_r \Delta r \leqslant -(n-1)\rho - \frac{1}{n-1}(\Delta r)^2.$$

Apply the Riccati comparison argument to deduce that

$$\Delta r \leqslant (n-1)\frac{s'_\rho}{s_\rho} = \Delta_\rho r_\rho$$

where r_ρ is the distance from a point in the model space with constant curvature ρ, and Δ_ρ is the Laplacian in (M_ρ, g_ρ).

(d) Assume (M, g) has $Ric(g) \geqslant 0$, $h : M \to \mathbb{R}$ is harmonic (ie. $\Delta h = 0$), and $|\nabla h|^2 = 1$. Show that $\text{Hess}h \equiv 0$. In particular, we can think of such functions as the analog of linear functions.

(e) Assume now that we have coordinates $(x_1, \ldots, x_n)$ defined in an open neighborhood of $p \in M$ such that $\Delta x_i = 0$. Such coordinates are called *harmonic coordinates* and will play a fundamental role in what we study this semester. Let g_{ij} be the components of the metric in this coordinate system, regarded as functions on our local patch. Using part (b), show that

$$\Delta g_{ij} = -2\text{Ric}_{ij} + Q_{ij}(g, \partial g)$$

where Q_{ij} is quadratic in g, and ∂g. In particular, this formula shows that Ricci curvature bounds can be thought of as bounds for the Laplacian of the metric. Once we have developed a little bit of elliptic regularity theory, we will see why such bounds should imply good regularity properties for the metric.

(3) This problem works out the completeness of the Gromov-Hausdorff metric by proving an (a priori) simpler result which goes back to Hausdorff. Let (X, d) be a metric space. For any subset $A \subset X$, and any $\varepsilon > 0$ we set

$$B_\varepsilon(A) = \bigcup_{p \in A} B_\varepsilon(p),$$

where $B_\varepsilon(p)$ is the ball of radius ε around p. This is the "ε- fattening" of A. For Y, Z compact subsets of X define the *Hausdorff distance* between Y and Z by

$$d_H(A, B) := \inf \left\{ \varepsilon > 0 \middle| Y \subset B_\varepsilon(Z), \quad Z \subset B_\varepsilon(Y) \right\}.$$

(a) Show that d_H defines a metric on the set

$$\tilde{X} := \{A \subset X \big| \ A \ \text{is compact}\}.$$

(b) Show that $(\tilde{X}, d_H)$ is a complete metric space if and only if (X, d) is complete. As a hint, suppose X is complete and $\{A_j\}$ is a Cauchy sequence in $\tilde{X}$. Consider "discretizing" A_j at scale N^{-1} using a collection of points. Do this for a sequence of scales going to zero.

(c) For compact metric spaces $(X, d_X), (Y, d_Y)$ (or pointed, proper spaces $(X, d_X, p), (Y, d_Y, q)$), define an admissible metric d on $X \sqcup Y$ to be a metric such that $d|_X = d_X$, and $d|_Y = d_Y$. Define

$$\hat{d}_{GH} = \inf\{\varepsilon > 0 : \exists\, d \text{ admissible metric on } X \sqcup Y \text{ such that } d_H(X, Y) < \varepsilon\}$$

where d_H denotes the Hausdorff metric. In the pointed case, denote by $\overline{B_{\varepsilon^{-1}}}(p)$ the closed ball of radius ε^{-1} in X, and similarly for Y. Define $\hat{d}^p_{GH}$ to be the infimum over $\varepsilon > 0$ such that there exists an admissible metric d on $X \sqcup Y$ such that

$$d_H(\overline{B_{\varepsilon^{-1}}}(p), \overline{B_{\varepsilon^{-1}}}(q)) + d(p,q) < \varepsilon$$

Show that $\hat{d}_{GH}$ (resp. $\hat{d}^p_{GH}$) defines a notion of Gromov-Hausdorff (resp. pointed Gromov-Hausdorff) distance which is equivalent to the one defined in class. As a hint, given $f : X \to Y$ an ε-isometry, define an admissible metric d on $X \sqcup Y$ by

$$d(x,y) = \inf\{d_X(x,x') + d_Y(y,y') + \varepsilon : d(y', f(x')) < \varepsilon\}.$$

Prove that this defines an admissible metric and that $d_H(X,Y) < 2\varepsilon$.

(d) Now we can prove that the space of compact metric spaces (resp. proper, pointed metric spaces) equipped with the Gromov-Hausdorff metric is complete. I will explain how to do this in the non-pointed case. Your task is to check the details, and generalize everything to the case of pointed metric spaces. Given a Cauchy sequence of compact metric spaces (X_i, d_{X_i}) with

$$d_{GH}(X_i, X_{i+1}) < \frac{1}{2} 2^{-i}$$

define $Y = \sqcup_i X_i$. By part (c), for each i we have an admissible metric $d_{(i,i+1)}$ on $X_i \sqcup X_{i+1}$ such that

$$d_{(i,i+1),H}(X_i, X_{i+1}) < 2^{-i}.$$

This allows us to measure distances between nearest neighbors in Y. We extend this to an admissible metric on Y in the obvious way. For $x_i \in X_i, x_{i+k} \in X_{i+k}$ define

$$\begin{aligned} d_Y(x_i, x_{i+k}) = &\inf_{\{y_{i+j} \in X_{i+j} : 1 \leqslant j \leqslant k-1\}} d_{(i,i+1)}(x_i, y_{i+1}) \\ &+ \sum_{j=1}^{k-2} d_{(i+j,i+j+1)}(y_{i+j}, y_{i+j+1}) \\ &+ d_{(i+k-1,i+k)}(y_{i+k-1}, x_{i+k}) \end{aligned}$$

(it may help to draw a picture). Show that the compact sets X_i are Cauchy with respect to the Hausdorff metric on (Y, d_Y).

(e) We can invoke (a) to conclude that the X_i converge, once we show that (Y, d_Y) is complete. This is not the case, however, because of Cauchy sequences like $\{x_i\}$ where $x_i \in X_i$. But we can just take the completion of (Y, d_Y) in the usual way to get $(\overline{Y}, \overline{d}_Y)$, adding to Y equivalence classes of Cauchy sequences, and extending d_Y. With this detail taken care of, we can invoke the Hausdorff compactness result from (a) to conclude that the set of compact metric spaces is complete with the Gromov-Hausdorff distance. Note, in fact, that the Gromov-Hausdorff limit of the X_i is *precisely* $(\bar{Y}\backslash Y, \bar{d}_Y)$. Think about how this works for $X_k = \{\frac{n}{k} : n \in \mathbb{Z}, |n| \leqslant k\}$ with metric induced from $\mathbb{R}$.

1. Analysis

Finally, we're going to do a little analysis. Consider $\Omega \subset \mathbb{R}^n$ an open set. Let $C_c^\infty(\Omega)$ denote the space of smooth functions compactly supported in Ω, which is a vector space in the obvious way. A distribution T is a linear map $T : C_c^\infty(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) such that, for every compact set $K \subset \Omega$ there is a constant C_K and a number $N_K \geqslant 0$ with

$$|T(\varphi)| \leqslant C_K \left(\sum_{\ell=0}^{N_K} \sup_K |\nabla^\ell \varphi| \right)$$

for all functions φ with support contained in K, and where $\nabla^\ell \varphi$ denotes the ℓ-th derivative of φ.

(4) Show that the following give examples of distributions.
  (a) If $f \in L^1_{loc}(\mathbb{R}^n)$, then $T_f(\varphi) := \int f\varphi dx$.
  (b) The Dirac delta function $\delta_0(\varphi) = \varphi(0)$.
  (c) Any differential operator of the following form. Fix $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and let $|\alpha| = \sum_i \alpha_i$. Then

$$\partial^\alpha(f) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_i} \cdots \partial x_n^{\alpha_n}}$$

  (d) Any linear combination of distributions.

A particularly useful fact, which is essentially trivial, is that we can differentiate distributions using "integration by parts". Namely, with notation as above we define $\partial^\alpha T$ to be the distribution

$$\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(\partial^\alpha \varphi)$$

For example, this allows us to define the derivatives of any locally L^p functions for $p \geqslant 1$.

  (e) Let f be differentiable. Show that $\partial T_f = T_{\partial f}$.
  (f) Let $H(x)$ be the Heaviside function on $\mathbb{R}$. That is, $H(x) = 0$ if $x < 0$, and $H(x) = 1$ if $x \geqslant 0$. Clearly H is locally L^1 so we can define

$$T_H(\varphi) = \int H\varphi dx$$

Compute ∂T_H, which we denote by ∂H. Is ∂H the distribution associated to a function? Is it a measure?

(g) In $\mathbb{R}^2$ compute $\Delta \log(x^2 + y^2)$ as a distribution, where

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

One point to take away from this is that not all functions, even bounded functions, have distributional derivatives which are themselves associated with functions. For a domain $\Omega \subset \mathbb{R}^n$ we define

$$W^{k,p}(\Omega) = \{f \in L^p : \nabla^\ell f \in L^p(\Omega) \text{ for all } 0 \leqslant \ell \leqslant k\}.$$

That is, $W^{k,p}$ consists of those L^p functions whose distributional derivatives up to, and including order k, are associated with L^p functions. This space comes with a norm

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{\ell=0}^{k} \|\nabla^\ell f\|_{L^p(\Omega)}$$

and this norm makes $W^{k,p}$ into a Banach space. If $p = 2$, this space is a Hilbert space. These spaces play an important role in the elliptic regularity, and Hodge theory.