18.965: HOMEWORK 7

DUE: NEVER

This is an optional homework. The goal is to define, and work out some of the basic properties of Gromov-Hausdorff convergence. The Gromov-Hausdorff metric gives us a way of measuring distances between metric spaces. We are going to begin with a (seemingly) simpler situation, which goes back to work of Hausdorff. Let (X, d) be a metric space. For any subset $A \subset X$, and any $\varepsilon > 0$ we set

$$B_\varepsilon(A) = \bigcup_{p \in A} B_\varepsilon(p),$$

where $B_\varepsilon(p)$ is the ball of radius ε around p. This is the "ε- fattening" of A.

Definition 0.1. *For Y, Z compact subsets of X define the* Hausdorff distance *between Y and Z by*

$$d_H(Y, Z) := \inf \left\{ \varepsilon > 0 \middle| Y \subset B_\varepsilon(Z), \quad Z \subset B_\varepsilon(Y) \right\}.$$

The Hausdorff metric defines a metric structure on the set of all compact subsets of (X, d), which is furthermore a complete metric when (X, d) is a complete metric space. This is what you will prove in the next two problems

(1) Show that d_H defines a metric on the set

$$\tilde{X} := \{A \subset X \big| \, A \text{ is compact}\}.$$

(2) Show that $(\tilde{X}, d_H)$ is a complete metric space if and only if (X, d) is complete. As a hint, suppose X is complete and $\{A_j\}$ is a Cauchy sequence in $(\tilde{X}, d_H)$. Consider "discretizing A_j at scale N^{-1}" for $N \gg 1$ using a collection of points which are at most distance N^{-1} apart. Construct a Hausdorff limit for these discretized sets. Do this for a sequence of scales going to zero.

(3) Show that if (X, d) is compact, then so is $(\tilde{X}, d_H)$.

The Gromov-Hausdorff distance is an elaboration on the Hausdorff distance which allows us to measure distances between metric spaces. We are going to give several definitions of the distance, useful for various purposes, which all define equivalent metric structures, and hence equivalent notions of convergence (which is what we are interested in).

Definition 0.2. *Let (X, d_X) and (Y, d_Y) be compact metric spaces. We say that a metric $\hat{d}$ on $X \sqcup Y$ is* admissible *if $\hat{d}|_X = d_X$ and $\hat{d}|_Y = d_Y$. We* *define the Gromov-Hausdorff distance* $d_{GH}(X,Y)$ *to be*

$$d_{GH}(X,Y) = \inf\{\varepsilon > 0 : \exists \text{ an admissible metric } \hat{d} \text{ such that } d_H(X,Y) < \varepsilon\}$$

That is, we take admissible metrics on $X \sqcup Y$, and then measure the Hausdorff distance between X, Y in this metric. The Gromov-Hausdorff distance is the infimum of all such distances. There is an extension of this notion to non-compact metric spaces called the *pointed* Gromov-Hasudorff distance. In essence, one fixes a point $p_X \in X$ and a point $p_Y \in Y$. Assuming that $\overline{B_R(p_X)}, \overline{B_R(p_Y)}$ are compact for all $R > 0$, we then consider the infimum over all admissible metrics and $\varepsilon > 0$ such that

$$d_H(\overline{B(p_X, \varepsilon^{-1})}, \overline{B(p_Y, R)}) + d(p_X, p_Y) < \varepsilon$$

For the most part we will focus on the setting of compact metric spaces, but you should think about the appropriate generalizations to the case of pointed metric spaces.

(4) Prove that there is a always an admissible metric on $X \sqcup Y$, and that

$$d_{GH}(X,Y) \leqslant \frac{1}{2}\max\{\text{diam}X, \text{diam}Y\}$$

Here is another way to define the Gromov-Hasudorff distance

Definition 0.3. *A map* $f : (X, d_X) \to (Y, d_Y)$ *is said to be an* ε*-isometry if*

(i) $|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon$
(ii) $Y \subset B^Y_\varepsilon(f(X))$

Note that an ε-isometry need not be injective, nor continuous.

Definition 0.4. *We define the Gromov-Hausdorff distance between* $(X, d_X), (Y, d_Y)$ *to be*

$$d_{GH}(X,Y) = \inf\{\varepsilon > 0 : \exists \ \varepsilon\text{-isometries } f : X \to Y, \quad h : Y \to X\}$$

(5) This definition really only needs f. The ε-isometry h is included only to make the distance symmetric. To see this, show that if $f : X \to Y$ is an ε-isometry, then there is a 3ε-isometry $h : Y \to X$. This exercise should indicate to you how far from continuous maps these ε-isometries can be.

(6) Prove that this notion of Gromov-Hausdorff distance is equivalent to the first definition. As a hint, given $f : X \to Y$ an ε-isometry, define an admissible metric d on $X \sqcup Y$ by

$$d(x,y) = \inf\{d_X(x,x') + d_Y(y,y') + \varepsilon : d(y', f(x')) < \varepsilon\}.$$

Prove that this defines an admissible metric and that $d_H(X,Y) < 2\varepsilon$.

Note that if (X, d_X) and (Y, d_Y) are isometric, in the sense that there is a bijection $f : X \to Y$ with $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$ then we clearly have $d_{GH}(X, Y) = 0$. Thus, in order for the Gromov-Hausdorff distance to define a genuine metric, we clearly need to consider the set of compact metric spaces modulo isometry. Define $\mathcal{M}$ to be the set of equivalence classes of compact metric spaces. We are going to show that $(\mathcal{M}, d_{GH})$ is a metric space.

(7) Prove that d_{GH} satisfies the triangle inequality.

(8) Prove that if $d_{GH}(X, Y) = 0$ then X, Y are isometric. Here's a hint. For all $k \in \mathbb{N}$ we have a $\frac{1}{k}$-isometry $f_k : X \to Y$. Take $\{x_\ell\} \subset X$ a countable dense subset. Then $\{f_k(x_1)\}$ is a sequence in Y. Passing to a subsequence we can extract a limit y_1. Repeat this for all $\ell \in \mathbb{N}$ and take a diagonal subsequence to get a map $f : \{x_\ell\} \to Y$. Show that this can be extended to an isometry from X to Y.

(9) Next we will prove that the metric space $(\mathcal{M}, d_{GH})$ is *complete*. I will sketch how to do this, and your task is to check the details. Given a Cauchy sequence of compact metric spaces (X_i, d_{X_i}) with

$$d_{GH}(X_i, X_{i+1}) < \frac{1}{2} 2^{-i}$$

define $Y = \sqcup_i X_i$. By part (c), for each i we have an admissible metric $d_{(i,i+1)}$ on $X_i \sqcup X_{i+1}$ such that

$$d_{(i,i+1),H}(X_i, X_{i+1}) < 2^{-i}.$$

This allows us to measure distances between nearest neighbors in Y. We extend this to an admissible metric on Y in the obvious way. For $x_i \in X_i, x_{i+k} \in X_{i+k}$ define

$$\begin{aligned} d_Y(x_i, x_{i+k}) = &\inf_{\{y_{i+j} \in X_{i+j} : 1 \leqslant j \leqslant k-1\}} d_{(i,i+1)}(x_i, y_{i+1}) \\ &+ \sum_{j=1}^{k-2} d_{(i+j,i+j+1)}(y_{i+j}, y_{i+j+1}) \\ &+ d_{(i+k-1,i+k)}(y_{i+k-1}, x_{i+k}) \end{aligned}$$

(it may help to draw a picture). Show that the compact sets X_i are Cauchy with respect to the Hausdorff metric on (Y, d_Y). We can invoke problem (2) to conclude that the X_i converge, once we show that (Y, d_Y) is complete. This is not the case, however, because of Cauchy sequences like $\{x_i\}$ where $x_i \in X_i$. But we can just take the completion of (Y, d_Y) in the usual way to get $(\overline{Y}, \overline{d}_Y)$, adding to Y equivalence classes of Cauchy sequences, and extending d_Y. With this detail taken care of, we can invoke the Hausdorff completeness result from (2) to conclude that the set of compact metric spaces is complete with the Gromov-Hausdorff distance. Note, in fact, that the Gromov-Hausdorff limit of the X_i is *precisely* $(\bar{Y}\backslash Y, \overline{d}_Y)$. Think about how this works for $X_k = \{\frac{n}{k} : n \in \mathbb{Z}, |n| \leqslant k\}$ with metric induced from $\mathbb{R}$.

We've now established the basic properties of the Gromov-Hausdorff distance. Our next task is to put these to use in some way. For example, when does a sequence of compact Riemannian manifolds converge in the Gromov-Hausdorff sense?

Definition 0.5. *Given a compact metric space (X, d_X) we define the* covering number $N(X,\varepsilon)$ *to be the smallest integer N such that there are points $x_i \in X, 1 \leqslant i \leqslant N$ with*

$$X \subset \bigcup_{i=1}^{N} B_\varepsilon(x_i)$$

Definition 0.6. *Let $C(\varepsilon)$ be any positive function of ε and let $D \geqslant 0$. Let*

$$\mathcal{M}_{C,D} := \{(X, d_X) \in \mathcal{M} : \mathrm{diam}(X) \leqslant D, \quad N(X,\varepsilon) < C(\varepsilon)\}$$

We will prove

Theorem 0.7. *The set $\mathcal{M}_{C,D}$ is precompact in $(\mathcal{M}, d_{GH})$. Conversely, any pre-compact set $K \subset (\mathcal{M}, d_{GH})$ is contained in $\mathcal{M}_{C,D}$ for some C, D.*

The idea of the proof is that any metric space in $\mathcal{M}_{C,D}$ can be discretized at a scale N^{-1} using a controlled number of points. The proof is then an elaboration of the technique used in problem (2).

(10) Suppose (X_k, d_k) is a sequence in $\mathcal{M}_{C,D}$. For each N let $\{x_i^k\}_{i=1}^{L_k(N)}$ be a $\frac{1}{N}$-dense net in X_k. By assumption we can find such a net with $L_k(N) < C(N^{-1})$, and hence by taking a subsequence we may assume that $L_k(N) = L(N) < C(N^{-1})$ is constant. For each k let

$$d_{ij}^k = d_k(x_i^k, x_j^k).$$

This is just a finite list of numbers, all bounded by D. Therefore, after taking a subsequence k_0 we can assume that

$$d_{ij}^{k_{0,\ell}} \to \hat{d}_{ij}, \qquad |d_{ij}^{k_{0,\ell}} - d_{ij}^{k_{0,p}}| < \frac{1}{N} \qquad \forall \ell, p$$

Now consider the metric spaces $(X_{k_{0,\ell}}(N), d_{k_{0,\ell}}) := (\{x_i^{k_{0,\ell}}\}_{i=1}^{L(N)}, d_{k_{0,\ell}})$. These are the discretizations of $(X_{k_{0,\ell}}, d_{k_{0,\ell}})$ at scale N^{-1}. Show that we have

$$d_{GH}((X_{k_{0,\ell}}(N), d_{k_{0,\ell}}), (X_{k_{0,p}}(N), d_{k_{0,p}})) < \frac{1}{N}$$

and

$$d_{GH}((X_{k_{0,\ell}}(N), d_{k_{0,\ell}}), (X_{k_{0,\ell}}, d_{k_{0,\ell}})) < \frac{1}{N}$$

(11) Now repeat this for $N_1 > N = N_0$ to construct a subsequence $(X_{k_{1,\ell}}, d_{k_{1,\ell}})$ of the sequence $(X_{k_{0,\ell}}, d_{k_{0,\ell}})$. Show that by repeating this argument for a sequence $N_k \to \infty$ sufficiently quickly, and using a diagonal argument you can construct a subsequence (X_k, d_k) in $\mathcal{M}_{C,D}$ which is Cauchy. Now you can apply the completeness theorem in problem (9).

Next we will give some geometric criteria under which the compactness theorem applies.

Theorem 0.8. *Suppose (M_i, g_i) is a sequence of Riemannian manifolds with $\dim M_i = n, \mathrm{Ric}(g_i) \geqslant (n-1)kg_i$ and $\mathrm{diam} M_i \leqslant D$, for n, k, D fixed constants. Then, after taking a subsequence (M_i, g_i) converge to a limit (Z, d_Z) in the Gromov-Hausdorff sense.*

Note that if $k > 0$, then the diameter bound follows from Bonnet-Myers. In order to prove this theorem it suffices to prove that the counting number of any manifold (M^n, g) with $\mathrm{diam} M \leqslant D$ and $\mathrm{Ric}(g) \geqslant (n-1)kg$ is controlled.

(12) As a warm-up, prove that any set $K \subset \mathbb{R}^n$ with $\mathrm{diam} K \leqslant D$ can be covered by $C(n)\left(\frac{D}{\varepsilon}\right)^n$ balls of radius ε, where $C(n)$ is some universal constant that depends only on n.

(13) We now prove the general case. Fix $\varepsilon > 0$. Take $p_0 \in M$, and consider $B_\varepsilon(p)$. If $M \subset B_\varepsilon(p)$ then we're done. Otherwise, choose $p_1 \in \partial B_\varepsilon(p_0)$. If $M \subset B_\varepsilon(p_0) \cup B_\varepsilon(p_1)$ then we're done. Otherwise, choose $p_2 \in \partial(B_\varepsilon(p_0) \cup B_\varepsilon(p_1))$. Proceeding in this way we get points $p_1, \ldots, p_L$ such that

$$M \subset \bigcup_{i=0}^{L} B_\varepsilon(p_i)$$

and furthermore, $B_{\frac{\varepsilon}{2}}(p_i) \cap B_{\frac{\varepsilon}{2}}(p_j) = \emptyset$. We are going to show that L is bounded. By the Bishop-Gromov volume comparison theorem, for any $x \in M$ and $0 < r \leqslant D$ we have

$$1 \geqslant \frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}_k(B_r)} \geqslant \frac{\mathrm{Vol}(M)}{\mathrm{Vol}_k(B_D)}$$

Show that

$$\mathrm{Vol}(M) \geqslant L \min_{0 \leqslant i \leqslant L} \mathrm{Vol}(B_{\frac{\varepsilon}{2}}(p_i)) \geqslant L \frac{\mathrm{Vol}(M)\mathrm{Vol}_k(B_{\frac{\varepsilon}{2}})}{\mathrm{Vol}_k(B_D)}$$

and so

$$L \leqslant \frac{\mathrm{Vol}_k(B_D)}{\mathrm{Vol}_k(B_{\frac{\varepsilon}{2}})} = C(\varepsilon)$$

What does this give you when $k = 0$? How does this compare to the previous problem?

A major area in geometric analysis is to understand the properties of these limit spaces. In general the GH limits of manifolds with bounded Ricci curvature need not be manifolds. You should try to come up with some examples of Gromov-Hausdorff limits to help build your intuition. A typical question one might ask, for example, is whether GH limits of manifolds with bounded Ricci curvature are manifolds outside a small singular set. This is best understood when the manifolds are "non-collapsed", which is more or less a condition which prevents the Gromov-Hausdorff limit from being lower dimensional (e.g. $\mathrm{Vol}(M) \geqslant \nu > 0$, together with a diameter and lower Ricci bound, would suffice). There is a large body of work in this area, which is now called the Cheeger-Colding theory. A central tool in this theory is the use of harmonic coordinates, which are coordinate systems $(x^1, \ldots, x^n)$ such that $\Delta x^i = 0$. As a result, the properties of harmonic functions (both local and global) on Riemannian manifolds is of fundamental importance. We obviously won't have time to discuss any of these ideas, but if you're interested there is a (somewhat hard to find) book by Cheeger, called "Degeneration of Riemannian Metrics under Ricci Curvature Bounds", which covers a lot of the basic theory. The Riemannian Geometry text of Petersen also contains a nice treatment of some parts of the theory as well.