18.965: HOMEWORK 6

DUE: TUESDAY, DECEMBER 3

(1) Let $(M.g)$ be a complete Riemannian manifold, and $f : U \to \mathbb{R}$ a smooth function. Recall that we defined f to be a *generalized distance function* if $|\nabla f| \equiv 1$. Show that the Second fundamental form of the smooth submanifold $N_c = \{f = c\}$ with respect to the normal vector ∇f is $A = -\mathrm{Hess}_f$. Conclude that the shape operator of N_c is give by $(S_c)^i_k = -g^{ij}(\mathrm{Hess}_f)_{jk}$.

(2) We showed in class that the integral curves of $\mathrm{grad} f$ are geodesics. Prove the following generalization of the Bochner formula: Let $\gamma(t)$ be a geodesic with $\dot\gamma = \mathrm{grad} f$, and let S_t denote the shape operator of $N_t = \{f(x) = f(\gamma(t))\}$. Let $V = V(t)$ and $W = W(t)$ be parallel unit vector fields along $\gamma(t)$, such that W, V are orthogonal $\mathrm{grad} f$ (and hence $W(t), V(t) \in T_{\gamma(t)}N_t$. Then we have

$$\frac{d}{dt}\langle S_t V, W\rangle = R(\mathrm{grad} f, V, W, \mathrm{grad} f) + \langle S_t^2 V, W\rangle$$

where $S_t^2 = S_t \circ S_t : T_{\gamma(t)}N_t \to T_{\gamma(t)}N_t$. Using Cauchy-Schwarz show that if (M, g) has sectional curvature bounded from below by K, then

$$\frac{d}{dt}\langle S_t V, V\rangle \geqslant (\langle S_t V, V\rangle)^2 + K$$

and therefore $\langle S_t V, V\rangle$ is also a supersolution of the Ricatti equation. Let now $f(x) = r(x) = d(p, x)$. Using the preceding formula and the Ricatti comparison theorem show that if (M, g) has sectional curvature $K(g) \geqslant \kappa$ then

$$\langle SV(t), V(t)\rangle \geqslant -\frac{s'_\kappa(t)}{s_\kappa(t)}$$

(3) Using the previous problem, show that if J is a Jacobi field along an arc-length parametrized geodesic $\gamma(t)$, such that $J(0), \dot J(0)$ are orthogonal $\dot\gamma$, then

$$\nabla_{\dot\gamma} J(t) = -S_t J(t).$$

Using this and the preceding problem prove the *generalized Rauch theorem*: Suppose (M, g) has sectional curvature $K(g) \geqslant \kappa$ and $\gamma(t)$ is an arc-length parametrized geodesic with $\gamma(0) = p$, such that $\gamma(t)$

is disjoint from $\mathrm{Cut}(p)$ on $[0,T]$. Then, for any Jacobi field $J(t)$ along $\gamma(t)$ we have

$$\frac{d}{dt}\frac{|J(t)|}{s_\kappa(t)} \leqslant 0$$

on $[0,T]$.

(4) Show that the previous result implies the following theorem of Rauch (see e.g. do Carmo Chapter 10). Suppose (M,g) has sectional curvature $K(g) \geqslant \kappa$ and $\gamma(t)$ is an arc-length parametrized geodesic with $\gamma(0) = p$ such that $\gamma(t)$ is disjoint from $\mathrm{Cut}(p)$ on $[0,T]$. Let $J(t)$ be a Jacobi field along $\gamma(t)$ with $J(0) = 0$, $\langle \dot{J}(0), \dot{\gamma}(0)\rangle = 0$. Let (M_K, g_K) be the simply connected space form with constant curvature K, and let γ_K be a unit speed geodesic in (M_K, g_K). Let $J_K(t)$ be a Jacobi field along γ_K such that $J_K(0) = 0$, $\langle \dot{J}_K(0), \dot{\gamma}_K(0)\rangle = 0$, and $|\dot{J}_K(0)| = |\dot{J}(0)|$. Then

$$|J(t)| \leqslant |J_K(t)|$$

on $[0,T]$. This result says that geodesics tend to spread apart faster when the curvature is smaller (ie. more negative).

(5) Prove that the Laplacian has the following coordinate expression;

$$\Delta f = \frac{1}{\sqrt{\det g}}\partial_j\left(g^{ij}\sqrt{\det g}\,\partial_i f\right)$$

(6) It's possible to make sense of Δf for functions that have less than two derivatives worth of regularity (e.g. the distance function near the cut locus). We will explain one way of doing this. Let f be a *continuous* function defined on (M,g).

Definition 0.1. (i) *We say that a C^2 function u is a* upper barrier *for f at p if there is an open set $U \ni p$ such that $u(x) \geqslant f(x)$ for all $x \in U$, and $u(p) = f(p)$. We define the concept of a* lower barrier *similarly.*

(ii) *We say that $\Delta f(p) \leqslant c$ if, for all $\varepsilon > 0$ there is an upper barrier u_ε for f at p, and $\Delta u(p) \leqslant c + \varepsilon$. Using lower barriers we may similarly define $\Delta f(p) \geqslant c$, and hence also $\Delta f(p) = c$.*

(a) Show that if f is C^2 at p then $\Delta f = c$ in the barrier sense if and only if $\Delta f = c$ in the classical sense.

(b) Consider the function $f(x) = 1 - |x|$. Prove that $\Delta f(0) \leqslant 0$ in the sense of barriers. Show also that f does not admit a lower barrier at 0, and hence Δf has no lower bound in the sense of barriers.

(7) The function in the preceding problem is a model for the behavior of the distance function near the cut locus. In this problem we prove that the Laplacian comparison theorem extends over the cut locus, in the sense of barriers. Suppose (M, g) is a Riemannian manifold with $Ric(g) \geqslant (n-1)Kg$. Let $p \in M$, $r = d(p, \cdot)$. We have shown that if $x \notin \mathrm{Cut}(p)$ then we have

$$\Delta r(x) \leqslant \Delta_K r_K(|x|) = \frac{(n-1)s_K'(|x|)}{s_K(|x|)}$$

where $|x| = d(p, x)$. We will show this extends over the cut locus. Fix a point $x \in \mathrm{Cut}(p)$, and let $\gamma(t)$ be the length minimizing geodesic from p to x. The key is the following result. Fix $0 < \eta \ll 1$, and define

$$u_\eta(y) = \eta + d(\gamma(\eta), y) = d(p, \gamma(\eta)) + d(\gamma(\eta), y)$$

Show that $u_\eta(y)$ is an upper barrier for r at x, and that

$$\Delta u_\eta \leqslant \frac{(n-1)s_K'(d(\gamma(\eta), y))}{s_K(d(\gamma(\eta), y))}$$

Conclude that $\Delta r(x) \leqslant \frac{(n-1)s_K'(|x|)}{s_K(|x|)}$ on all of (M, g) in the barrier sense.

(8) In this problem we prove the maximum principle for uniformly elliptic operators. Let u be a C^2 function defined on a domain $\Omega \subset \mathbb{R}^n$. Define

$$Lu = a^{ij}u_{ij} + b^i u_i$$

where a^{ij} is a continuous $n \times n$-matrix valued function, and b^i is a continuous map taking values in $\mathbb{R}^n$. We furthermore assume this operator is *uniformly elliptic*; ie. there are constants $0 < \lambda < \Lambda < +\infty$ satisfying

$$0 < \lambda|\xi|^2 \leqslant a^{ij}\xi_i\xi_j < \Lambda|\xi|^2$$

for all vectors $\xi \in \mathbb{R}^n$. Note that, in local coordinates, the Laplacian on (M, g) is precisely of this form.

(i) Prove that if $Lu > 0$, then u cannot have a maximum in the interior of Ω. Similarly, if $Lu < 0$ then u cannot have an interior minimum.

(ii) Now suppose that f is a continuous function on a connected open set $\Omega \subset (M, g)$, $\Delta f \leqslant 0$ in the sense of barriers. We will show that if f has an interior minimum in Ω, then f is constant. Suppose not. By subtracting a constant we can assume the minimum of f is 0. Argue that there is a point $0 \in \Omega$ such that, for any $\delta > 0$ sufficiently small there is a point $p_\delta \in B_\delta(0) \backslash B_{\frac{1}{2}\delta}(0)$ with $f(p_\delta) > 0$.

(iii) Choosing coordinates and $\delta > 0$ we can assume that $0 = (0, \ldots, 0)$, and $p_\delta = (\delta, 0, \ldots, 0) \in \partial B_\delta(0)$. By continuity there is a $\tau > 0$ such that $0 < \frac{1}{2} f(p_\delta) \leqslant f(x)$ for all $x \in B_\tau(p_\delta)$. Consider the function

$$\varphi(x) = x_1 - A(x_2^2 + \cdots + x_n^2)$$

Argue that you can choose A so large that $\varphi(x) < 0$ whenever $d(x, p_\delta) > \tau$ and $x \in \partial B_\delta(0)$.

(iii) Define $\psi(x) = e^{a\varphi(x)} - 1$ where $a \gg 1$ is a constant to be determined. Show that you can choose a large so that $\Delta \psi \geqslant \varepsilon$ for some $\varepsilon > 0$.

(iv) Now consider $\hat{f} := f - \eta\psi$ for $\eta > 0$. Argue that you can choose $\eta > 0$ such that

$$\begin{aligned} \hat{f} &\geqslant 0 && \text{on } B_\delta(0) \\ \hat{f} &> 0 && \text{on } \partial B_\delta(0) \end{aligned}$$

(v) On the other hand, since $\hat{f}(0) = 0$, it follows that $\hat{f}$ has an interior minimum, while $\Delta \hat{f} \leqslant -\eta\varepsilon < 0$ in the sense of barriers. Show that this contradicts (i).