18.965: HOMEWORK 4

DUE: TUESDAY, NOVEMBER 5

1. The Learning Part

In this section we will discuss the notion of curvature for general vector bundles, and some applications. This is going to combine our discussions from the last two homeworks on vector bundles, and Lie groups in a way which I hope you find interesting. A good reference for this material is Taubes' book "Differential geometry: Bundles, Connections, Metrics and Curvature". Roughly, we will discuss Chapters 12 and 14 from Taubes' book, though some material from earlier chapters may also be useful. However, everything we're going to use (and indeed, essentially all the basic theory of vector bundles) boils down more or less to linear algebra. If you get stuck, write things out in a trivialization! Or look at a book! If you feel really stuck, send me an email.

First, recall that in Homework 2 we discussed the notion of a real vector bundle. In this homework we will use the notion of a *complex* vector bundle.

Definition 1.1. *A smooth complex vector bundle E of complex rank k over M is a manifold E with a surjective, continuous map $\pi : E \to M$ with the following properties*

(i) *For each $p \in M$ the set $\pi^{-1}(p)$ has the structure of a k-dimensional vector space over $\mathbb{C}$.*

(ii) *For each $p \in M$ there is an open neighborhood $U \subset M$ and a map $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$, called a local trivialization, such that*

$$\pi(\varphi^{-1}(x, v)) = x$$

and, for each $x \in U$ the map $\varphi^{-1}(x, \cdot) : \mathbb{C}^k \to \pi^{-1}(x)$ is a $\mathbb{C}$-linear isomorphism of vector spaces.

(iii) *For open sets $U, V \subset M$, with $U \cap V \neq \emptyset$, if φ_U, φ_V are local trivializations, then*

$$\varphi_U \circ \varphi_V^{-1} : U \cap V \to GL(k, \mathbb{C})$$

is a smooth map.

Definition 1.2. *Let $E \to M$ be a real vector bundle. An almost complex structure on E is a section $J \in \mathrm{End}(E)$ such that $J^2 = -I_E$ where $I_E \in \mathrm{End}(E)$ denotes the identity map.*

(1) Show that a complex vector bundle of complex rank k can be regarded as a real vector bundle of rank $2k$ together with an almost complex structure. Prove conversely that any real complex vector bundle with an almost complex structure has even rank $2k$ and can be given the structure of a complex vector bundle of complex rank k.

Note that if J is an almost complex structure then so is $-J$. If E is the complex vector bundle with almost complex structure J we denote by $\overline{E}$ the complex vector bundle with almost complex structure $-J$. Equivalently, $\overline{E}$ is the vector bundle constructed from E by taking the complex conjugate of the trivializations; $\overline{\varphi}_U : \pi^{-1}(U) \to \mathbb{C}$.

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $G \subset GL(k, \mathbb{F})$ be a Lie group. We introduce the notion of a structure group;

Definition 1.3. *We say that a $\mathbb{F}$-vector bundle E of rank k over M (where $\mathbb{F} = \mathbb{R},$ or $\mathbb{C}$) has* structure group $G \subset GL(k, \mathbb{F})$ *if M admits a cover by open sets U_α such that for each U_α there is a local trivialization $\varphi_\alpha : \pi^{-1}(U) \to U \times \mathbb{F}^k$ with the following property: if $U_\alpha \cap U_\beta \neq 0$ then the transition functions are valued in G; ie.*

$$\varphi_\alpha \circ \varphi_\beta^{-1} : U_\alpha \cap U_\beta \to G \subset GL(k, \mathbb{F}) \tag{1}$$

For simplicity, a bundle E with structure group G is always equipped with a maximal atlas of local trivializations satisfying (1). *A local frame $\{\sigma_1, \ldots, \sigma_k\}$ of E induced by an element of this atlas is called a G-frame.*

Here is an easy way to reduce the structure group.

Definition 1.4. (a) *Let E be a real vector bundle over M. A metric on E is a smooth section $H \in E^* \otimes E^*$ which, over each point $p \in M$ induces an inner product on the fiber $E_p = \pi^{-1}(p)$.*

(b) *Let E be a complex vector bundle over M. A hermitian metric on E is a smooth section $H \in E^* \otimes \overline{E^*}$ which, over each point $p \in M$ induces a non-degenerate hermitian inner product on the fiber $E_p = \pi^{-1}(p)$*

Using local trivializations and a partition of unity it is not hard to show that any real (resp. complex) vector bundle admits a metric (resp. hermitian metric).

(2) Prove that if E is a real (respectively complex) vector bundle of rank k which admits a metric (resp. hermitian metric) then the structure group reduces to $O(k)$ (resp. $U(k)$).

Definition 1.5. *Suppose E is a vector bundle with structure group G. We say that a connection $\nabla = d + A$ is compatible with the G structure if parallel transportation along any curve takes a G-frame to a G-frame.*

(3) Here is a simpler way to view the compatibility of $d + A$ with the G structure. Let $\{\sigma_1, \ldots, \sigma_k\}$ get a local G-frame for E near p, and

$(x^1, \dots, x^n)$ be local coordinates for M centered at p. Write the connect as $\nabla = d + A$ where A is now a endomorphism values 1 form, which we can write in the general form

$$A = A_i^{\alpha}{}_{\beta} dx^i \otimes \sigma_\alpha \otimes (\sigma_\beta)^*$$

where $1 \leqslant \alpha, \beta \leqslant k$. Let $\gamma(t)$ be a smooth curve, $\gamma(0) = p$ and $P(\gamma)_0^t$ denote parallel transportation along γ from $\gamma(0)$ to $\gamma(t)$. Since ∇ is compatible with the G structure, to $t \in (-\varepsilon, \varepsilon)$ sufficiently small we have

$$P(\gamma)_0^t \sigma_\alpha = (g(t))_\alpha^\beta \sigma_\beta$$

where $(g(t))_\alpha^\beta$ is a $k \times k$ matrix in G. Using this, prove that for each i we have $A_i^{\alpha}{}_{\beta} \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G.

(4) Consider the vector bundle $TM \to M$. Let g be a Riemannian metric on M. The choice of a Riemannian metric gives TM the structure of an $O(n)$-bundle. Show that the Levi-Civita connection ∇ is compatible with the $O(n)$ structure.

We now define the curvature of a connection ∇ as the failure of covariant derivatives to commute.

Definition 1.6. *Let $E \to M$ be a vector bundle (over $\mathbb{R}$ or $\mathbb{C}$), and $\nabla = d + A$ a connection. We define the curvature in local coordinates $(x^1, \dots, x^n)$ as*

$$F_{ij} = [\nabla_i, \nabla_j] dx^i \otimes dx^j \in \mathrm{End}(E) \otimes T^*M \otimes T^*M$$

Note that the curvature is anti-symmetric since $F_{ij} = -F_{ji}$, thus we may (and will) view F as a section of $\mathrm{End}(E) \otimes \Lambda^2 T^*M$.

In homework 3 we defined the de Rham differential as a map $d : \Lambda^k T^*M \to \Lambda^{k+1} T^*M$. Given a choice of connection there is a natural extension of this operator to a map

$$d_A : E \otimes \Lambda^k T^*M \to E \otimes \Lambda^{k+1} T^*M$$

defined in the following way: for a section σ of E and a 1-form β we define

$$d_A(\sigma \otimes \beta) = d_A \sigma \wedge \beta + \sigma \otimes d\beta$$

We can therefore compose the operators $d_A : E \to E \otimes T^*M$ and $d_A : E \otimes T^*M \to E \otimes \Lambda^2 T^*M$.

(5) Show that the curvature $F_A \in \mathrm{End}(E) \otimes \wedge^2 T^*M$ satsifies

$$d_A^2 \sigma = F_A \wedge \sigma$$

In particular, derive the formula $F_A = dA + A \wedge A$. Show that if E has structure group G, then in a G-frame we have

$$F_A \in \mathfrak{g} \otimes \wedge^2 T^*M$$

That is, F_A is valued in the Lie algebra of G.

Recall that in homework 3 we saw that the de Rham differential gave rise to the de Rham cohomology

$$H^k(M,\mathbb{R}) = \frac{\{\mathrm{Ker} d : \Lambda^k T^*M \to \Lambda^{k+1}T^*M\}}{\{\mathrm{Im} d : \Lambda^{k-1}T^*M \to \Lambda^k T^*M\}}$$

and we saw (at least in a simple example) that these groups encode some topological information. For bundle valued forms, the previous exercise shows that there is *no* natural extension of this theory *unless* E admits a connection with $F_A = 0$. Such a bundle is called a *flat* bundle. Nevertheless, we can get interesting topological data about our manifold from non-flat vector bundles.

(6) Let E be a vector bundle and d_A be a connection on E. Recall that d_A induces a natural connection on $\mathrm{End}(E)$ and hence using the above construction we have natural operators

$$d_A : \mathrm{End}(E) \otimes \Lambda^k T^*M \to \mathrm{End}(E) \otimes \Lambda^{k+1}T^*M.$$

locally, a section of τ of $\mathrm{End}(E)$ can be written as a matrix. Show that, in this notation the covariant derivative on $\mathrm{End}(E)$ can be written as

$$d_A\tau = d\tau + A\tau - \tau A$$

where $A\tau$ denotes left matrix multiplication by A, while τA denotes right matrix multiplication by A. Conclude that if $\hat{\tau} \in \mathrm{End}(E) \otimes \Lambda^k T^*M$ then

$$d_A\hat{\tau} = d\hat{\tau} + A \wedge \hat{\tau} - \hat{\tau} \wedge A \in \mathrm{End}(E) \otimes \Lambda^{k+1}T^*M$$

Here we make use of the natural composition (ie. matrix multiplication) on endomorphism valued forms. If we have endomorphism $\tau_0, \tau_1 \in \mathrm{End}(E)$, and α_0 is a p-form, and α_1 is a q-form, then

$$(\tau_0 \otimes \alpha_0) \wedge (\tau_1 \otimes \alpha_1) = \tau_0\tau_1 \otimes (\alpha_0 \wedge \alpha_1)$$

where $\tau_0\tau_1$ denotes the composition.

(7) Using the previous problem, show that $d_A F_A = 0$. This is the *second Bianchi identity*. You proved a special case of this on Homework 3, (do Carmo, chapter 4, problem 7), when $E = TM$, and d_A is the Levi-Civita connection on (M, g).

(8) Let E be a vector bundle (over $\mathbb{R}$ or $\mathbb{C}$). Show that if $\nabla^0 = d + A_0$ and $\nabla^1 = d + A_1$ are two connections on E, then

$$\nabla^0 - \nabla^1 = A_0 - A_1$$

is a globally defined section of $\mathrm{End}(E) \otimes T^*M$. In particular, while the connection coefficient is *not* a globally defined section of $\mathrm{End}(E) \otimes T^*M$, the difference of any two connection coefficients *is* globally defined, and transforms as a section of $\mathrm{End}(E) \otimes T^*M$. In particular, the space of connections on E is an affine space modeled on the global sections $\Gamma(M, \mathrm{End}(E) \otimes T^*M)$

Recall that there is a well-defined trace map $\mathrm{Tr} : \mathrm{End}(E) \to C^\infty(M)$ given explicitly in a local frame by

$$\mathrm{Tr}\left(T^\alpha_\beta \sigma_\alpha \otimes (\sigma_\beta)^*\right) = \sum_\alpha T^\alpha_\alpha$$

This is well defined since under a change of frame $T \mapsto g^{-1}Tg$, and $\mathrm{Tr}(g^{-1}Tg) = \mathrm{Tr}(Tgg^{-1}) = \mathrm{Tr}(T)$. It also extends in the obvious way to a map

$$\mathrm{Tr} : \mathrm{End}(E) \otimes \Lambda^k T^*M \to \Lambda^k T^*M.$$

(9) Let E be a vector bundle with a connection d_A. Show that, for any endomorphism $\tau \in \mathrm{End}(E)$ we have

$$d\mathrm{Tr}(\tau) = \mathrm{Tr}(d_A \tau)$$

Hint: The point here is that if A, B are $n\times n$ matrices then $\mathrm{Tr}([A, B]) = 0$.

(10) Let E be a vector bundle with a connection d_A, and let F_A be the curvature 2-form. Define

$$ch_1(A) = \sqrt{-1}2\pi \mathrm{Tr}(F_A) \in \Lambda^2 T^*M$$

Using the previous problems show that $dch_1(A) = 0$. Hence $c_1(A)$ defines a cohomology class $[ch_1(A)] \in H^2(M, \mathbb{C})$– in fact, it turns out $[ch_1(A)] \in H^2(M, \mathbb{R})$ as we will see below. Furthermore, show that if $d + A_0, d + A_1$ are two connections on E, then

$$ch_1(A_1) = ch_1(A_0) + d\mathrm{Tr}(A_1 - A_0).$$

Since $A_1 - A_0$ is a globally defined endomorphism, $\mathrm{Tr}(A_1 - A_0)$ is a smooth function and so $[ch_1(A_0)] = [ch_1(A_1)]$. Thus, the bundle E gives rise to a cohomology class $ch_1(E) \in H^2(M, \mathbb{R})$ which is independent of the choice of connection. This is called the first Chern class.

A few remarks are in order. First of all, this construction is not interesting for real vector bundles. The reason is that we can always find a metric H on E, and a connection d_A compatible with the resulting $O(n)$ structure. Therefore the curvature $F_A \in \mathfrak{o}(n) \otimes \Lambda^2 T^*M$. But on the other hand, any matrix in $B \in \mathfrak{o}(n)$ has $\mathrm{Tr}(B) = 0$. Thus, for a real vector bundle $ch_1(E) = 0$. However, this construction applied to complex vector bundles produces interesting non-trivial cohomology classes. Note that if E is complex with a hermitian metric then the curvature is valued in $\mathfrak{u}(n) \otimes \Lambda^2 T^*M$ where $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$. Since $\mathfrak{u}(n)$ consists of skew-hermitian matrices, it follows that for any $B \in \mathfrak{u}(n)$ the diagonal entries of B are pure imaginary, and hence we have $\mathrm{Tr}(B) \in \sqrt{-1}\mathbb{R}$. This explains why, for any complex vector bundle E, we introduced the factor of $\sqrt{-1}$ in the definition of $ch_1(E)$; it is necessary to have $ch_1(E)$ a real 2-form and hence $[ch_1(A)] \in H^2(M, \mathbb{R})$.

Secondly, there is a more general version of this construction, by defining

$$ch_k(A) = \left(\frac{\sqrt{-1}}{2\pi}\right)^k \operatorname{Tr}(F_A \wedge \cdots \wedge F_A) \in \Lambda^{2k} T^* M$$

where we wedge F_A with itself k-times. It is not too hard to show that $ch_k(A)$ are also close $2k$-forms whose cohomology class is independent of the choice of connection.

2. The Practicing Part

(1) do Carmo, Chapter 5, problem 3
(2) do Carmo, Chapter 7, problems 1, 2, 6, 8, 12