18.965: HOMEWORK 3

DUE: TUESDAY, OCTOBER 22

1. The Learning Part

A special class of manifolds are Lie groups.

Definition 1.1. *A Lie group is a manifold M with the structure of a group such that*

- *The multiplication map $M \times M \ni (a, b) \mapsto a \cdot b \in M$ is smooth.*
- *The inverse map $M \ni a \mapsto a^{-1} \in M$ is smooth.*

(1) Prove that $GL(n, \mathbb{R})$, the set of invertible $n \times n$ matrices with real coefficients is a Lie group. Note that $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and hence has a natural smooth structure.

(2) Prove that $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | \det A = 1\}$ is a Lie group (**Hint**: Recall Homework 1).

(3) More generally, prove that if G is a Lie group, and $H \subset G$ is a subgroup which is also a smooth submanifold of G, then H is a Lie group. H is then called a **Lie subgroup**

The solutions to problems (2), and (3) give rise to lots of examples. Many of your favorite matrix groups, like $O(n), SO(n), U(n), SU(n)$ are all examples of Lie groups.

(4) A Lie group G comes equipped with several natural maps $G \to G$. For each $g \in G$ we define the left/right multiplication maps by

$$L_g(a) = g \cdot a, \qquad R_g(a) = a \cdot g.$$

We define a vector field $V \in \Gamma(G, TG)$ to be left (resp. right) invariant if $V(g \cdot a) = dL_g V(a)$ (resp. $V(a \cdot g) = dR_g V(a)$). Show that the set of left/right invariant vector fields is isomorphic to $T_1 G$, where $1 \in G$ is the identity.

Remark 1.2. Note that we can make the same definition for any tensor, like a 1-form.

(5) Prove the following general result. If $f : M \to N$ is a smooth map, and $X, Y \in \Gamma(M, TM)$, then $df[X, Y] = [dfX, dfY]$. Be careful! dfX, dfY may not be defined in a open neighborhood of a point $f(p) \in N$.

By problem (5), the Lie bracket of left invariant vector fields is again left invariant. This induces a lie bracket on T_1G as follows. If X, Y are left invariant, let $x = X(1), y = Y(1)$ so that $x, y \in \mathfrak{g}$. We define $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by.

$$[x, y] = [X, Y](1)$$

Definition 1.3. *If G is a Lie group, we define the Lie algebra of G to be $\mathfrak{g} = T_1G$, where $1 \in G$ denotes the identity, equipped with the Lie bracket $[\cdot,\cdot]$ induced by the bracket on left invariant vector fields.*

Remark 1.4. Why not use right invariant vector fields instead? You can check that the bracket induced by right invariant vector fields is precisely *minus* the bracket induced by left invariant vector fields.

(7) Show that $gl(n, \mathbb{R})$, the Lie algebra of $GL(n, \mathbb{R})$, is nothing but the $n \times n$ matrices with real entries. Show that the Lie bracket on matrices A, B is

$$[A, B] = AB - BA$$

(8) If $H \subset G$ is a Lie subgroup, then $\mathfrak{h} := T_1H \subset T_1G = \mathfrak{g}$. Prove that the Lie bracket on $\mathfrak{h}$ agrees with the restriction of the Lie bracket on $\mathfrak{g}$ (in particular, the Lie bracket on $\mathfrak{g}$ preserves $\mathfrak{h}$). This, in particular, shows that the Lie bracket on Lie subgroups of $GL(n, \mathbb{R})$ agrees with the commutator.

2. The practicing part

(9) do Carmo, Chapter 1, problem 7

(10) do Carmo, Chapter 3, problem 3

(11) do Carmo, Chapter 4, problem 1

(12) do Carmo, Chapter 4, problem 7

(13) do Carmo, Chapter 4, problem 10

(14) This problem introduces the de Rham differential. Define an operator $d : \Lambda^k T^*X \to \Lambda^{k+1} T^*X$ by the following axioms.

(i) d is $\mathbb{R}$-linear.

(ii) for smooth functions $f \in \Lambda^0 T^*X$, df is the 1-form $df(V) = V(f)$.

(iii) for smooth function $d(df) = 0$.

(iv) for any p form α, and $k - p$ form β we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p(\alpha \wedge d\beta).$$

(a) Show that if $\alpha = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ in local coordinates, then

$$d\alpha = \sum_j \frac{\partial f}{\partial x_j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Note that this formula, by linearity, specifies the action of d in general (though it's not completely obvious that this formula glues to a globally defined operator).

(b) Show that $d(d\alpha) = 0$ for all α.

(c) Show that, if $X_0, \ldots, X_k$ are smooth vector fields on M, then for any smooth k-form ω we have

$$\begin{aligned} d\omega(X_0, X_1, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ &+ \sum_{0 \leqslant i \leqslant j \leqslant k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \end{aligned}$$

where $\hat{X}_i$ means that we omit X_i.

(d) Show that there is a 1-form α on S^1 so that $d\alpha = 0$, but $\alpha \neq df$ for any smooth function $f : S^1 \to \mathbb{R}$. In other words

$$\frac{\text{Kernel } d : \Lambda^1 T^* S^1 \to \Lambda^2 T^* S^1}{\text{Image } d : \Lambda^0 T^* S^1 \to \Lambda^1 T^* S^1} \neq 0.$$

What if we replace S^1 with $\mathbb{R}$?

(e) More generally, show that if M is any compact manifold, and α is a 1-form so that $\alpha(p) \neq 0 \in T_p^* M$ for any $p \in M$, the $\alpha \neq df$ for any smooth function f.