18.965: HOMEWORK 2

DUE: TUESDAY, OCTOBER 8

1. The Learning Part

This will develop the basics of vector bundles, generalizing the discussion of the tangent bundle from class.

Definition 1.1. *A smooth vector bundle E of rank k over M is a manifold E with a surjective, continuous map $\pi : E \to M$ with the following properties*

(i) *For each $p \in M$ the set $\pi^{-1}(p)$ has the structure of a k-dimensional vector space over $\mathbb{R}$.*

(ii) *For each $p \in M$ there is an open neighborhood $U \subset M$ and a map $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a local trivialization, such that*

$$\pi(\varphi^{-1}(x, v)) = x$$

and, for each $x \in U$ the map $\varphi^{-1}(x, \cdot) : \mathbb{R}^k \to \pi^{-1}(x)$ is a linear isomorphism of vectorspaces.

(iii) *For open sets $U, V \subset M$, with $U \cap V \neq \emptyset$, if φ_U, φ_V are local trivializations, then*

$$\varphi_U \circ \varphi_V^{-1} : U \cap V \to GL(k, \mathbb{R})$$

is a smooth map.

Definition 1.2. *If $E \to M$ is a smooth vector bundle, and $U \subset M$ an open set, a section of E over U is a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma(p) = p$. We denote by $\Gamma(U, E)$ the collection of smooth sections.*

(1) Let $U \subset M$ be an open set. Prove that a trivialization $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is equivalent to a choice of k sections $\{\sigma_1, \ldots, \sigma_k\} \in \Gamma(U, E)$ such that, for each $p \in U$, $\{\sigma_1(p), \ldots, \sigma_k(p)\}$ spaces $\pi^{-1}(p)$. A collection of sections $\{\sigma_1, \ldots, \sigma_k\}$ as above is called a local **trivialization** or a local **frame** for E.

(2) Suppose $\{U_\alpha\}_{\alpha \in A}$ is a cover of M by open sets. Suppose that for each $\alpha, \beta \in A$ we have a smooth map

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}),$$

which we call a *transition function* satisfying the following:

(i) $h_{\alpha\alpha} = Id$ where $Id \in GL(k, \mathbb{R})$ denotes the identity.

(ii) $h_{\alpha\beta} h_{\beta\alpha} = Id$,

(iii) For each triple $U_\alpha, U_\beta, U_\gamma$ of open sets we have

$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = Id$$

on $U_\alpha \cap U_\beta \cap U_\gamma$.

Prove that this data is equivalent to a vector bundle $E \to M$ of rank k. (**Hint**: Take the disjoint union $U_\alpha \times \mathbb{R}^k$ and use the $h_{\alpha\beta}$ to define an equivalence relation on the overlaps).

(3) Given a vector bundle $\pi : E \to M$ we want to define a bundle $\pi^* : E^* \to M$ whose fiber $(\pi^*)^{-1}(p)$ is the dual of $\pi^{-1}(p)$. Explain how to make E^* into a vector bundle.

(4) Let $\pi_E : E \to M$, $\pi_F : F :\to M$ be vector bundles, and denote by $E_p = \pi_E^{-1}(p)$, $F_p = \pi_F^{-1}(p)$. Explain briefly how to define $E \otimes F, E \wedge E, \mathrm{Hom}(E, F)$ as vector bundles whose fibers over $p \in M$ are, respectively $E_p \otimes F_p, E_p \wedge E_p, \mathrm{Hom}(E_p, F_p)$. If E has rank r, what are the transition functions of $\wedge^r E$?

(5) Prove that a manifold M of dimension n is orientable if and only if $\wedge^n T^*M$ has a global, non-vanishing section.

Next we will define the notion of a connection on a vector bundle E.

Definition 1.3. *A connection on a vector bundle $E \to M$ is a map*

$$\nabla : E \to E \otimes T^*M$$

such that, for sections $\sigma_1, \sigma_2 \in \Gamma(M, E)$, and a smooth function $f : M \to \mathbb{R}$ we have

(i) $\nabla(\sigma_1 + \sigma_2) = \nabla\sigma_1 + \nabla\sigma_2$.
(ii) $\nabla(f\sigma_1) = df \otimes \sigma_1 + f\nabla\sigma_1$

(6) Suppose ∇ is a connection on E. Let $U \subset M$ be an open set such that there are local coordinates $(x^1, \dots, x^n)$ on U and a local frame $\{\sigma_1, \dots, \sigma_r\}$ for E over U. Show that ∇ is determined by

$$\nabla_{\frac{\partial}{\partial x_i}} \sigma_\alpha = A_i^\beta{}_\alpha \sigma_\beta$$

where $A = A_i^\beta{}_\alpha dx^i$ can be viewed as a locally defined 1-form valued in $\mathrm{Hom}(E, E)$: that is, a locally defined section of $Hom(E, E) \otimes T^*M$. For this reason poeple often write

$$\nabla = d + A$$

where A is the above matrix valued one-form. However, we must remember that this expression only makes sense once we choose a frame, as explained in the next problem.

(7) Suppose $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_r\}$ is another frame for E over U. We can write
$$\tilde{\sigma}_\gamma = h_\gamma^\alpha \sigma_\alpha$$
or more schematically $\tilde{\sigma} = h\sigma$ where h is the change of basis matrix. Then we define
$$\nabla_{\frac{\partial}{\partial x_i}} \tilde{\sigma}_\alpha = \tilde{A}_i^\beta{}_\alpha \tilde{\sigma}_\beta.$$
Show that
$$\tilde{A} = (dh)h^{-1} + hAh^{-1}$$
In particular, A does **not** define a section of $Hom(E, E) \otimes T^*M$.

(8) A connection on a vector bundle E induces a connection on E^* in the following way. If σ is a local section of E, and τ is a local section of E^*, then $\tau(\sigma)$ is a locally defined smooth function. Therefore, we can impose the Leibniz rule
$$d\tau(\sigma) = (\nabla\tau)(\sigma) + \tau(\nabla\sigma)$$
If $\nabla = d + A$ is the connection on E, what is the connection on E^*? This problem is important, since it tells us how to covariantly differentiate one-forms with respect to the Levi-Civita connection.

(9) If E, F are bundles with connections ∇^E, ∇^F we get an induced connection on $E \otimes F$ as follows. If σ is a local section of E and τ is a local section of F then
$$\nabla^{E\otimes F} \sigma \otimes \tau = (\nabla^E \sigma) \otimes \tau + \sigma \otimes \nabla^F \tau$$
This in particular implies we know how to differentiate $\wedge^r E$ etc.

(10) Let (M, g) be a Riemannian manifold. Explain why g can be viewed as a section of $T^*M \otimes T^*M$. Show that, with respect to the Levi-Civita connection we have
$$\nabla g = 0$$

2. The practicing part

(1) Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.

(2) Show that if γ is a geodesic, then $g(\gamma', \gamma')$ is constant.

(3) Let (M, g), and (N, h) be two Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an isometry if $f^*h = g$. Show that if f is an isometry, and γ is a geodesic in M, then $f(\gamma)$ is a geodesic in N.

(4) Consider the round sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ with metric induced by restricting the euclidean metric. Describe the geodesics. (**Hint**: there is a large isometry group.)

(5) do Carmo, Chapter 2, problem 3.

(6) Let $S \subset \mathbb{R}^3$ be a surface with metric induced by restricting the euclidean metric. Show that a curve $\gamma(t)$ in $\mathbb{R}^3$ whose image lies entirely on S is a geodesic of (S, g) if and only if $\ddot{\gamma}(t)$ is orthogonal to $T_{\gamma(t)}S$ for all t.

(7) Let $\mathcal{H}^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Equip $\mathcal{H}^2$ with a metric

$$g_\alpha := \frac{dx^2 + dy^2}{y^2}.$$

(i) Show that the lines $x = constant$ are geodesics. (**Hint**: use symmetry!)

(ii) Identify $z = x + iy$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, consider the map $z \mapsto \frac{az+b}{cz+d}$. Show that this defines an isometry of $(\mathcal{H}^2, g_2)$.

(iii) By using the action of $SL(2, \mathbb{R})$, show that the geodesics of $(\mathcal{H}^2, g)$ are the lines with $x = constant$, or semi-circles with center lying on the x-axis.