18.786. Algebraic Number Theory - Galois Representations (Spring 2014)[1]

- Instructor: Sug Woo Shin (`swshin@math.mit.edu`)
- Class hours: TR 9:30-11
- Classroom: E17-133
- Instructor's Office: E18-426
- Office hours: F 3-4 or by appointment
- Course website:
- Homework: Weekly (exemption: MIT/Harvard grad students who passed the quals; but they are still welcome to submit their homework)
- Exams: None
- Grade (for those who need it): Solely based on homework

What to expect The goal of this course is to understand how to prove the automorphy of Galos representations, as initiated by Wiles and substantially developed by many others, mainly in the two dimensional case perhaps under some simplifying hypotheses. Along the way I hope to get to some details in Wiles's proof of Fermat's Last Theorem (as a consequence of his work on the Shimura-Taniyama-Weil conjecture that "every elliptic curve over $\mathbb{Q}$ is modular", which can be rephrased as the automorphy of certain Galois representations). Prerequisites are basic knowledge of algebraic number theory and commutative algebra. Familiarity with class field theory will be helpful but not strictly required. There is little overlap with what used to be taught in the old 18.786 before the recent revision of the catalogue.

Tentatively a nonempty subset of the following topics will be discussed.

- Basic facts about Galois representations
- (Hilbert) modular forms, automorphic representations and Jacquet-Langlands correspondence for $GL(2)$,
- Langlands-Kottwitz method for $GL(2)$ (from automorphic to Galois) - very briefly
- Deformation of Galois representations
- Taylor-Wiles-Kisin method (from Galois to automorphic)
- Application to modularity of elliptic curves over $\mathbb{Q}$ and Fermat's Last Theorem

Prerequisite

- Reasonable knowledge of general algebra, algebraic number theory, and commutative algebra. It would be helpful to be familiar with elliptic curves, modular forms, and class field theory. (If you are not familiar, it's a good idea to start to teach yourself from early on. Here are some suggestions. For general background on algebraic number theory and class field theory, there are many standard texts but Milne's lecture notes might be a friendly option for self-study. A canon is Cassels-Fr"ohlich "Algebraic Number Theory"[2]. You only need to understand the main theorems and some of the essential tools such as Galois cohomology. A natural place to read about Galois cohomology (and local fields, of course) is [Ser79]. For elliptic curves and modular forms, a quick introduction with a view towards our goal is found in [Sai13] or survey articles in the volumes [CSS97], [CY97], [Mur95], and [CR01].)

Some useful references There is no single textbook, but I intend to closely follow Toby Gee's notes below (with several elaborations and digressions) for a good part of the course. Also see the references therein.

`http://www2.imperial.ac.uk/~tsg/Index_files/ArizonaWinterSchool2013.pdf`

The original papers on Fermat's Last Theorem and modularity of elliptic curves over $\mathbb{Q}$ are [Wil95] and [TW95]; a complete proof of the modularity appeared in [BCDT01]. (See [Ser96], [Oes96] and [Dar99] for Bourbaki talks and an announcement in Notices of AMS.) On the work of Wiles and Taylor-Wiles, and more, there are books containing many survey articles: [CSS97], [CY97], [Mur95], and [CR01]. Some more survey/expository articles are [Kha10], [Kis07b], [Kis07a], [Rib95], and [Tay04].

[1] Last update: February 5, 2014

[2] Available on Amazon: http://www.amazon.com/Algebraic-Number-Theory-William-Cassels/dp/0950273422

Recently T. Saito published a friendly introductory book [Sai13]. Also visit Standford's learning seminar webpage:

`http://math.stanford.edu/~conrad/modseminar/`

More references for specific materials will be added on the course website as we proceed. It should be clear to you but I don't mean that you should read most of the above to follow the course. (I have not done so.) It suffices to just pick up the references that work for you.

Other remarks

- This course is designed as a graduate course. An undergraduate with proper background knowledge is welcome to attend my class but keep in mind that you may be required to invest a lot to follow the course.

References

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939 (electronic). MR 1839918 (2002d:11058)

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[CSS97] Gary Cornell, Joseph H. Silverman, and Glenn Stevens (eds.), *Modular forms and Fermat's last theorem*, Springer-Verlag, New York, 1997, Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, August 9–18, 1995. MR 1638473 (99k:11004)

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[Oes96] Joseph Oesterlé, *Travaux de Wiles (et Taylor, ...). II*, Astérisque (1996), no. 237, Exp. No. 804, 5, 333–355, Séminaire Bourbaki, Vol. 1994/95. MR 1423631 (97m:11077)

[Rib95] Kenneth A. Ribet, *Galois representations and modular forms*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 4, 375–402. MR 1322785 (96b:11073)

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[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 (82e:12016)

[Ser96] ———, *Travaux de Wiles (et Taylor, ...). I*, Astérisque (1996), no. 237, Exp. No. 803, 5, 319–332, Séminaire Bourbaki, Vol. 1994/95. MR 1423630 (97m:11076)

[Tay04] R. Taylor, *Galois representations*, Ann. Fac. Sci. Toulouse **13** (2004), 73–119.

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[Wil95] Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551. MR 1333035 (96d:11071)