18.786. (Spring 2014) Problem set # 9 (due Thu Apr 29)

1. Do [Gee, Exercise 4.25].

2. In [Gee, 5.4] the proof of Theorem 5.1 is reduced to the case where five conditions are satisfied. Justify the reduction step for any two of the first four conditions there.[1] Feel free to appeal to Facts 4.22 and 4.26 of the notes.

3. Let the notation be basically as in [Gee, 5.5]. (In particular $\overline{\rho} : G_F \to GL_2(\mathbb{F})$ is absolutely irreducible. If ρ_0 has weight $(\mathbf{k}, \eta)$ then the Hecke algebra $\mathbb{T}_U$ is made out of the automorphic forms of the same weight $(\mathbf{k}, \eta)$.) Let $\rho_{\mathfrak{m}}^{\mathrm{mod}} : G_F \to GL_2(\mathbb{T}_{U,\mathfrak{m}})$ be the representation constructed in [Gee, 5.2]. Let

$$\mathscr{S} = (F, T, \overline{\rho}, \chi, \{\mathscr{D}_v\}_{v \in T})$$

be a global deformation problem (deformation of $\overline{\rho}$ unramified outside T with determinant χ and local deformation problem $\mathscr{D}_v$ at $v \in T$). Denote by $R := R_{\mathscr{S}}^{\mathrm{univ}}$ the universal deformation ring (without framing). By checking that $\rho_{\mathfrak{m}}^{\mathrm{mod}}$ is a deformation of $\overline{\rho}_{\mathfrak{m}}$ of type $\mathscr{S}$, one obtains a surjection $R \to \mathbb{T}_{U,\mathfrak{m}}$. (No need for you to check this in your solution.) Let $\rho : G_F \to GL_2(\mathcal{O}) \subset GL_2(L)$ is a deformation of $\overline{\rho}$ of type $\mathscr{S}$, corresponding to $f_\rho : R \to \mathcal{O}$. If f_ρ factors through the surjection $R \to \mathbb{T}_{U,\mathfrak{m}}$, show that ρ is automorphic, i.e. arises from a regular automorphic cuspidal representation of $GL_2(\mathbb{A}_F^\infty)$.[2]

4. (This exercise will be useful next week.) Let A be a commutative ring with unity. For an A-module M define
$$\mathrm{Supp}_A M := \{\mathfrak{p} \in \mathrm{Spec}\, A \,:\, M_{\mathfrak{p}} \neq 0\}.$$

   (a) Show that if M is finitely generated then $\mathrm{Supp}_A M$ is equal to $\mathrm{Spec}\,(A/\mathrm{Ann}_A(M))$ (as a set), where $\mathrm{Ann}_A(M)$ denotes the annihilator ideal of M in A.

   (b) Suppose that R and T are commutative rings with unity with a surjection $\xi : R \twoheadrightarrow T$. Let S be a finitely generated ideal over T, viewed also as an R-module. If $\mathrm{Supp}_R S = \mathrm{Spec}\, R$ and T is reduced[3] then deduce that ξ induces an isomorphism $R^{\mathrm{red}} \simeq T$, where R^{red} is the maximal reduced quotient of R.

References

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.

[1] Again you need not deal with all the four conditions as there won't be extra credit.
[2] I sketched the idea in class but please write out the details.
[3] Thanks to Ka Yu Tam for reporting that I forgot this condition in an earlier version.