18.786. (Spring 2014) Problem set # 8 (due Thu Apr 17)

Let K be a p-adic field. As usual $\mathrm{Irr}(GL_n(K))$ denote the set of isomorphism classes of irreducible admissible representations of $GL_n(K)$. Let D be a division algebra over K with center K such that $[D : K] = n^2$.

1. Let Γ be a locally profinite group such that $\Gamma/Z(\Gamma)$ is compact (where $Z(\Gamma)$ denotes the center of Γ). Show that every irreducible admissible representation of Γ is finite-dimensional. Deduce that the same is true for $\Gamma = D^\times$ by verifying that $D^\times/K^\times$ is compact.

2. Let K'/K be a cyclic extension of p-adic fields so that $\mathrm{Gal}(K'/K) = \langle\sigma\rangle$. We have the following properties of the local base change map

$$BC = BC_{K'/K} : \mathrm{Irr}(GL_n(K)) \to \mathrm{Irr}(GL_n(K')).$$

   (a) The image of BC is the subset of $\mathrm{Irr}(GL_n(K'))$ consisting of π' such that $\pi' \simeq \pi' \circ \sigma$. (Here $\pi' \circ \sigma$ is the representation such that each $g \in GL_n(K')$ acts by $\pi'(\sigma(g))$.)

   (b) $BC(\pi_1) \simeq BC(\pi_2)$ if and only if $\pi_1 \simeq \pi_2 \otimes (\chi \circ \det)$ for some nontrivial smooth character $\chi : K^\times \to \mathbb{C}^\times$ factoring through $K^\times/N_{K'/K}(K')^\times$ (which is isomorphic to $\mathbb{Z}/[K' : K]\mathbb{Z}$ via local Artin map).

   (c) $BC(\pi)$ is supercuspidal if and only if π is supercuspidal and $\pi \not\simeq \pi \otimes (\chi \circ \det)$ for any nontrivial χ of the form as in (b).

   (d) $\omega_{BC(\pi)}(z) = \omega_\pi(N_{K'/K}(z))$ for $z \in (K')^\times$. (As usual ω_π denotes the central character of π.)

   Choose any three out of the four assertions at your will[1] and verify them, *assuming*[2] that $BC_{K'/K}$ corresponds to the restriction map from W_K to $W_{K'}$ on Weil-Deligne representations via the local Langlands correspondences $\mathrm{rec}_{K'}$ and rec_K. (Note: See [AC89, 1.6.1, 1.6.2] for more on the local base change including its unique characterization by means of trace characters. Also compare with the global cyclic base change, cf. [Gee, 4.22] and [AC89, 3.4.2, 3.5.1]. The references are unlikely to help you solve the problem though.)[3]

3. Do [Gee, Exercise 4.9].[4]

References

[AC89] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Princeton University Press, no. 120, Annals of Mathematics Studies, Princeton, New Jersey, 1989.

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.

[1] Write up only three and you'll get full credit. Feel free to check all four, but there's no extra credit.

[2] This exercise is just for the sake of better understanding. As I explained in class, this is not the correct order of logic: base change was established well before the local Langlands correspondence.

[3] I thank Fan Zheng for pointing out the missing condition in an earlier version that χ above should be nontrivial.

[4] The phrase "can naturally identified" should be taken with a grain of salt. Because the definition of τ_v in 4.8 has exponent $\eta_v - 1$ rather than η_v, you'll see that the identification may not look as natural as the one you would think of at first.