18.786. (Spring 2014) Problem set # 7 (due Thu Apr 10)

1. (a) Suppose that a locally profinite topological group Γ is unimodular. Fix a bi-invariant Haar measure μ on Γ. Let $\mathcal{H}(\Gamma)$ denote the Hecke algebra of locally constant compactly supported functions $\Gamma \to \mathbb{C}$ with convolution product $*$. Let (π, V) be a smooth representation of Γ. Recall that $\pi(f)$ is a linear operator on V via

$$(\pi(f))(v) := \int_\Gamma f(\gamma)\pi(\gamma)v \cdot d\mu_\gamma, \quad v \in V.$$

   Verify that $f \mapsto \pi(f)$ is a homomorphism from $\mathcal{H}(\Gamma)$ to $\mathrm{End}_{\mathbb{C}}(V)$, i.e. $\pi(f * g) = \pi(f)\pi(g)$.

   (b) The above construction associates to (π, V) an $\mathcal{H}(\Gamma)$-module structure on V. Show that this gives a fully faithful functor from the category of smooth representations of Γ to the category of $\mathcal{H}(\Gamma)$-modules.

2. Fix a Haar measure $\mu^\times$ (resp. μ) on $\mathbb{Q}_p^\times$ (resp. $\mathbb{Q}_p$).[1] Let B and T be the subgroups of $GL_2(\mathbb{Q}_p)$ consisting of upper-triangular and diagonal matrices, resp (denoted $P_{(1,1)}$ and $M_{(1,1)}$ in class). Let N be the group of all unipotent matrices in B so that $B = N \rtimes T = T \ltimes N$. Using the natural isomorphism $T \simeq \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$, we can define a Haar measure ν on T by transporting the product measure $\mu^\times \cdot \mu^\times$ on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. The natural isomorphism $N \simeq \mathbb{Q}_p$ allows to view μ as a Haar measure on N. Now define two measures μ' and μ'' such that for $f \in C_c^\infty(B)$,

$$\int_B f(b) \cdot d\mu'_b := \int_N \int_T f(tn) \cdot d\nu_t d\mu_n, \qquad \int_B f(b) \cdot d\mu''_b := \int_N \int_T f(nt) \cdot d\nu_t d\mu_n.$$

   Prove that μ' (resp. μ'') is a left-invariant (resp. right-invariant) Haar measure. Show that $\mu'' = \delta_B \mu'$, i.e. $\int_B f(b) d\mu''_b = \int_B f(b)\delta_B(b) d\mu'_b$, where $\delta_B \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |a/d|_p$.

3. Do [Gee, Exercise 4.7]. Repeat the same for the double coset $GL_2(\mathcal{O}_K) \begin{pmatrix} \varpi_K^2 & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathcal{O}_K)$. Can you identify the corresponding operator as an element of $\mathbb{C}[T, S^{\pm 1}]$?

References

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.

[1] For convenience you may assume that $\mu^\times(\mathbb{Z}_p^\times) = \mu(\mathbb{Z}_p) = 1$ to pin down $\mu^\times$ and μ if you like. There are several texts explaining more details about these $\mu^\times$ and μ, e.g. Tate's thesis in [CF].