18.786. (Spring 2014) Problem set # 6 (due Tue Apr 1)

1. Prove the lower bound for the Krull dimension of R_S^{univ} as in Proposition 3.24.(3) of [Gee]. (Make sure you have the latest version of Gee's notes.) Compute the lower bound, by explicitly computing the various dimensions appearing in the formula, assuming (in addition to $\ell \nmid n$, $\ell > 2$) that[1]

   - $n = 2$ and F is a totally real field,
   - at every $v|\infty$, $\det \overline{\rho}(c_v) = -1$, where c_v denotes the image of complex conjugation c under $G_{\mathbb{R}} = \{1, c\} \hookrightarrow G_F$ (induced by any fixed F-embedding $\overline{F} \hookrightarrow \mathbb{C}$),[2]
   - at every $v|\ell$, $R^{\square}_{\overline{\rho}|_{G_{F_v}}}/I(\mathcal{D}_v)$ is equal to $R^{\square}_{\overline{\rho}|_{G_{F_v}},\chi,\mathrm{cr},\{H_\sigma\}}$ as in Theorem 3.28 of [Gee],
   - at every $v \nmid \ell$, $\mathcal{D}_v$ consists of all liftings (so that $I(\mathcal{D}_v) = (0)$).

2. Do [Gee, Exercise 3.34].

3. Let K be a finite extension of $\mathbb{Q}_p$.

   (a) Let $r : W_K \to GL_2(\mathbb{C})$ be an *irreducible* representation such that the kernel of $r|_{I_K}$ is open in I_K. Show that $r|_{I_K}$ is reducible if and only if there exists an unramified quadratic extension K'/K such that $r \simeq \mathrm{Ind}_{W_{K'}}^{W_K}(\psi)$ for some character $\psi : W_{K'} \to \mathbb{C}^\times$. (This provides an example where a WD-rep is (absolutely) irreducible but its type is reducible as an I_K-representation.)

   (b) Now let $\psi_i : I_K \to \mathbb{C}^\times$ be tame continuous characters for $i = 1, 2$. (Tame means that ψ_i are trivial on the wild inertia subgroup.) Find a necessary and sufficient condition[3] for $\psi_1 \oplus \psi_2$ to be the restriction to I_K of some WD-representation of W_K. (In other words, find the condition for $(\psi_1 \oplus \psi_2, 0)$ to be an inertial type.) When is it the restriction of an *irreducible* WD-representation?

4. This exercise is meant to supply a heuristic explanation for why there are two irreducible components (which are disjoint) in a certain split ramified case, cf. Part 2 of Theorem 4.1.5 of [Pil].[4] Let $\psi : G_K \to \mathcal{O}^\times$ be a continuous character and $\overline{\psi} := \psi \otimes_{\mathcal{O}} \mathbb{F}$. Put $\overline{\rho} = \mathbf{1} \oplus \overline{\psi}$, where $\mathbf{1}$ is the trivial character of G_K. Consider *liftings* of $\overline{\rho}$ of the form

$$\text{either} \quad \rho = \eta_1\psi \oplus \eta_2 \quad \text{or} \quad \rho = \eta_1 \oplus \eta_2\psi \quad \eta_i : G_K \to \mathcal{O}^\times,\ i = 1, 2$$

   (so that in the former case the reduction of $\eta_1\psi$ is $\mathbf{1}$ and the reduction of η_2 is $\overline{\psi}$, and similarly in the latter case). Put $\overline{\eta}_i := \eta_i \otimes_{\mathcal{O}} \mathbb{F}$. Assume that τ is a split ramified type (in the current situation this means $\psi|_{I_K}$ is nontrivial, i.e. ψ is ramified).

   Now let ρ run over all liftings as above under the constraint that ρ has inertial type $\tau = [(\psi|_{I_K} \oplus \mathbf{1}, 0)]$, i.e. $\rho|_{I_K} \simeq \psi|_{I_K} \oplus \mathbf{1}$. Show that the ordered pair $(\overline{\eta}_1, \overline{\eta}_2)$ is uniquely determined except when

   - $\overline{\psi}$ is unramified and $\overline{\psi} \neq \mathbf{1}$,

   in which case exactly two distinct pairs occur for $(\overline{\eta}_1, \overline{\eta}_2)$.[5]

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[1] Again my ℓ (resp. p) is Gee's p (resp. ℓ). I'm sticking to my convention but it's fine if you decide to follow Gee's notation in your homework.

[2] In this case we say that $\overline{\rho}$ is totally odd. Compare with Fact 4.20.(4) of [Gee].

[3] Of course I'm asking for a non-tautological condition, without any reference to W_K or anything external to I_K, ψ_1, and ψ_2.

[4] One could consider the apparently more general case $\overline{\rho} = \overline{\psi} \oplus \overline{\psi}'$ but quickly reduces to the case $\overline{\psi}' = \mathbf{1}$ by twisting by a character.

[5] In this case the two irreducible components (which are also connected components) correspond to the two $(\overline{\eta}_1, \overline{\eta}_2)$ in that the family of characters (η_1, η_2) occurring on each component have the same mod ℓ reduction.

References

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, http://www2.imperial.ac.uk/~tsg.

[Pil] V. Pilloni, *The study of 2-dimensional p-adic Galois deformations in the ell not p case*, draft, http://perso.ens-lyon.fr/vincent.pilloni/Defo.pdf.