18.786. (Spring 2014) Problem set # 5 (due Tue Mar 18)

1. Prove Lemma 3.17.(2) in [Gee]. (Hint: The explanation immediately following Definition 2.2.4 in [CHT08] is that it is true "because $\mathcal{I}(\mathcal{D})$ is $1 + M_n(\mathfrak{m}_{R^{\square}_{\overline{\rho}}})$ invariant".[1] Your mission is to elaborate on this sentence. The bijection in Exercise 3.11 might be useful.)

2. Also prove the following assertions in part (1) and part (3) of [Gee, Lemma 3.17]:

   - Recall that we constructed the ideal $I(\mathcal{D})$ of $R^{\square}_{\overline{\rho}}$ as the unique minimal ideal in the set of ideals $I \subset R^{\square}_{\overline{\rho}}$ such that $(R^{\square}_{\overline{\rho}}/I, \rho^{\square}_{\overline{\rho}} \bmod I) \in \mathcal{D}$. Show that it is $\ker(GL_n(R^{\square}_{\overline{\rho}}) \to GL_n(\mathbb{F}))$-invariant.
   - Verify that $\mathcal{D}(I)$ as defined in part (3) satisfies the first five conditions of [Gee, Definition 3.16]. (As I said during the class the last condition is trickier. See [BLGHT11, Lemma 3.2] for the argument if you are interested.)

3. Do [Gee, Exercise 3.19] in the following instances: State the analogues of the bijection between (1)-(4) in Exercise 3.11 (only for the universal lifting ring) as well as the analogues of Corollaries 3.12 and 3.14. Give lower bounds for the Krull dimensions of $R^{\square}_{\overline{\rho}}$ and $R^{\mathrm{univ}}_{\overline{\rho}}$ similar to the ones below Corollary 3.14. You need not provide details – it suffices to briefly indicate how to modify the previous arguments.

4. Give a proof of [Gee, Proposition 3.26]. (Feel free to borrow ideas from the proof of [BLGGT, Lemma 1.2.3] but write up in your own language.)

References

[BLGGT] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, to appear in Ann. of Math., `math.ias.edu/~rtaylor`.

[BLGHT11] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, P.R.I.M.S. **98** (2011), 29–98.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687 (2010j:11082)

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.

[1] I changed the notation to be consistent with [Gee].