18.786. (Spring 2014) Problem set # 3 (due Tue Mar 4)

1. Let Γ be a profinite group. Show that the following are equivalent.[1] (For any group G, its commutator subgroup, a.k.a. derived subgroup, is denoted $[G,G]$. We write G^ℓ for the set $\{g^\ell : g \in G\}$. The maximal pro-ℓ quotient of G is the quotient group of G which is pro-ℓ and satisfies the requirement that every surjection with open kernel from G to a finite ℓ-power order group factor through it.)

   (a) The quotient of Γ by the closure of $\langle [\Gamma,\Gamma], \Gamma^\ell \rangle$ is finite.

   (b) The maximal pro-ℓ quotient of Γ is topologically finitely generated.

   (You may take for granted the following fact, a version of Burnside's basis theorem: Let G be a finite group of ℓ-power order and let $g_1, ..., g_r \in G$. If the images of g_i's in $G/\langle [G,G], G^\ell \rangle$ generate the quotient group then g_i's generate G. Actually it's not terribly difficult to prove this fact but you need not write it up, cf. Exercise 26(c) on p.199 of Dummit and Foote, Abstract Algebra, 3rd ed.)

2. In the proof that the functor $\mathscr{R}^{\square}_{\overline{\rho}}$ is representable, we had the following situation: Let Γ be a profinite group satisfying Hyp(Γ). Let Γ_0 be an open **normal**[2] finite index subgroup of Γ (so that the two conditions in problem 1 above hold true for Γ_0). Let Δ be the kernel of the natural surjection from Γ_0 onto its maximal pro-ℓ quotient group. Show that Δ is a closed normal subgroup of Γ. (Note: This implies that Γ/Δ is a profinite group; the quotient of a profinite group by a closed normal subgroup is again a profinite group.)

3. Do [Gee] Exercise 3.8.

4. Do [Gee] Exercise 3.9.

References

[Boe11] Gebhard Boeckle, *Deformations of Galois representations*, draft, http://www.iwr.uni-heidelberg.de/groups/arith-geom/home/members/gebhard-boeckle/ (2011).

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.

[1] See [Boe11, Exercise 1.8.(b)(c)] if you are interested in seeing other rephrases.

[2] Thanks to Amol for pointing out that this condition was missing in the earlier version. When proving $\mathscr{R}^{\square}_{\overline{\rho}}$ is representable, the normality condition is clearly satisfied because Γ_0 is the kernel of a homomorphism out of Γ.