18.786. (Spring 2014) Problem set # 2 (due Tue Feb 25)

Let ℓ, p be distinct primes. Let K (resp. L) be a finite extension of $\mathbb{Q}_p$ (resp. $\mathbb{Q}_\ell$). (Unfortunately the roles of ℓ and p are swapped in Gee's notes; there he considers p-adic representations of the Galois group over an ℓ-adic field. My justification is that our convention here seems a bit more popular.)

1. Do Parts (2), (4), (5), and (6) of [Gee, Exercise 2.11].
2. Do [Gee, Exercise 2.16].
3. Let ℓ, p be distinct primes. Recall that $\mathrm{WD}_{\zeta,\varphi}$ is a map

$$(V, \rho) \mapsto (V, r, N)$$

(may also be written as $\rho \mapsto (r, N)$ for simplicity) from the set of continuous finite-dimensional representations of $G_K := \mathrm{Gal}(\overline{K}/K)$ on L-vector spaces to the set of Weil-Deligne representations of W_K on finite dimensional L-vector spaces, by the recipe in class or in [Gee, Prop 2.17]. (It is a private exercise for you to check that (r, N) is indeed a WD-rep, cf. [Gee, Exercise 2.19.(4)]; no need to write this up.)

- Show that $\mathrm{WD}_{\zeta,\varphi}$ induces a fully faithful *functor* with the obvious notion of morphisms on the source and target *categories* of Galois and Weil-Deligne representations.[1] (Namely a morphism is an L-linear map of vector spaces compatible with either the data ρ or (r, N).[2])
- Prove that the functors $\mathrm{WD}_{\zeta,\varphi}$ and $\mathrm{WD}_{\zeta',\varphi'}$ as defined in [Gee, Prop 2.17] are isomorphic.

References

[Gee] Toby Gee, *Modularity lifting theorems - notes for arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.

[1] The essential image of this functor can be described as in the statement of [Gee, Prop 2.17]. Checking this is left as another private exercise, cf. Exercise 2.19.(4) therein.

[2] Explicitly, a morphism from (V, r, N) to (V', r', N') is an L-linear map $f : V \to V'$ such that $r'(\tau) \circ f = f \circ r(\tau)$ for every $\tau \in W_K$ and $f \circ N = N' \circ f$. A morphism from (V, ρ) to (V', ρ') is an L-linear $f : V \to V'$ such that $\rho'(\tau) \circ f = f \circ \rho(\tau)$ for every $\tau \in W_K$.