18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1,2,3,4,5,6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

1. Compute the condition probability $P[X = 4|Z = 6]$. **ANSWER:** 1/5

2. Compute the conditional expectation $E[Z|Y]$ as a function of Y. **ANSWER:** $Y + 7/2$.

2. (10 points) Janet is standing outside at time zero when it starts to drizzle. The times at which raindrops hit her are a Poisson point process with parameter $\lambda = 2$. In expectation, she is hit by 2 raindrops in each given second.

(a) What is the expected amount of time until she is first hit by a raindrop? **ANSWER:** 1/2 second

(b) What is the probability that she is hit by exactly 4 raindrops during the first 2 seconds of time? **ANSWER:** $e^{-2\lambda}(2\lambda)^k/k! = e^{-4}4^4/4!$.

3. (10 points) Let X be a random variable with density function f, cumulative distribution function F, variance V and mean M.

(a) Compute the mean and variance of $3X + 3$ in terms of V and M. **ANSWER:** Mean $3M + 3$, variance $9V$.

(b) If $X_1, \ldots, X_n$ are independent copies of X. Compute (in terms of F) the cumulative distribution function for the largest of the X_i. **ANSWER:** $F(a)^n$. This is the probability that all n values are less than a.

4. (10 points) Suppose that X_i are i.i.d. random variables, each uniform on $[0, 1]$. Compute the moment generating function for the sum $\sum_{i=1}^n X_i$. **ANSWER:** $M_{X_1}(a) = E^{aX_1} = \int_0^1 e^{ax}dx = (e^a - 1)/a$. Moment generating function for sum is $(e^a - 1)^n/a^n$.

5. (10 points) Suppose that X and Y are outcomes of independent standard die rolls (each equal to $\{1,2,3,4,5,6\}$ with equal probability). Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $\log 6$ and $\log 6$

(b) Compute $H(X, Z)$. **ANSWER:** $\log 36 = 2\log 6$.

(c) Compute $H(10X + Y)$. **ANSWER:** $\log 36 = 2\log 6$ (since 36 sums all distinct).

(d) Compute $H(Z) + H_Z(Y)$. (Hint: you shouldn't need to do any more calculations.) **ANSWER:** $\log 36$

6. (10 points) Elaine's not-so-trusty old car has three states: broken (in Elaine's possession), working (in Elaine's possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the car starts out B, it has a .5 chance of staying B and a .5 chance of switching to S by the next morning.

(ii) Each morning the car starts out W, it has .5 chance of staying W, and a .5 chance of switching to B by the next morning.

(iii) Each morning the car starts out S, it has a .5 chance of staying S and a .5 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem. **ANSWER:** Markov chain matrix is

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .5 & .5 & 0 \\ 0 & .5 & .5 \end{pmatrix}$$

(b) If the car starts out B on one morning, what is the probability that it will start out B two days later? **ANSWER:** 1/4

(c) Over the long term, what fraction of mornings does the car start out in each of the three states, B, S, and W? **ANSWER:** Row vector π such that $\pi M = \pi$ (with components of π summing to one) is $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

7. Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability 1/3 and .5 with probability 2/3. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^{n} X_i$ for $n \geq 1$.

(a) What is the the probability that Y_n reaches 8 before the first time that it reaches $\frac{1}{8}$? **ANSWER:** sequences is martingale, so $1 = EY_T = 8p + (1/8)(1-p)$. Solving gives $1 - 8p = (1-p)/8$, so $8 - 64p = 1 - p$ and $63p = 7$. Answer is $p = 1/9$.

(b) Find the mean and variance of $\log Y_{10000}$. **ANSWER:** Compute for $\log Y_1$, multiply by 10000.

(c) Use the central limit theorem to approximate the probability that $\log Y_{10000}$ (and hence Y_{10000}) is greater than its median value. **ANSWER:** About .5.

8. (10 points) Eight people toss their hats into a bin and the hats are redistributed, with all of the 8! hat permutations being equally likely. Let N be the number of people who get their own hat. Compute the following:

(a) $\mathbb{E}[N]$ **ANSWER:** 1

(b) $\mathrm{Var}[N]$ **ANSWER:** 1

9. (10 points) Let X be a normal random variable with mean μ and variance σ^2.

(a) $\mathbb{E}e^X$. **ANSWER:** $e^{\mu+\sigma^2/2}$.

(b) Find μ, assuming that $\sigma^2 = 3$ and $E[e^X] = 1$. **ANSWER:** $\mu + \sigma^2/2 = 0$ so $\mu = -3/2$.

10. (10 points)

1. Let $X_1, X_2, \ldots$ be independent random variables, each equal to 1 with probability 1/2 and -1 with probability 1/2. In which of the cases below is the sequence Y_n a martingale? (Just circle the corresponding letters.)

   (a) $Y_n = X_n$ **NO**
   (b) $Y_n = 1 + X_n$ **NO**
   (c) $Y_n = 7$ **YES**
   (d) $Y_n = \sum_{i=1}^n iX_i$ **YES**
   (e) $Y_n = \prod_{i=1}^n (1 + X_i)$ **YES**

2. Let $Y_n = \sum_{i=1}^n X_i$. Which of the following is necessarily a stopping time for Y_n?

(a) The smallest n for which $|Y_n| = 5$. **YES**

(b) The largest n for which $Y_n = 12$ and $n < 100$. **NO**

(c) The smallest value n for which $n > 100$ and $Y_n = 12$. **YES**