NAME: ______________________________

Fall 2012 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Lisa's truck has three states: broken (in Lisa's possession), working (in Lisa's possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the truck starts out B, it has a 1/2 chance of staying B and a 1/2 chance of switching to S by the next morning.

(ii) Each morning the truck starts out W, it has 9/10 chance of staying W, and a 1/10 chance of switching to B by the next morning.

(iii) Each morning the truck starts out S, it has a 1/2 chance of staying S and a 1/2 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem.

(b) If the truck starts out W on one morning, what is the probability that it will start out B two days later?

(c) Over the long term, what fraction of mornings does the truck start out in each of the three states, B, S, and W?

2. (10 points) Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of independent random variables which are each equal to 1 with probability 1/2 and -1 with probability 1/2. Write $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

(a) What is the probability that Y_n reaches 10 before the first time that it reaches -30?

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = X_n + Y_n$

(ii) $Z_n = \prod_{i=1}^{n} (2X_i + 1)$

(iii) $Z_n = \prod_{i=1}^{n} (-X_i + 1)$

(iv) $Z_n = \sum_{i=1}^{n} Y_i$

(v) $Z_n = \sum_{i=2}^{n} X_i X_{i-1}$

3. (10 points) Ten people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 10! permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N^2]$

(b) $P(N = 8)$

4. (10 points) When Harry's cell phone is on, the times when he receives new text messages form a Poisson process with parameter $\lambda_T = 3/\text{minute}$. The times at which he receives new email messages form an independent Poisson process with parameter $\lambda_E = 1/\text{minute}$. He receives personal messages on Facebook as an independent Poisson process with rate $\lambda_F = 2/\text{minute}$.

(a) After catching up on existing messages one morning, Harry begins to wait for new messages to arrive. Let X be the amount of time (in minutes) that Harry has to wait to receive his first text message. Write down the probability density function for X.

(b) Compute the probability that Harry receives 10 new messages total (including email, text, and Facebook) during his first two minutes of waiting.

(c) Let Y be the amount of time elapsed before the third email message. Compute $\text{Var}(Y)$.

(d) What is the probability that Harry receives no messages of any kind during his first five minutes of waiting?

5. (10 points) Suppose that X and Y have a joint density function f given by

$$f(x,y) = \begin{cases} 1/\pi & x^2+y^2<1 \\ 0 & x^2+y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X.

(b) Express $E[\sin(XY)]$ as a double integral. (You don't have to explicitly evaluate the integral.)

6. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1, 2, 3, 4, 5, 6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

(a) Compute the conditional probability $P[X = 6|Z = 8]$.

(b) Compute the conditional expectation $E[Y|Z]$ as a function of Z (for $Z \in \{2, 3, 4, \ldots, 12\}$).

7. (10 points) Suppose that X_i are i.i.d. random variables, each of which assumes a value in $\{-1, 0, 1\}$, each with probability 1/3.

(a) Compute the moment generating function for X_1.

(b) Compute the moment generating function for the sum $\sum_{i=1}^{n} X_i$.

8. (10 points) Let X and Y be independent random variables. Suppose X takes values in $\{1, 2\}$ each with probability $1/2$ and Y takes values in $\{1, 2, 3, 4\}$ each with probability $1/4$. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$.

(b) Compute $H(X, Z)$.

(c) Compute $H(X + Y)$.

9. (10 points) Let X be a normal random variable with mean 0 and variance 1.

(a) Compute $\mathbb{E}[e^X]$.

(b) Compute $\mathbb{E}[e^X 1_{X>0}]$.

(c) Compute $\mathbb{E}[X^2 + 2X - 5]$.

10. (10 points) Let X be uniformly distributed random variable on $[0, 1]$.

(a) Compute the variance of X.

(b) Compute the variance of $3X + 5$.

(c) If $X_1, \ldots, X_n$ are independent copies of X, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function F_Z?