18.440 Midterm 1, Spring 2014: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

1. (10 points) How many quintuples $(a_1, a_2, a_3, a_4, a_5)$ of non-negative integers satisfy $a_1 + a_2 + a_3 + a_4 + a_5 = 100$?

2. (20 points) Thirty people are invited to a party. Each person accepts the invitation, independently of all others, with probability $1/3$. Let X be the number of accepted invitations. Compute the following:

(a) $E[X]$

(b) $\mathrm{Var}[X]$

(c) $E[X^2]$

(d) $E[X^2 - 4X + 5]$

3. (20 points) Bob has noticed that during every given minute, there is a $1/720$ chance that the Facebook page for his dry cleaning business will get a "like", independently of what happens during any other minute. Let L be the total number of likes that Bob receives during a 24 hour period.

(a) Compute $\mathbb{E}[L]$ and $\text{Var}[L]$. (Give exact answers, not approximate ones.)

(b) Compute the probability that $L = 0$. (Give an exact answer, not an approximate answer.)

(c) Bob is really hoping to get at least 2 more likes during the next 24 hours (because this would boost his cumulative total to triple digits). Use a Poisson random variable calculation to *approximate* the probability that $L \geq 2$.

4. (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p. Compute (in terms of p) the probability that the fifth head occurs on the tenth toss.

5. (20 points) Let X be the number on a standard die roll (assuming values in $\{1, 2, 3, 4, 5, 6\}$ with equal probability). Let Y be the number on an independent roll of the same die. Compute the following:

(a) The expectation $E[X^2]$.

(b) The expectation $E[XY]$.

(c) The covariance $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$.

6. (20 points) Three hats fall out of their assigned bins and are randomly placed back in bins, one hat per bin (with all 3! reassignments being equally likely). Compute the following:

(a) The expected number of hats that end up in their own bins.

(b) The probability that the third hat ends up in its own bin.

(b) The conditional probability that the third hat ends up in its own bin *given* that the first hat does *not* end up in its own bin.