18.600 Midterm 1, Fall 2023: solutions

1. (20 points) The online commentators on Opinion Planet routinely promote 20 specific opinions. (Opinion 1: Income tax should be optional. Opinion 2: We need more squirrels....) Alice decides to select 10 of the 20 to passionately support, 5 to passionately oppose, and 5 to be indifferent about.

(a) Let N be the total number of ways she can do that. Compute N. **ANSWER:** $\binom{20}{10,5,5} = \frac{20!}{10!5!5!}$.

(b) Alice randomly adopts one of the N possibilities (all N being equally likely). Bob is contrarian. He independently chooses 12 of the 20 opinions to strongly oppose (all sets of 12 equally likely) and stays indifferent on the rest. Let D be the number of *passionate disagreements* (i.e., opinions that Alice passionately supports *and* Bob passionately opposes). Compute $E(D)$. **Note:** If it helps, you can define D_i to be 1 if they passionately disagree on the ith opinion, 0 otherwise. **ANSWER:** $E[D_i] = P(\text{Alice supports, Bob opposes opinion } i)$ which is $\frac{10}{20} \cdot \frac{12}{20} = \frac{3}{10}$. Then $E[D] = \sum E[D_i] = 20 \cdot \frac{3}{10} = 6$.

(c) Compute the probability p that Alice and Bob passionately disagree on *both* Opinions 1 and 2. **ANSWER:** Since Alice's and Bob's choices are independent we have $p = \frac{10}{20} \cdot \frac{9}{19} \cdot \frac{12}{20} \cdot \frac{11}{19}$

(d) Compute the expectation $E(D^2)$. If it helps, you can express your answer in terms of the p from part (c). **ANSWER:** $E(D^2) = E[\sum_{i=1}^{20} D_i \sum_{j=1}^{20} D_j] = \sum_{i=1}^{20}\sum_{j=1}^{20} E[D_iD_j]$. If $i = j$ then the summand is $E[D_iD_i] = E[D_i] = 3/10$. If $i \neq j$ then the summand is the p from (c). Since there are 20 terms of the first type and 380 of the second the answer is $20 \cdot \frac{3}{10} + 380p = 6 + 380p$.

2. (15 points) Ten bankers gather at a Swiss retreat. The first has a watch worth \$1000, the second has a watch worth \$2000, and generally the nth banker has a watch worth \$1000n. They remove their watches to use the spa. Each banker retrieves a watch afterward, but since the watches appear identical at first glance, they get all mixed up, with each of the 10! permutations being equally likely.

(a) Compute the expected number of bankers who end up with a watch that is *more valuable* than the one they started out with.**ANSWER:** Let M_j be 1 if jth person gets a more valuable watch and 0 otherwise. Then $M = \sum_{j=1}^{10} [M_j]$ is the total number of people getting a more valuable watch. Then $E[M_1] = .9$, $E[M_2] = .8$, etc., so $E[M] = \sum E[M_j] = .9 + .8 + \ldots + .1 + 0 = 4.5$.

ALTERNATIVE ANSWER: If L is number getting less valuable watch and S is number getting same watch, then $M + L + S = 10$ with probability 1, so $E[M] + E[L] + E[S] = 10$. Check that $E[S] = 1$ and observe by symmetry that $E[M] = E[L]$ which implies $E[M] = 4.5$.

(b) As a function of $k \in \{1, 2, \ldots, 10\}$, find the probability p_k that *all* of the first k bankers get their own watches. **ANSWER:** $p_k = \frac{(10-k)!}{10!}$.

(c) Find the probability that at least one *of the first four* bankers gets their own watch back. If it helps, you can express your answer in terms of the p_k from (b). **ANSWER:** Let E_i be the event the ith person gets own hat back. Then

$$P[\cup_{i=1}^4 E_i] = \sum_{i=1}^4 P[E_i] - \sum_{1 \le i < j \le 4} P[E_iE_j] + \sum_{1 \le i < j < k \le 4} P[E_iE_jE_k] - \sum_{1 \le i < j < k < \ell} P[E_iE_jE_kE_\ell] =$$

$$\binom{4}{1}p_1 - \binom{4}{2}p_2 + \binom{4}{3}p_3 - \binom{4}{4}p_4 = 4p_1 - 6p_2 + 4p_3 - p_4$$

3. (20 points) Bob prepares two online dating photos: Photo 1 (pensively strumming a guitar) and Photo 2 (tenderly cuddling a puppy). His dating service sends Photo 1 to 1000 potential partners, each of whom responds (independently of all else) with probability 1/500. The service sends Photo 2 to 1000 different people, each of whom responds (independently of all else) with probability 1/250. Let R_1 and R_2 be the number of responses to Photo 1 and Photo 2, respectively.

(a) Compute $P(R_1 = k)$ for $k \in \{0, 1, \dots, 1000\}$. Given an exact formula, not a Poisson approximation. **ANSWER:** R_1 is a binomial random variable so $P(R_1 = k) = \binom{1000}{k}(\frac{1}{500})^k(\frac{499}{500})^{1000-k}$.

(b) Use Poisson approximations to estimate $P(R_1 = 1)$ and $P(R_2 = 0)$. **ANSWER:** First is $e^{-\lambda}\lambda^k/k!$ with $\lambda = 2$ and $k = 1$, which is $2e^{-2}$. Second answer is same with $\lambda = 4$ and $k = 0$ which gives e^{-4}.

(c) Use Poisson approximations to estimate the probability that Bob receives one response total. That is, estimate $P(R_1 + R_2 = 1)$. **ANSWER:**
$P(R_1 + R_2 = 1) = P(R_1 = 1, R_2 = 0) + P(R_1 = 0, R_2 = 1) \approx 2e^{-2} \cdot e^{-4} + e^{-2} \cdot 4e^{-4} = 6e^{-6}$.

(d) Use Poisson approximations to estimate the *conditional probability* that the response is from somebody who viewed the puppy picture, *given* that Bob received exactly one response overall. In other words, estimate $P(R_2 = 1 | R_1 + R_2 = 1)$. Express your answer as a simple rational number. **ANSWER:** Note that the event "$R_2 = 1$ and $R_1 + R_2 = 1$" is equivalent to the event "$R_2 = 1$ and $R_1 = 0$." The answer is $P(R_2 = 1, R_1 = 0)/P(R_1 + R_2 = 1) \approx 4e^{-6}/6e^{-6} = 2/3$.

4. (10 points) Six teenagers trick-or-treating late at night encounter a bucket containing 30 candy bars and a sign that reads "Help yourself!" Let c_i represent the number of candy bars the ith teenager takes.

(a) How many 6-tuples $(c_1, \dots, c_6)$ are possible? In other words, how many ordered 6-tuples $(c_1, c_2, \dots, c_6)$ of *non-negative* integers satisfy $c_1 + c_2 + \ldots + c_6 \le 30$? (Caution: don't mistake the $\le$ for an equality. Not all bars have to be taken.) **ANSWERS:** Treating the bucket as a separate person, this is equivalent to counting ordered 7-tuples of non-negative integers summing to 30. Using stars and bars this is $\binom{30+6}{6}$.

(b) Suppose the teenagers decide to take all the candy bars but that (to be fair) they will independently roll a 6-sided die for each candy bar to decide which person gets it. Let $A_1, A_2, \dots, A_{30}$ be the numbers that come up on the 30 successive die rolls. (So each A_i is a random integer in $\{1, 2, \dots, 6\}$.) The ith bar goes to the A_ith teenager. What is the probability that the candy bars end up evenly divided (i.e. the probability that each number between 1 and 6 appears in the sequence $A_1, \dots, A_{30}$ exactly five times)? **ANSWER:** $\frac{\binom{30}{5,5,5,5,5,5}}{6^{30}} = \frac{30!}{(5!)^6 \cdot 6^{30}}$.

5.(15 points) Compute the following:

(a) $\displaystyle\sum_{k=0}^{6} \binom{6}{k} 9^k$ **ANSWER:** This is the binomial expansion of $(9+1)^6 = 10^6$.

(b) $\displaystyle\lim_{n\to\infty} \left(1 - \frac{3}{2n}\right)^n$ **ANSWER:** $e^{-3/2}$.

(c) $\displaystyle\sum_{k=0}^{\infty} e^{-5} \left(\frac{5^k}{k!} \cdot k^2\right)$ **ANS:** If X is Poisson, $\lambda = 5$, then $E[X^2] = \mathrm{Var}(X) + E[X]^2 = \lambda + \lambda^2 = 30$.

6. (10 points) How many "two pair" poker hands are there? **Recall:** there are 13 face values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, K, Q, A). A deck has 4 distinct cards of each face value. A "two pair" hand consists of 5 cards from the same deck, with two cards of one face value, two cards of a second (distinct) face value, and one card of a third (distinct) face value. **ANSWER:** $\binom{13}{2} \cdot 11 \cdot \binom{4}{2}^2 \cdot \binom{4}{1}$. (Choose face values for pairs, face value for single, suits for lower-face-value pair, suits for other pair, suit for single.)

7.(10 points) Let X be a random variable with $P(X = -2) = 1/4$ and $P(X = 2) = 3/4$. Compute:

(a) $\text{Var}(X)$ **ANSWER:** $E[X^2] - E[X]^2 = 4 - 1 = 3$.

(b) $\text{Var}(19 + 20X + 21X^2 + 20X^3 + 19X^4)$ **ANSWER:** The terms 19, $21X^2$ and $19X^4$ are constant on $\{-2, 2\}$ and don't impact the variance. And $20X^3 + 20X$ takes values in $\{-200, 200\}$ with probabilities $\{1/4, 3/4\}$ so $\text{Var}(20X^3 + 20X) = \text{Var}(100X) = 10000\text{Var}(X) = 30000$.