18.600: Lecture 39

Review: practice problems

Scott Sheffield

MIT

Martingales

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  - $Y_n = \prod_{i=1}^n (X_i - 1)$

Martingales

- Yes, no, yes, no.

Calculations like those needed for Black-Scholes derivation

- Let X be a normal random variable with mean 0 and variance 1. Compute the following (you may use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answers):

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  - $E[e^{3X-3}]$.
  - $E[e^X 1_{X \in (a,b)}]$ for fixed constants $a < b$.

Calculations like those needed for Black-Scholes derivation – answers

$$
\begin{aligned}
E[e^{3X-3}] &= \int_{-\infty}^{\infty} e^{3x-3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-6x+6}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-6x+9}{2}} e^{3/2} dx \\
&= e^{3/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-3)^2}{2}} dx \\
&= e^{3/2}
\end{aligned}
$$

Calculations like those needed for Black-Scholes derivation – answers

$$
\begin{aligned}
E[e^X 1_{X \in (a,b)}] &= \int_a^b e^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \int_a^b e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2x+1-1}{2}} dx \\
&= e^{1/2} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx \\
&= e^{1/2} \int_{a-1}^{b-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= e^{1/2}(\Phi(b-1) - \Phi(a-1))
\end{aligned}
$$

Call functions

Let X be a uniform random variable on the interval $[0, 1]$. For each real number K write $C(K) = E[\max\{X - K, 0\}]$.

- Compute $C(K)$ as a function of K for $K \in [0, 1]$.

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Let X be a uniform random variable on the interval $[0,1]$. For each real number K write $C(K) = E[\max\{X - K, 0\}]$.

- Compute $C(K)$ as a function of K for $K \in [0,1]$.
- Compute the derivatives C' and C'' on the interval $[0,1]$.

Call functions – answers

- $C(K) = \int_0^1 \max\{x - K, 0\} dx = \int_K^1 (x - K) dx = (1 - K)^2/2.$

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- $C(K) = \int_0^1 \max\{x - K, 0\} dx = \int_K^1 (x - K) dx = (1 - K)^2/2.$
- Taking derivative directly gives $C'(x) = -(1 - x) = x - 1$ and $C''(x) = 1$. Alternatively, one could remember the call function formulas derived in lecture: $C'(x) = F_X(x) - 1$ and $C''(x) = f_X(x)$.

If you want *more* probability and statistics...

- **UNDERGRADUATE:**
  (a) 18.615 Introduction to Stochastic Processes
  (b) 18.642 Topics in Math with Applications in Finance
  (c) 18.650 Statistics for Applications
  (d) 18.???/6.265/15.070 Advanced Stochastic Processes

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  (a) 18.675 (formerly 18.175) Theory of Probability
  (b) 18.676 (formerly 18.176) Stochastic calculus
  (c) 18.677 (formerly 18.177) Topics in stochastic processes (topics vary, can be pretty much anything in probability, repeatable)

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- **OUTSIDE OF MATH DEPARTMENT**
  (a) Look up MIT minor in statistics and data sciences.
  (b) Look up longer lists of probability/statistics courses at `https://stat.mit.edu/academics/minor-in-statistics/`
  (c) Ask MIT faculty how they use probability/statistics in research.

Thanks for taking the course!

- Considering previous generations of mathematically inclined MIT students, and adopting a frequentist point of view...

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- Happy exam day!
- And may the odds be ever in your favor.