18.600: Lecture 38

Review: practice problems

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Entropy

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  - Compute $H(X+Y)$.
  - Which is larger, $H(X+Y)$ or $H(X,Y)$? Would the answer to this question be the same for any discrete random variables X and Y? Explain.

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- $H(X,Y)$ is larger, and we have $H(X,Y) \geq H(X+Y)$ for any X and Y. To see why, write $a(x,y) = P\{X = x, Y = y\}$ and $b(x,y) = P\{X + Y = x + y\}$. Then $a(x,y) \leq b(x,y)$ for any x and y, so
  $H(X,Y) = E[-\log a(x,y)] \geq E[-\log b(x,y)] = H(X+Y)$.

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- When Alice showers, she first checks to see if at least one towel is present. If a towel is present, she dries off with that towel and returns it to the bathroom towel rack. Otherwise, she cheerfully retrieves both towels from the walk-in closet, then showers, dries off and leaves both towels on the rack.

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- **Problem:** describe towel-distribution evolution as a Markov chain and determine (over the long term) on what fraction of days Bob emerges from the shower to find no towel.

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- Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel $\frac{2}{9} \times \frac{1}{2} = \frac{1}{9}$ fraction of the time.

Optional stopping, martingales, central limit theorem

Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and -1 with probability $1/2$. Let $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

- What is the the probability that Y_n reaches -25 before the first time that it reaches 5?

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- What is the the probability that Y_n reaches -25 before the first time that it reaches 5?
- Use the central limit theorem to approximate the probability that $Y_{9000000}$ is greater than 6000.

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- $p_{-25}25 + p_5 5 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.

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- $p_{-25}25 + p_5 5 = 0$ and $p_{-25} + p_5 = 1$. Solving, we obtain $p_{-25} = 1/6$ and $p_5 = 5/6$.
- One standard deviation is $\sqrt{9000000} = 3000$. We want probability to be 2 standard deviations above mean. Should be about $\int_2^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

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- Find the probability *at least one player* has an epic collapse.

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- The probability for Alice is $(4/23) \cdot (1/24)$ and summing over six players gives $6(4/23) \cdot (1/24) = 1/23$.
- You might think to use inclusion-exclusion... but on inspection... it is impossible to two players to have an epic collapse. So the probability of an epic collapse is the same as the expected number of epic collapses.