18.600: Lecture 37

Review: practice problems

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Expectation and variance

- Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let N be the number of teams whose rank does not change from season one to season two. Let N_+ the number of teams whose rank improves by exactly two spots. Let N_- be the number whose rank declines by exactly two spots. Compute the following:

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Expectation and variance — answers

- Let N_i be 1 if team ranked ith first season remains ith second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$

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- $\mathrm{Var}[N] = E[N^2] - E[N]^2$ and $E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2$.

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- N_+^i be 1 if team ranked ith has rank improve to $(i-2)$th for second seasons. Then $E[(N_+)^2] = E[\sum_{j=1}^{8} \sum_{3=1}^{8} N_+^i N_+^j] = 6 \cdot \frac{1}{8} + 30 \cdot \frac{1}{56} = 9/7$, so $\mathrm{Var}[N_+] = 9/7 - (3/4)^2$.

Conditional distributions

- Roll ten dice. Find the conditional probability that there are exactly 4 ones, given that there are exactly 4 sixes.

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- Alternate solution: first condition on location of the 6's and then use binomial theorem.

Poisson point processes

- Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The V be length of time (in decades) until the first volcano eruption and E the length of time (in decades) until the first earthquake. Compute the following:
  - $\mathbb{E}[E^2]$ and $\mathrm{Cov}[E, V]$.

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  - $\mathbb{E}[E^2]$ and $\mathrm{Cov}[E, V]$.
  - The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
  - The probability density function of $\min\{E, V\}$.

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- Probability density function of $\min\{E, V\}$ is $3e^{-(2+1)x}$ for $x \geq 0$, and 0 for $x < 0$.

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  - If $X_1, \ldots, X_n$ are independent copies of X, what is the probability density function for the smallest of the X_i

Order statistics — answers

- $$\begin{aligned}\mathrm{Var}[X^2] &= E[X^4] - (E[X^2])^2 \\ &= \int_{-1}^{1} \frac{1}{2}x^4 dx - (\int_{-1}^{1} \frac{1}{2}x^2 dx)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.\end{aligned}$$

Order statistics — answers

- $$\mathrm{Var}[X^2] = E[X^4] - (E[X^2])^2$$
$$= \int_{-1}^{1} \frac{1}{2} x^4 dx - (\int_{-1}^{1} \frac{1}{2} x^2 dx)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$
- Note that for $x \in [-1, 1]$ we have
$$P\{X > x\} = \int_{x}^{1} \frac{1}{2} dx = \frac{1-x}{2}.$$
If $x \in [-1, 1]$, then
$$P\{\min\{X_1, \ldots, X_n\} > x\}$$
$$= P\{X_1 > x, X_2 > x, \ldots, X_n > x\} = (\frac{1-x}{2})^n.$$
So the density function is
$$-\frac{\partial}{\partial x}(\frac{1-x}{2})^n = \frac{n}{2}(\frac{1-x}{2})^{n-1}.$$

Moment generating functions

- Suppose that X_i are independent copies of a random variable X. Let $M_X(t)$ be the moment generating function for X. Compute the moment generating function for the average $\sum_{i=1}^n X_i/n$ in terms of $M_X(t)$ and n.

Moment generating functions — answers

- Write $Y = \sum_{i=1}^{n} X_i/n$. Then
$$M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} X_i/n}] = (M_X(t/n))^n.$$