18.600 Midterm 1, Spring 2022: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials and $\binom{n}{k}$ expressions — no need to multiply them out).**

NAME: ____

1. (20 points) 6 married couples—numbered 1 to 6—are preparing to board a small airplane. The plane has 6 rows of seats, with 2 seats per row. However, instead of seating one couple per row in the usual way, the airline is assigning seating randomly — with all 12! ways of assigning the 12 people to the 12 seats being equally likely. Let X be the number of married couples seated together (i.e. both spouses in the same row). Note that X is a random integer taking values between 0 and 6. Let E_i be the event that the ith couple is seated together.

(a) Compute the probabilities $P(E_1)$ and $P(E_2E_3)$.

(b) Compute $P(E_4 \cup E_5)$.

(c) Compute the expectation $E[X]$.

(d) Compute the expectation $E[X^2]$.

2.(20 points) Compute the following:

(a) $\sum_{k=0}^{\infty} \frac{1}{5^k \cdot k!}$

(b) $\lim_{n\to\infty} (1 - \frac{1}{3n})^{5n}$

(c) $\sum_{k=0}^{\infty} kq^{k-1}p$ where p and q are both positive with $p + q = 1$.

(d) $\sum_{k=0}^{20} \frac{20!}{k!(20-k)!}$

3. (20 points) Alice loves basketball. She practices free throw shots at a court by the river. Each time she shoots, there is a 1/3 probability that she makes the shot and retrieves the ball, a 1/3 probability that she misses and retrieves the ball, and a 1/3 probability that she misses badly enough that the ball goes down the river and is lost forever. Alice starts the day with 10 basketballs and plans to keep shooting until all the balls are lost.

(a) What is the probability that Alice makes at least 10 shots before all 10 of her balls are lost?

(b) What is the probability that Alice's first 9 shots include 3 she makes, 3 she misses but retrieves, and 3 she loses to the river?

(c) Let X be the total number of shots Alice *takes* (whether she makes them or not) before all of her balls are lost forever. Compute $P(X = k)$ as a function of k.

(d) Compute $E[X]$.

4. (15 points) Carol is taking a test to determine if she has a certain disease. She thinks *a priori* that she has a 1/2 chance of having the disease. If she has the disease, the test will return positive with probability 7/10, negative with probability 3/10. If she doesn't have the disease, the test will return positive with probability 1/10 and negative with probability 9/10.

(a) Compute the probability that the test result is positive.

(b) Compute the conditional probability that Carol has the disease given that the test result is positive.

(c) Compute the conditional probability that Carol has the disease given that the test result is negative.

5. (10 points) Bob is an aspiring musician who uses a streaming service that gives him .5 cents every time somebody listens his song. During each second of the day there is (independently of all other seconds) a probability of 1/3600 that somebody starts listening to his song that second. So on average one person listens to his song every hour.

(a) Bob has a lunch break from his day job in 3 hours, and he was hoping to buy a $10 sandwich for lunch, but his cash account currently only has $9.99. In order to buy the sandwich he needs to get at least 2 more listens during the next 3 hours. Use a Poisson approximation to estimate the probability that this happens.

(b) Using the same Poisson approximation, estimate the mean and variance of the total number of listens during the next 10 hours.

6. (15 points) Answer the following:

(a) Let S_n be the number of heads in n independent tosses of a coin that comes up heads with probability p and tails with probability $q = 1 - p$. Compute the $n \to \infty$ limit of the probability that $\frac{S_n - np}{\sqrt{npq}}$ lies in the interval $(-1, 1)$. You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer.

(b) Suppose that X is a random variable with probability density given by $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Compute the expectation $E[X^3 + X + 1]$.

(c) Suppose that X is a random variable with density

$$f_X(x) = \begin{cases} x/2 & x \in [0,2] \\ 0 & x \notin [0,2] \end{cases}.$$

Compute the probability $P(X > 1)$.