18.600 Midterm 1, Spring 2021: Solutions

1. (20 points) Toss 3 independent fair coins. Let X be the number of heads.

(a) Compute the expectation $E[X]$ and variance $\mathrm{Var}[X]$. **ANSWER:** X is a binomial random variable with $n = 3$ and $p = 1/2$. So $E[X] = np = 3/2$ and $\mathrm{Var}[X] = np(1 - p) = 3/4$.

(b) Compute the expectation $E[(X - 1)^3]$. **ANSWER:** We first compute the probability X is equal to 0, 1, 2 or 3. We find $p(0) = 1/8, p(1) = 3/8, p(2) = 3/8, p(3) = 1/8$. So $E[(X-1)^3] = \sum_{j=0}^{3} p(j)(j-1)^3 = (1/8)\cdot(-1)+(3/8)\cdot(0)+(3/8)\cdot(1)+(1/8)\cdot 8 = 10/8 = 5/4$.

(c) For $i \in \{1, 2, 3\}$ let E_i be the event that the ith coin is heads. **Use inclusion-exclusion** to compute $P(E_1 \cup E_2 \cup E_3)$ and show your work. (This problem is testing inclusion-exclusion, so you have to use inclusion-exclusion to get full credit.) **ANSWER:** $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_1E_3) - P(E_2E_3) + P(E_1E_2E_3) = 1/2 + 1/2 + 1/2 - 1/4 - 1/4 - 1/4 + 1/8 = 7/8$. One gets the same answer by observing that the probability of 3 tails is $1/8$ so the probability of at least one heads is $1 - 1/8 - 7/8$.

2. (15 points) A small college dorm has 6 (distinguishable) double occupancy rooms and 12 students. The students are assigned to the six rooms randomly (two students per room) with all such assignments being equally likely. Six of these students — Alice, Beatrice, Carol, Deborah, Eve and Francis — are especially close friends who are *really* hoping to be roomed together.

(a) Compute the probability that Alice and Beatrice end up in the same room. **ANSWER:** Alice must be equally likely to be paired with any of the other 11 students, so the answer is $1/11$ by symmetry.

(b) Compute the conditional probability that Alice and Beatrice end up in the same room *given* that Carol and Deborah end up in the same room. **ANSWER:** Given that Alice and Beatrice are together, Carol is equally likely to be paired with any of the other 9 students, so the answer is $1/9$ by symmetry.

(c) Compute the probability that Alice, Beatrice, Carol, Deborah, Eve and Francis end up in six different rooms. **ANSWER:** If we insist on separate rooms for the named students, there are $6!$ ways to assign them rooms, and then $6!$ ways to choose their roommates: so we get a numerator of $(6!)^2$. The denominator is the total number of ways to divide up the students, which is $\binom{12}{2,2,2,2,2,2} = 12!/2^6$. So the answer is $\frac{(6!)^2}{12!/2^6} = 2^6(6!)^2/12!$.

3. (20 points) Suppose that during a global pandemic a virus infects 10^9 people and within each person it makes 10^{11} copies of itself. (The individual copies are called *virions.*) Overall 10^{20} new virions are produced. Each time a new virion is produced, there is a 10^{-20} probability that a Terrible Mutation occurs (independently). Let X be the number of Terrible Mutations that occur.

(a) Compute the expectation $E(X)$. **ANSWER:** $E(X) = 10^{20} \cdot 10^{-20} = 1$ by additivity of expectation.

(b) Use a Poisson random variable to approximate the probability $P(X = 2)$. **ANSWER:** X is roughly Poisson with $\lambda = 1$ so probability is roughly $e^{-\lambda}\lambda^2/2! = e^{-1}/2 = \frac{1}{2e}$.

(c) Use a Poisson random variable to approximate the conditional probability $P(X \geq 2|X \geq 1)$. In other words, compute the conditional probability that there are *multiple* terrible mutations *given* that there is at least one. **ANSWER:**

$$P(X \geq 2|X \geq 1) = \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{1 - P(X = 0) - P(X = 1)}{1 - P(X = 0)} = \frac{1 - 2e^{-1}}{1 - e^{-1}}$$

(d) Suppose that instead of infecting 10^9 people, the virus infects $c \cdot 10^9$ people for some moderate-sized $c > 0$. Let p_c be the probability that at least one terrible mutation occurs in this scenario. Use a Poisson random variable to approximate p_c (as a function of c). **ANSWER:** X is roughly Poisson with $\lambda = c$. So $P(X > 0) = 1 - P(X = 0) \approx 1 - e^{-c}$.

4.(10 points) Let $E_1, E_2, \ldots$ be an infinite sequence of independent events such that each E_n occurs with probability $1/n!$ and let X be the number of these events that occur.

(a) Compute the expectation $E[X]$. **ANSWER:**
$E[X] = E[1_{E_1} + 1_{E_2} \ldots] = P(E_1) + P(E_2) + \ldots = 1/1 + 1/2! + 1/3! + \ldots = e - 1.$

(b) Compute the conditional probability $P(E_1E_2E_3|E_3E_4E_5)$. **ANSWER:**
$\frac{P(E_1E_2E_3E_4E_5)}{P(E_3E_4E_5)} = \frac{P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)}{P(E_3)P(E_4)P(E_5)} = P(E_1)P(E_2) = 1/2$

5. (15 points) Alice has a pile of 30 green jelly beans, 30 blue jelly beans, and 30 red jelly beans. She wants to make jelly bean gift bags as party favors for her friends. So she picks 9 jelly beans (uniformly at random) to put into the first bag; then she picks another 9 jelly beans (uniformly at random from the 81 remaining in the pile) to put in the second bag, and so on, until she has made 10 gift bags (each containing 9 jelly beans).

(a) Compute the probability the first gift bag is perfectly balanced, in the sense that it has exactly three jelly beans of each color. **ANSWER:** $\binom{90}{9}$ ways to fill bag; $\binom{30}{3}^3$ are perfectly balanced. So answer is $\binom{30}{3}^3/\binom{90}{9}$.

(b) Compute the *expected number* of perfectly balanced gift bags. **ANSWER:** By additivity of expectation, this is 10 times answer in (a) or $10 \cdot \binom{30}{3}^3/\binom{90}{9}$

(c) Compute the probability that *all ten* bags are perfectly balanced. **ANSWER:** $\binom{30}{3,3,3,3,3,3,3,3,3,3}^3 / \binom{90}{9,9,9,9,9,9,9,9,9,9} = \left(\frac{30!}{(3!)^{10}}\right)^3 / \left(\frac{90!}{(9!)^{10}}\right)$. The numerator is number of ways to divide the green jelly beans evenly, then the blue, then the red. The denominator is the total number of ways to divide the jelly beans into 10 bags.

6. (10 points) How many ways are there to divide 15 (indistinguishable) pieces of sushi among 4 (distinguishable) people? In other words, now many sequences of non-negative integers a_1, a_2, a_3, a_4 satisfy $a_1 + a_2 + a_3 + a_4 = 15$? **ANSWER:** Stars and bars argument (3 bars, 15 stars) gives $\binom{18}{3}$.

7. (10 points) Each time Bob casts his fishing line into the water, he has a $p = .01$ chance of catching a fish, independently of all else. Let X be the number of times Bob has to cast his fishing line in order to catch 3 fish.

(a) Compute the expectation $E(X)$ and variance $\mathrm{Var}(X)$. **ANSWER:** X is negative binomial with $p = .01$, $n = 3$. So $E[X] = n/p = 300$ and $\mathrm{Var}[X] = 3(1-p)/p^2 = 3 \cdot .99/(.01)^2 = 29700$.

(b) Compute the probability $P(X > 200)$. **ANSWER:** This is the probability Bob gets 0, 1 or 2 fish among the first 200 tries, so $\binom{200}{0}(1-p)^{200} + \binom{200}{1}p(1-p)^{199} + \binom{200}{2}p^2(1-p)^{198}$.