

18.600: Lecture 26

**Moment generating functions and
characteristic functions**

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Outline

Moment generating functions

Characteristic functions

Continuity theorems and perspective

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Continuity theorems and perspective

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- ▶ If $b > 0$ and $t > 0$ then $E[e^{tX}] \geq E[e^{t \min\{X, b\}}] \geq P\{X \geq b\} e^{tb}$.
- ▶ If X takes both positive and negative values with positive probability then $M(t)$ grows at least exponentially fast in $|t|$ as $|t| \rightarrow \infty$.

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$$e^{tX} = 1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots$$

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- ▶ Another way to think of this: write
$$e^{tX} = 1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots$$
- ▶ Taking expectations gives
$$E[e^{tX}] = 1 + tm_1 + \frac{t^2 m_2}{2!} + \frac{t^3 m_3}{3!} + \dots$$
, where m_k is the k th moment. The k th derivative at zero is m_k .

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- ▶ By independence, $M_Z(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$ for all t .

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- ▶ In other words, adding independent random variables corresponds to multiplying moment generating functions.

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- ▶ Answer: M_X^n . Follows by repeatedly applying formula above.
- ▶ This a big reason for studying moment generating functions. It helps us understand what happens when we sum up a lot of independent copies of the same random variable.

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- ▶ Latter answer is the special case of $M_Z(t) = M_X(t)M_Y(t)$ where Y is the constant random variable b .

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- ▶ Answer: $M_X(t) = E[e^{tX}] = \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda} e^{\lambda e^t} = \exp[\lambda(e^t - 1)]$.

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- ▶ We know that if you add independent Poisson random variables with parameters λ_1 and λ_2 you get a Poisson random variable of parameter $\lambda_1 + \lambda_2$. How is this fact manifested in the moment generating function?

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- ▶ Answer: Z has same law as $\sigma X + \mu$, so
 $M_Z(t) = M(\sigma t)e^{\mu t} = \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}.$

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- ▶ Then Z has the law of a sum of n independent copies of X .
So $M_Z(t) = M_X(t)^n = \left(\frac{\lambda}{\lambda-t}\right)^n$.

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- ▶ Exponential calculation above works for $t < \lambda$. What happens when $t > \lambda$? Or as t approaches λ from below?
- ▶ $M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda-t)x} dx = \infty$ if $t \geq \lambda$.

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- ▶ Informal statement: moment generating functions are not defined for distributions with fat tails.

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- ▶ And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.
- ▶ And if X has an m th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

Characteristic functions

- ▶ Let X be a random variable.
- ▶ The **characteristic function** of X is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$. Like $M(t)$ except with i thrown in.
- ▶ Recall that by definition $e^{it} = \cos(t) + i \sin(t)$.
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- ▶ And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.
- ▶ And if X has an m th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- ▶ But characteristic functions have a distinct advantage: they are always well defined for all t even if f_X decays slowly.

Outline

Moment generating functions

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Continuity theorems and perspective

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- ▶ Moment generating functions are central to so-called *large deviation theory* and play a fundamental role in statistical physics, among other things.
- ▶ Characteristic functions are *Fourier transforms* of the corresponding distribution density functions and encode “periodicity” patterns. For example, if X is integer valued, $\phi_X(t) = E[e^{itX}]$ will be 1 whenever t is a multiple of 2π .

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- ▶ **Moment generating analog:** if moment generating functions $M_{X_n}(t)$ are defined for all t and n and $\lim_{n \rightarrow \infty} M_{X_n}(t) = M_X(t)$ for all t , then X_n converge in law to X .