18.600 Midterm 1, Fall 2019, Solutions

1. (15 points) A *super-eruption* is a volcanic eruption producing more than 1000 cubic kilometers of deposits (and maybe enough ash to change the global climate for several years). Assume that each year (independently of all other years) there is a $\frac{1}{25{,}000}$ probability that there will be a single super-eruption somewhere in the world. (To simplify matters, assume that the probability of more than one super-eruption during the same year is zero.)

(a) Compute the expected number of super-eruptions that will take place during the next 100,000 years. **ANSWER:** Binomial with $n = 100,000$, $p = \frac{1}{25{,}000}$ has expectation $np = 4$.

(b) Use a Poisson approximation to estimate the probability that there will be exactly 3 super-eruptions during the next 100,000 years. **ANSWER:** Setting $\lambda = np = 4$, probability is approximately $e^{-\lambda}\lambda^k/k! = e^{-4}4^3/3!$.

(c) Use a Poisson approximation to estimate the probability that there will be *at least* one super-eruption at some point during the next 100 years. **ANSWER:** This is binomial with $n = 100$ and $p = \frac{1}{25{,}000}$, hence roughly Poisson with $\lambda = np = 1/250$. Chance of at least one super-eruption is one minus chance of zero. So roughly $1 - e^{-\lambda}\lambda^k/k!$ with $k = 0$. That is, $1 - e^{-1/250}$. If interested, note by Taylor that this is roughly $1 - (1 - 1/250) = 1/250$ to say somebody living 100 years has roughly $1/250$ chance of living through super-eruption.

2. (15 points) Two teams are playing a soccer game (in a league with no overtime or shootouts). The first team's score is a Poisson random variable X with parameter $\lambda_X = 1$. The second team's score is an independent Poisson random variable Y with parameter $\lambda_Y = 2$.

(a) Compute the probability that the game ends in a tie. That is, compute $P(X = Y)$. (You can leave your answer as an infinite sum.) **ANSWER:** $P\sum_{k=0}^{\infty} P(X = k, Y = k)$ is

$$\sum_{k=0}^{\infty}\bigl(e^{-\lambda_X}\lambda_X^k/k!\bigr)\bigl(e^{-\lambda_Y}\lambda_Y^k/k!\bigr) = \sum_{k=0}^{\infty}\bigl(e^{-1}/k!\bigr)\bigl(e^{-2}2^k/k!\bigr).$$

FYI, summing this on computer gives about 21 percent. Ties happen.

(b) Compute the probability the underdog team wins. That is, compute $P(X > Y)$. (You can leave your answer as a double infinite sum.) **ANSWER:** $\sum_{j=1}^{\infty}\sum_{k=0}^{j-1} P(X = j, Y = k)$ is

$$\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\bigl(e^{-1}/j!\bigr)\bigl(e^{-2}2^k/k!\bigr).$$

FYI, summing this on a computer gives about 18 percent. Underdog has a fighting chance!

(c) Compute the probability that exactly two goals are scored overall. That is compute $P(X + Y = 2)$. **ANSWER:** $P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0)$ is

$$\bigl(e^{-1}/0!\bigr)\bigl(e^{-2}2^2/2!\bigr) + \bigl(e^{-1}/1!\bigr)\bigl(e^{-2}2^1/1!\bigr) + \bigl(e^{-1}/2!\bigr)\bigl(e^{-2}2^0/0!\bigr) = (2 + 2 + 1/2)/e^3 = 9/(2e^3)$$

3. (20 points) 14 students are taking a chemistry class, and the professor plans to assign each person a partner — so that there are 7 (unordered) partnerships with two people per partnership.

(a) How many ways are there to do that? **ANSWER:** If pairs were ordered (pair one, pair two, etc.) number would be $\binom{14}{2,2,2,2,2,2,2} = 14!/2^7$. Dividing by 7! gives answer $14!/(2^7 \cdot 7!)$. Alternatively, line people up in a row. First person has 13 choices for partner, next unpartnered person in row has 11 choices for partner, etc. So answer is $13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$.

(b) Two of the students are good friends and are hoping they will get to be partners. Assuming the professor chooses the partner division randomly (with all possible ways of forming the 7 partnerships being equally likely) what is the probability that they will be partners? **ANSWER:** 1/13 (since first student equally likely to be paired with any of 13 others)

(c) Suppose that 7 of the people are men and 7 are women. Let N be the number of partnerships with exactly one man and one woman and compute the expectation $E[N]$. (If it helps, you can write N_i for the random variable that is 1 if the ith female has a male partner and 0 otherwise.) **ANSWER:** Expectation additivity gives $\sum_{j=1}^{7} E[N_j] = 49/13$.

(d) Compute the expectation $E[N^2]$. **ANSWER:** Additivity of expectation gives $E\big[(\sum_{j=1}^{7} N_j)(\sum_{k=1}^{7} N_k)\big] = \sum_{j=1}^{7}\sum_{k=1}^{7} E[N_j N_k]$. The 7 terms with $j = k$ contribute 7/13 each, and the 42 with $j \neq k$ contribute (7/13)(6/11) each. Answer is $49/13 + 42(7/13)(6/11)$

4.(20 points) Compute the following:

(a) $\lim_{n\to\infty}(1 - \frac{1}{4n})^n$ **ANSWER:** Generally $\lim_{n\to\infty}(1 + x/n)^n = e^x$ so this is $e^{-1/4}$

(b) $\sum_{k=0}^{9}\Big(2^k 8^{9-k}\binom{9}{k}\Big)$ **ANSWER:** Binomial theorem gives $(2 + 8)^9 = 10^9$.

(c) $\sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!}$ **ANSWER:** Generally $\sum_{k=0}^{\infty} x^k/k! = e^x$ so this is $e^{1/2}$.

(d) $\sum_{k=0}^{\infty}\Big((\frac{5}{6})^{k-1}(\frac{1}{6})k\Big)$ **ANSWER:** This is expectation of geometric random variable with $p = 1/6$ which is $1/p = 6$.

5. (15 points) A baking competition has ten contestants. The judges are allergic to most baking ingredients, so instead of tasting the food, they select three winners at random (with all possible three-person subsets of the 10 contestants being equally likely). Contestants Alice, Bob and Carol are good friends who are hoping to all be winners, but who feel it will be awkward if two of them are winners and the third one isn't. Let A be the event that Alice, Bob and Carol are all winners and let B be the event that *at least two* of these three people are winners.

(a) Compute $P(A)$. **ANSWER:** $1/\binom{10}{3}$

(b) Compute $P(B)$. **ANSWER:** Either all three are chosen or one of three left out and one of other seven chosen. So $1 + 3 \cdot 7 = 22$ possibilities. Answer is $22/\binom{10}{3}$

(c) Compute the conditional probability $P(A|B)$. **ANSWER:** $P(AB)/P(B) = P(A)/P(B) = 1/22$. Note: it may seem surprising that even though each person has a $3/10$ chance of winning separately, it is still the case that *given* that two win, the chance that all three win is only $1/22$. This is similar to the powerball problem (where the chance of matching all five white balls is surprisingly small compared to chance of matching 4 or 3).

6. (15 points) Janet thinks she might have a fever. Or maybe just a headache or a cold. She is not really sure. She prepares to take her temperature with a digital thermometer which reports Fahrenheit temperature (rounded to the nearest integer) and she thinks she will see one of the values in $\{98, 99, 100, 101, 102\}$ each with probability $1/5$. Let X be number she actually sees.

(a) Compute the variance $\text{Var}(X)$. Simplify your expression to give an exact value (i.e., an explicit rational number). **ANSWER:** $E[X] = 100$ so $\text{Var}(X) = E[(X - 100)^2]$ which is $\frac{1}{5}(4 + 1 + 0 + 1 + 4) = 2$.

(b) Use the answer in (a) to compute $E(X^2)$. **ANSWER:** $2 = \text{Var}(X) = E[X^2] - E[X]^2 = E[X^2] - 10000$ so $E[X^2] = 10002$.

(c) After taking her temperature, regardless of what the thermometer shows, Janet plans to a roll a three-sided die (which takes values in $\{0, 1, 2\}$ each with equal probability). If N is the number that comes up, Janet will take N ibuprofen tablets. Compute the expectation $E\big[2N^3 + 3X^2\big]$. **ANSWER:** $E[N^3] = \frac{1}{3}(0 + 1 + 8) = 3$. Additivity of expectation gives answer: $2 \cdot 3 + 3 \cdot 10002 = 30012$.