18.600 Midterm 1, Fall 2019: 50 minutes, 100 points

1. Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.

2. No calculators, books, or notes may be used.

3. Simplify your answers as much as possible (but answers may include factorials and $\binom{n}{k}$ expressions — no need to multiply them out).

NAME: _____

1. (15 points) A *super-eruption* is a volcanic eruption producing more than 1000 cubic kilometers of deposits (and maybe enough ash to change the global climate for several years). Assume that each year (independently of all other years) there is a $\frac{1}{25,000}$ probability that there will be a single super-eruption somewhere in the world. (To simplify matters, assume that the probability of more than one super-eruption during the same year is zero.)

(a) Compute the expected number of super-eruptions that will take place during the next 100,000 years.

(b) Use a Poisson approximation to estimate the probability that there will be exactly 3 super-eruptions during the next 100,000 years.

(c) Use a Poisson approximation to estimate the probability that there will be *at least* one super-eruption at some point during the next 100 years.

2. (15 points) Two teams are playing a soccer game (in a league with no overtime or shootouts). The first team's score is a Poisson random variable X with parameter $\lambda_X = 1$. The second team's score is an independent Poisson random variable Y with parameter $\lambda_Y = 2$.

(a) Compute the probability that the game ends in a tie. That is, compute $P(X = Y)$. (You can leave your answer as an infinite sum.)

(b) Compute the probability the underdog team wins. That is, compute $P(X > Y)$. (You can leave your answer as a double infinite sum.)

(c) Compute the probability that exactly two goals are scored overall. That is compute $P(X + Y = 2)$.

3. (20 points) 14 students are taking a chemistry class, and the professor plans to assign each person a partner — so that there are 7 (unordered) partnerships with two people per partnership.

(a) How many ways are there to do that?

(b) Two of the students are good friends and are hoping they will get to be partners. Assuming the professor chooses the partner division randomly (with all possible ways of forming the 7 partnerships being equally likely) what is the probability that they will be partners?

(c) Suppose that 7 of the people are men and 7 are women. Let N be the number of partnerships with exactly one man and one woman and compute the expectation $E[N]$. (If it helps, you can write N_i for the random variable that is 1 if the ith female has a male partner and 0 otherwise.)

(d) Compute the expectation $E[N^2]$.

4.(20 points) Compute the following:

(a) $\lim_{n \to \infty}(1 - \frac{1}{4n})^n$

(b) $\sum_{k=0}^{9}\left(2^k 8^{9-k}\binom{9}{k}\right)$

(c) $\sum_{k=0}^{\infty}\frac{1}{2^k \cdot k!}$

(d) $\sum_{k=0}^{\infty}\left((\frac{5}{6})^{k-1}(\frac{1}{6})k\right)$

5. (15 points) A baking competition has ten contestants. The judges are allergic to most baking ingredients, so instead of tasting the food, they select three winners at random (with all possible three-person subsets of the 10 contestants being equally likely). Contestants Alice, Bob and Carol are good friends who are hoping to all be winners, but who feel it will be awkward if two of them are winners and the third one isn't. Let A be the event that Alice, Bob and Carol are all winners and let B be the event that *at least two* of these three people are winners.

(a) Compute $P(A)$.

(b) Compute $P(B)$.

(c) Compute the conditional probability $P(A|B)$.

6. (15 points) Janet thinks she might have a fever. Or maybe just a headache or a cold. She is not really sure. She prepares to take her temperature with a digital thermometer which reports Fahrenheit temperature (rounded to the nearest integer) and she thinks she will see one of the values in $\{98, 99, 100, 101, 102\}$ each with probability $1/5$. Let X be number she actually sees.

(a) Compute the variance $\text{Var}(X)$. Simplify your expression to give an exact value (i.e., an explicit rational number).

(b) Use the answer in (a) to compute $E(X^2)$.

(c) After taking her temperature, regardless of what the thermometer shows, Janet plans to a roll a three-sided die (which takes values in $\{0, 1, 2\}$ each with equal probability). If N is the number that comes up, Janet will take N ibuprofen tablets. Compute the expectation $E\left[2N^3 + 3X^2\right]$.