18.600: Lecture 6

Conditional probability

Scott Sheffield

MIT

Outline

Definition: probability of A given B

Examples

Multiplication rule

Outline

Definition: probability of A given B

Examples

Multiplication rule

Conditional probability

- Suppose I have a sample space S with n equally likely elements, representing possible outcomes of an experiment.

Conditional probability

- Suppose I have a sample space S with n equally likely elements, representing possible outcomes of an experiment.
- Experiment is performed, but I don't know outcome. For some $F \subset S$, I ask, "Was the outcome in F?" and receive answer yes.

Conditional probability

- Suppose I have a sample space S with n equally likely elements, representing possible outcomes of an experiment.
- Experiment is performed, but I don't know outcome. For some $F \subset S$, I ask, "Was the outcome in F?" and receive answer yes.
- I think of F as a "new sample space" with all elements equally likely.

Conditional probability

- Suppose I have a sample space S with n equally likely elements, representing possible outcomes of an experiment.
- Experiment is performed, but I don't know outcome. For some $F \subset S$, I ask, "Was the outcome in F?" and receive answer yes.
- I think of F as a "new sample space" with all elements equally likely.
- Definition: $P(E|F) = P(EF)/P(F)$.

Conditional probability

- Suppose I have a sample space S with n equally likely elements, representing possible outcomes of an experiment.
- Experiment is performed, but I don't know outcome. For some $F \subset S$, I ask, "Was the outcome in F?" and receive answer yes.
- I think of F as a "new sample space" with all elements equally likely.
- Definition: $P(E|F) = P(EF)/P(F)$.
- Call $P(E|F)$ the "conditional probability of E given F" or "probability of E conditioned on F".

Conditional probability

- Suppose I have a sample space S with n equally likely elements, representing possible outcomes of an experiment.
- Experiment is performed, but I don't know outcome. For some $F \subset S$, I ask, "Was the outcome in F?" and receive answer yes.
- I think of F as a "new sample space" with all elements equally likely.
- Definition: $P(E|F) = P(EF)/P(F)$.
- Call $P(E|F)$ the "conditional probability of E given F" or "probability of E conditioned on F".
- Definition makes sense even without "equally likely" assumption.

Outline

Definition: probability of A given B

Examples

Multiplication rule

Outline

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is p.

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is p.
- $S = \{\text{disease}, \text{no disease}\} \times \{\text{positive, negative}\}$.

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is p.
- $S = \{\text{disease}, \text{no disease}\} \times \{\text{positive, negative}\}$.
- $P(\text{positive}) = .9p + .1(1-p)$ and $P(\text{disease, positive}) = .9p$.

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is p.
- $S = \{\text{disease}, \text{no disease}\} \times \{\text{positive, negative}\}$.
- $P(\text{positive}) = .9p + .1(1-p)$ and $P(\text{disease, positive}) = .9p$.
- $P(\text{disease}|\text{positive}) = \frac{.9p}{.9p+.1(1-p)}$. If p is tiny, this is about $9p$.

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is p.
- $S = \{\mathrm{disease}, \mathrm{no\ disease}\} \times \{\text{positive, negative}\}$.
- $P(\text{positive}) = .9p + .1(1-p)$ and $P(\text{disease, positive}) = .9p$.
- $P(\text{disease}|\text{positive}) = \frac{.9p}{.9p+.1(1-p)}$. If p is tiny, this is about $9p$.
- Probability suspect guilty of murder given a particular suspicious behavior.

More examples

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is p.
- $S = \{\text{disease}, \text{no disease}\} \times \{\text{positive, negative}\}$.
- $P(\text{positive}) = .9p + .1(1-p)$ and $P(\text{disease, positive}) = .9p$.
- $P(\text{disease}|\text{positive}) = \frac{.9p}{.9p+.1(1-p)}$. If p is tiny, this is about $9p$.
- Probability suspect guilty of murder given a particular suspicious behavior.
- Probability plane will come eventually, given plane not here yet.

Another famous Tversky/Kahneman study (Wikipedia)

- Imagine you are a member of a jury judging a hit-and-run driving case. A taxi hit a pedestrian one night and fled the scene. The entire case against the taxi company rests on the evidence of one witness, an elderly man who saw the accident from his window some distance away. He says that he saw the pedestrian struck by a blue taxi. In trying to establish her case, the lawyer for the injured pedestrian establishes the following facts:

Another famous Tversky/Kahneman study (Wikipedia)

- Imagine you are a member of a jury judging a hit-and-run driving case. A taxi hit a pedestrian one night and fled the scene. The entire case against the taxi company rests on the evidence of one witness, an elderly man who saw the accident from his window some distance away. He says that he saw the pedestrian struck by a blue taxi. In trying to establish her case, the lawyer for the injured pedestrian establishes the following facts:
  - There are only two taxi companies in town, "Blue Cabs" and "Green Cabs." On the night in question, 85 percent of all taxis on the road were green and 15 percent were blue.

Another famous Tversky/Kahneman study (Wikipedia)

- Imagine you are a member of a jury judging a hit-and-run driving case. A taxi hit a pedestrian one night and fled the scene. The entire case against the taxi company rests on the evidence of one witness, an elderly man who saw the accident from his window some distance away. He says that he saw the pedestrian struck by a blue taxi. In trying to establish her case, the lawyer for the injured pedestrian establishes the following facts:
  - There are only two taxi companies in town, "Blue Cabs" and "Green Cabs." On the night in question, 85 percent of all taxis on the road were green and 15 percent were blue.
  - The witness has undergone an extensive vision test under conditions similar to those on the night in question, and has demonstrated that he can successfully distinguish a blue taxi from a green taxi 80 percent of the time.

Another famous Tversky/Kahneman study (Wikipedia)

- Imagine you are a member of a jury judging a hit-and-run driving case. A taxi hit a pedestrian one night and fled the scene. The entire case against the taxi company rests on the evidence of one witness, an elderly man who saw the accident from his window some distance away. He says that he saw the pedestrian struck by a blue taxi. In trying to establish her case, the lawyer for the injured pedestrian establishes the following facts:
    - There are only two taxi companies in town, "Blue Cabs" and "Green Cabs." On the night in question, 85 percent of all taxis on the road were green and 15 percent were blue.
    - The witness has undergone an extensive vision test under conditions similar to those on the night in question, and has demonstrated that he can successfully distinguish a blue taxi from a green taxi 80 percent of the time.
- Study participants believe blue taxi at fault, say witness correct with 80 percent probability.

Outline

Definition: probability of A given B

Examples

Multiplication rule

Outline

Definition: probability of A given B

Examples

Multiplication rule

Multiplication rule

- $P(E_1E_2E_3\dots E_n) =$
  $P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\dots P(E_n|E_1\dots E_{n-1})$

Multiplication rule

- $P(E_1E_2E_3\ldots E_n) =$
  $P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1\ldots E_{n-1})$
- Useful when we think about multi-step experiments.

Multiplication rule

- $P(E_1 E_2 E_3 \ldots E_n) =$
  $P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \ldots P(E_n|E_1 \ldots E_{n-1})$
- Useful when we think about multi-step experiments.
- For example, let E_i be event ith person gets own hat in the n-hat shuffle problem.

Multiplication rule

- $P(E_1 E_2 E_3 \dots E_n) =$
  $P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \dots P(E_n|E_1 \dots E_{n-1})$
- Useful when we think about multi-step experiments.
- For example, let E_i be event ith person gets own hat in the n-hat shuffle problem.
- Another example: roll die and let E_i be event that the roll *does not lie* in $\{1, 2, \dots, i\}$. Then $P(E_i) = (6 - i)/6$ for $i \in \{1, 2, \dots, 6\}$.

Multiplication rule

- $P(E_1 E_2 E_3 \dots E_n) =$
  $P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \dots P(E_n|E_1 \dots E_{n-1})$
- Useful when we think about multi-step experiments.
- For example, let E_i be event ith person gets own hat in the n-hat shuffle problem.
- Another example: roll die and let E_i be event that the roll *does not lie* in $\{1, 2, \dots, i\}$. Then $P(E_i) = (6 - i)/6$ for $i \in \{1, 2, \dots, 6\}$.
- What is $P(E_4|E_1 E_2 E_3)$ in this case?

Monty Hall problem

- Prize behind one of three doors, all equally likely.

Monty Hall problem

- Prize behind one of three doors, all equally likely.
- You point to door one. Host opens either door two or three and shows you that it doesn't have a prize. (If neither door two nor door three has a prize, host tosses coin to decide which to open.)

Monty Hall problem

- Prize behind one of three doors, all equally likely.
- You point to door one. Host opens either door two or three and shows you that it doesn't have a prize. (If neither door two nor door three has a prize, host tosses coin to decide which to open.)
- You then get to open a door and claim what's behind it. Should you stick with door one or choose other door?

Monty Hall problem

- Prize behind one of three doors, all equally likely.
- You point to door one. Host opens either door two or three and shows you that it doesn't have a prize. (If neither door two nor door three has a prize, host tosses coin to decide which to open.)
- You then get to open a door and claim what's behind it. Should you stick with door one or choose other door?
- Sample space is $\{1, 2, 3\} \times \{2, 3\}$ (door containing prize, door host points to).

Monty Hall problem

- Prize behind one of three doors, all equally likely.
- You point to door one. Host opens either door two or three and shows you that it doesn't have a prize. (If neither door two nor door three has a prize, host tosses coin to decide which to open.)
- You then get to open a door and claim what's behind it. Should you stick with door one or choose other door?
- Sample space is $\{1,2,3\} \times \{2,3\}$ (door containing prize, door host points to).
- We have $P\big((1,2)\big) = P\big((1,3)\big) = 1/6$ and $P\big((2,3)\big) = P\big((3,2)\big) = 1/3$. Given host points to door 2, probability prize behind 3 is $2/3$.

Another popular puzzle (see Tanya Khovanova's blog)

- Given that your friend has exactly two children, one of whom is a son born on a Tuesday, what is the probability the second child is a son.

Another popular puzzle (see Tanya Khovanova's blog)

- Given that your friend has exactly two children, one of whom is a son born on a Tuesday, what is the probability the second child is a son.
- Make the obvious (though not quite correct) assumptions. Every child is either boy or girl, and equally likely to be either one, and all days of week for birth equally likely, etc.

Another popular puzzle (see Tanya Khovanova's blog)

- Given that your friend has exactly two children, one of whom is a son born on a Tuesday, what is the probability the second child is a son.
- Make the obvious (though not quite correct) assumptions. Every child is either boy or girl, and equally likely to be either one, and all days of week for birth equally likely, etc.
- Make state space matrix of $196 = 14 \times 14$ elements

Another popular puzzle (see Tanya Khovanova's blog)

- Given that your friend has exactly two children, one of whom is a son born on a Tuesday, what is the probability the second child is a son.
- Make the obvious (though not quite correct) assumptions. Every child is either boy or girl, and equally likely to be either one, and all days of week for birth equally likely, etc.
- Make state space matrix of $196 = 14 \times 14$ elements
- Easy to see answer is $13/27$.