18.600: Lecture 5

Problems with all outcomes equally likely, including a famous hat problem

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Outline

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Equal likelihood

A few problems

Hat problem

A few more problems

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- Answer: $|A|/|S|$, where $|A|$ is the number of elements in A.

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- $1 - \prod_{i=0}^{22} \frac{365-i}{365}$

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Recall the inclusion-exclusion identity

- $$\begin{aligned} P(\cup_{i=1}^{n} E_i) &= \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \dots \\ &+ (-1)^{(r+1)} \sum_{i_1 < i_2 < \dots < i_r} P(E_{i_1} E_{i_2} \dots E_{i_r}) \\ &= + \dots + (-1)^{n+1} P(E_1 E_2 \dots E_n). \end{aligned}$$

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  $$= + \ldots + (-1)^{n+1} P(E_1E_2\ldots E_n).$$
- The notation $\sum_{i_1<i_2<i_r}$ means a sum over all of the $\binom{n}{r}$ subsets of size r of the set $\{1, 2, \ldots, n\}$.

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- $1 - P(\cup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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- What is the probability of a two-pair hand in poker?
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