18.600: Lecture 22

Sums of independent random variables

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- Differentiating both sides gives $f_{X+Y}(a) = \frac{d}{da} \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$.

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- Latter formula makes some intuitive sense. We're integrating over the set of x, y pairs that add up to a.

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- Worth memorizing.

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- $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy = \int_0^1 f_X(a-y)$ which is the length of $[0,1] \cap [a-1,a]$.

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- That's a when $a \in [0,1]$ and $2-a$ when $a \in [1,2]$ and 0 otherwise.

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- We can interpret Z as time slot where nth head occurs in i.i.d. sequence of p-coin tosses.
- So Z is negative binomial (n, p). So $P\{Z = k\} = \binom{k-1}{n-1} p^{n-1}(1-p)^{k-n}p$.

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- By induction, would suffice to show that a gamma $(\lambda, 1)$ plus an independent gamma (λ, n) is a gamma $(\lambda, n+1)$.

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- Up to constant factor (not depending on a) this is

$$\int_0^a e^{-\lambda(a-y)}(a-y)^{s-1} e^{-\lambda y} y^{t-1} dy = e^{-\lambda a} \int_0^a (a-y)^{s-1} y^{t-1} dy.$$

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- Yes: letting $x = y/a$, becomes $\int_0^1 (a-x/a)^{s-1}(ax)^{t-1}(adx) = a^{s+t-1}\int_0^1 (1-x)^{s-1}x^{t-1}dx$.

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- So $f_{X+Y}(a)$ is (constant times) $e^{-\lambda a}a^{s+t-1}$. Conclude that $X+Y$ is gamma $(\lambda, s+t)$.

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- If X, Y standard normal, then $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2-y^2)/2}$. Argue by rotational invariance that $\cos(\theta)X + \sin(\theta)Y$ is standard normal. Hence $r\cos(\theta)X + r\sin(\theta)Y$ is Gaussian with mean 0, variance $r^2 = \big(r\cos(\theta)\big)^2 + \big(r\sin(\theta)\big)^2$.

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- Or use fact that if $A_i \in \{-1, 1\}$ are i.i.d. coin tosses then $\frac{1}{\sqrt{N}} \sum_{i=1}^{\sigma^2 N} A_i$ is roughly normal with variance σ^2 when N large.

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- Generally: if independent random variables X_j are normal (μ_j, σ_j^2) then $\sum_{j=1}^n X_j$ is normal $(\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2)$.

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- Yes, Poisson $\lambda_1 + \lambda_2$. Can be seen from Poisson point process interpretation.