18.600: Lecture 11

Binomial random variables and repeated trials

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Outline

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Bernoulli random variables

Properties: expectation and variance

More problems

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- $1 = 1^n = (p+q)^n = \sum_{k=0}^n \binom{n}{k}p^k q^{n-k}$.
- Number of heads is **binomial random variable with parameters** (n, p).

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- If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal's triangle, but the probability that $X = i$ gets multiplied by $p^i(1-p)^{n-i}$.

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- Let $n = 100$. Compute the probability that nobody was born on a Tuesday.
- What is the probability that exactly 15 people were born on a Tuesday?

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- Perhaps the prior row $(1, 4, 6, 4, 1)$?

Useful Pascal's triangle identity

- Recall that $\binom{n}{i} = \frac{n\times(n-1)\times\ldots\times(n-i+1)}{i\times(i-1)\times\ldots\times(1)}$. This implies a simple but important identity: $i\binom{n}{i} = n\binom{n-1}{i-1}$.

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- Using this identity (and $q = 1 - p$), we can write

$$E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i q^{n-i}.$$

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- Substitute $j = i - 1$ to get

$$E[X] = np\sum_{j=0}^{n-1}\binom{n-1}{j}p^j q^{(n-1)-j} = np(p+q)^{n-1} = np.$$

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- Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} p = np.$$

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- Thus $E[X^k] = npE[(Y+1)^{k-1}]$ where Y is binomial with parameters $(n-1, p)$.

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- Commit to memory: variance of binomial (n, p) random variable is npq.
- This is n times the variance you'd get with a single coin. Coincidence?

Compute variance with decomposition trick

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- $\sum_{j=26}^{50} \binom{50}{j} (1/3)^j (2/3)^{50-j}$