**18.600 Midterm 2 Solutions, Spring 2019: 50 minutes, 100 points**

1. (10 points) Ramona enters a basketball free throw shooting contest and takes 100 shots. She makes each shot independently with probability .8 and misses with probability .2 Let $X$ be the number of shots she makes.

(a) Compute the expectation and variance of $X$. **ANSWER:** $E[X] = np = 80$ and $\text{Var}(X) = npq = 16$

(b) Use a normal random variable to estimate the probability that she makes between 76 and 84 shots total. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** $\text{SD}(X) = 4$ and 76 is one SD below mean, 84 one SD above mean, so normal approximation gives $\Phi(1) - \Phi(-1) \approx .68$.

2. (20 points) Becky's Bagel Bakery does a brisk business. Customers arrive at random times, and each customer immediately purchases one type of bagel. The times $C_1, C_2, \ldots$ at which cinnamon raisin bagels are sold form a Poisson point process with a rate of 1 per minute. The times $P_1, P_2, \ldots$ at which pumpernickel bagels are sold form an independent Poisson point process with rate 2 per minute. And the times $E_1, E_2, \ldots$ at which everything bagels are sold form a Poisson point process with rate 3 per minute. Compute the following:

(a) The probability density function for $C_3$. **ANSWER:** Sum of three exponentials is Gamma with parameter $n = 3$ and $\lambda = 1$. So answer is $x^2 e^{-x}/2$ on $[0, \infty)$.

(b) The probability density function for $X = \min\{C_1, P_1, E_1\}$. **ANSWER:** Minimum of exponentials with rates $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ is itself exponential with rate $\lambda_1 + \lambda_2 + \lambda_3 = 6$. So answer is $6e^{-6x}$ on $[0, \infty)$.

(c) The probability that *exactly* 10 bagels (altogether) are sold during the first 2 minutes the bakery is open. **ANSWER:** The set of all bagels sale times is a Poisson point process with parameter 6. So number of points sold in first two minutes is Poisson with $\lambda = 12$. Probability to sell 10 is $e^{-\lambda}\lambda^k/k! = e^{-12}12^{10}/10!$.

(d) The expectation of $\cos(P_1 + E_1^2)$. (You can leave this as a double integral — no need to evaluate it.) **ANSWER:** $P_1$ exponential with parameter 2, and $E_1$ is exponential with parameter 3. So joint density is $2e^{-2x}3e^{-3y}$. So for general function $g(x, y)$ we can write

$$E[g(x,y)] = \int_0^\infty \int_0^\infty 2e^{-2x}3e^{-3y} g(x,y) dx dy$$

which in our case gives

$$\int_0^\infty \int_0^\infty 2e^{-2x}3e^{-3y} \cos(x + y^2) dx dy$$

3. (10 points) Suppose that the pair of real random variables $X, Y$ has joint density function $f(x, y) = \frac{1}{\pi^2(1+x^2)(1+y^2)}$.

(a) Compute the probability density function for $\frac{X+Y}{2}$. **ANSWER:** $f(x,y) = \left(\frac{1}{\pi(1+x^2)}\right)\left(\frac{1}{\pi(1+y^2)}\right)$ so $X$ and $Y$ are independent Cauchy random variables. Hence their average is also Cauchy, with density $\frac{1}{\pi(1+x^2)}$.

(b) Compute the probability $P(X > Y + 2)$. **ANSWER:** Note that $(X-Y)/2$ has same probability density function as $(X+Y)/2$ (since density function for $Y$ is symmetric) so it is Cauchy. Hence $P(X > Y+2) = P(X - Y > 2) = P(\frac{X+Y}{2} > 1)$ is the probability that a Cauchy random variable is greater than 1. Recalling spinning flashlight story, this is probability that $\theta > \pi/4$ when $\theta$ is uniform on $[-\pi/2, \pi/2]$, and this is $1/4$.

4. (20 points) Suppose that $X_1, X_2, X_3, X_4$ are independent random variables, each of which has density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Compute the following:

(a) The correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$. **ANSWER:**

$$\mathrm{Cov}(X_i, X_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so bilinearity of covariance gives $\mathrm{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4) = 2$. Variance additivity for independent random variables gives $\mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_2 + X_3 + X_4) = 3$. So

$$\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4) = \frac{2}{\sqrt{3 \cdot 3}} = \frac{2}{3}.$$

(b) The probability that $\min\{X_1, X_2\} > \max\{X_3, X_4\}$. **ANSWER:** This is the probability that $X_1$ and $X_2$ are the "top two". There are $\binom{4}{2}$ pairs which could be "top two" and by symmetry each such pair is equally likely, so answer is $1/\binom{4}{2} = 1/6$. Alternatively, one may consider that of 24 permutations of $X_1, X_2, X_3, X_4$, exactly four satisfy the constraint.

(c) The probability density function for $X_1 + X_2 + X_3$. **ANSWER:** Sum of independent normals is also normal (with mean and variance given by the sum of the respective means and variances of the individual terms). Thus $X_1 + X_2 + X_3$ is normal with mean $\mu = 0$, variance $\sigma^2 = 3$. So answer is $\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} = \frac{1}{\sqrt{3}\sqrt{2\pi}} e^{-x^2/6}$.

(d) The probability $P(X_1^2 + X_3^2 \leq 2)$. Give an explicit value. **ANSWER:** The joint density of $X_1$ and $X_3$ is $f_{X_1,X_3}(x,y) = f_{X_1}(x) f_{X_3}(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$. We have to integrate this over region where $x^2 + y^2 \leq 2$ which is the disk of radius $\sqrt{2}$. This can be done in polar coordinates: answer is

$$\int_0^{\sqrt{2}} \int_0^{2\pi} e^{-r^2/2} d\theta r dr = \int_0^{\sqrt{2}} e^{-r^2/2} r dr = -e^{-r^2/2}\Big|_0^{\sqrt{2}} = 1 - e^{-1}.$$

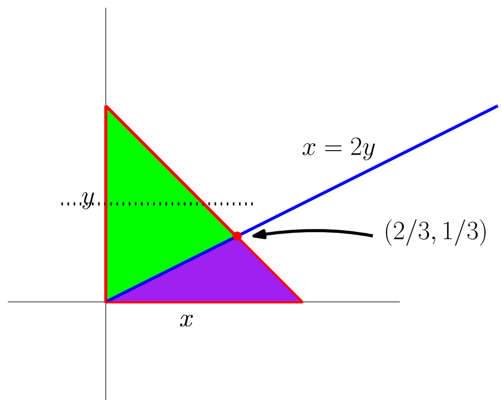
5. (10 points) Imagine that $A, B, C$ and $D$ are independent uniform random variables on $[0,1]$. You then find out that $A$ is the third largest of those random variables.

(a) Given this new information, give a revised probability density function $f_A$ for $A$ (i.e., a Bayesian posterior). **NOTE:** If you remember what this means, you may use the fact that a Beta $(a,b)$ random variable has expectation $a/(a+b)$ and density $x^{a-1}(1-x)^{b-1}/B(a,b)$, where $B(a,b) = (a-1)!(b-1)!/(a+b-1)!$. **ANSWER:** Answer is Beta with $a-1$ equal to number of points below $A$ (that's 1) and $b-1$ equal to number of points above $A$ (that's 2). So $a = 2$ and $b = 3$ and answer is $x(1-x)^2/B(2,3)$ on $[0,1]$. Can compute $B(2,3) = 1!2!/4! = 1/12$, so answer is $12x(1-x)^2$.

(b) According to your Bayesian prior, the expected value of $A$ was $1/2$. *Given* that $A$ was the third largest of the random variables, what is your *revised expectation* of the value $A$? **ANSWER:** $a/(a+b) = 2/5$, by the expectation formula given.

6. (15 points) Suppose that the pair $(X, Y)$ is uniformly distributed on the triangle $T = \{(x, y) : 0 \le x, 0 \le y, x + y \le 1\}$. That is, the joint density function is given by

$$f_{X,Y}(x,y) = \begin{cases} 2 & (x,y) \in T \\ 0 & (x,y) \notin T \end{cases}.$$




(a) Compute the marginal density function $f_X$. **ANSWER:** $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$. If $x \in [0,1]$, this value is 2 times length of intersection of vertical line through $(x, 0)$ with $T$, which is $1 - x$. So answer is $f_X(x) = \begin{cases} 2 - 2x & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$

(b) Compute the probability $P(X < 2Y)$. **ANSWER:** Using figure shown, area of whole triangle is $1/2$, area of subtriangle on which $X < 2Y$ is $1/3$, so answer is $(1/3)/(1/2) = 2/3$.

(c) Compute the conditional density function $f_{X|Y=.5}(x)$. **ANSWER:** $f_Y(1/2) = f_X(1/2) = 1$ so

$$f_{X|Y=.5}(x) = f(x, 1/2)/f_Y(1/2) = f(x, 1/2) = \begin{cases} 2 & x \in [0, 1/2] \\ 0 & x \notin [0, 2] \end{cases}.$$

(Visually, given that $(X, Y)$ is on horizontal dotted line, $X$ is uniform on $[0, 1/2]$.)

7. (15 points) Suppose that $X$ is an exponential random variable with parameter 1 and set $Z = X^5$.

(a) Compute the cumulative distribution function $F_Z(a)$ in terms of $a$. **ANSWER:** $F_X(a) = \int_0^a e^{-x} dx = 1 - e^{-a}$. And $F_Z(a) = P(Z \le a) = P(X \le a^{1/5}) = F_X(a^{1/5}) = 1 - e^{-a^{1/5}}$

(b) Compute the expectation $E[Z^2]$. **ANSWER:** $E[Z^2] = E[X^{10}] = \int_0^\infty e^{-x} x^{10} dx = 10!$. (Recall this is one of our definitions for $10!$.)

(c) Compute the conditional probability $P(Z > 32 | Z > 1)$. **ANSWER:**

$$P(Z > 32 | Z > 1) = P(X > 2 | X > 1) = P(X > 1) = e^{-1}.$$