18.600 Midterm 1, Spring 2019 solutions

1. (20 points) A town has 2000 residents. An obscure film is playing in its only theater. Each resident decides independently whether to view the film, and each resident views the film with probability 1/1000. Let X be the number of people who view the film.

(a) Compute $E[X]$. Given an exact answer, not an approximation. **ANSWER:** X is binomial with $n = 2000$ and $p = 1/1000$, so $E[X] = np = 2$.

(b) Compute $\mathrm{Var}[X]$. Give an exact answer, not an approximation. **ANSWER:** $\mathrm{Var}(X) = np(1-p) = 2 \cdot 999/1000 = 1.998$ (which is approximately 2, for what it's worth)

(c) Compute $E[X^2]$. Give an exact answer, not an approximation. **ANSWER:** We know $1.998 = \mathrm{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - 4$. Hence $E(X^2) = 5.998$. Alternatively, write $X = \sum_{i=1}^n X_i$ where X_i is 1 if ith person shows, 0 otherwise. Then

$$E(X^2) = E\big(\sum_{i=1}^n X_i \sum_{j=1}^n X_j\big) = \sum_{i=1}^n \sum_{i=1}^n E[X_i X_j].$$

Note that $E[X_i X_j] = p$ if $i = j$ and p^2 otherwise. Of the n^2 terms in the sum, we have n equal to p and $n^2 - n$ equal to p^2. So answer is

$$np + (n^2 - n)p^2 = 2 + (4000000 - 2000)/1000000 = 2 + 3.998 = 5.998.$$

(d) Use a Poisson random variable to *approximate* $P(X = 4)$. **ANSWER:** X should be approximately Poisson with $\lambda = E[X] = 2$. So $P(X = 4) \approx e^{-\lambda}\lambda^k/k! = e^{-2}2^4/4!$.

2.(10 points) Suppose that X is a Poisson random variable with parameter 2 and Y is a Poisson random variable with parameter 3.

(a) Compute the expectation $E(3X + 4Y + 5)$. **ANSWER:** By linearity of expectation, and fact Poisson of parameter λ has expectation λ, the answer is $3E[X] + 4E[Y] + 5 = 6 + 12 + 5 = 23$.

(b) Compute the variance $\mathrm{Var}(5X + 7)$. **ANSWER:** If a and b are constants, we have $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$. Poisson of parameter λ has variance λ so answer is $25\mathrm{Var}(X) = 50$.

3. (20 points) Alice, Bob, Carol, Dave, Eve, and Frank are gathered together for a night of pizza and dungeons and dragons. They order two large pizzas, each cut into 12 pieces, so there are 24 pieces altogether.

(a) How many ways are there to divide the 24 (indistinguishable) pieces among the six people? in other words, how many sequences $a_1, a_2, \ldots, a_6$ of *non-negative* integers satisfy $\sum_{i=1}^6 a_i = 24$? **ANSWER:** This is the stars and bars problem with $n = 24$ and $k = 6$, so answer is $\binom{29}{5}$.

(b) Eve proposes that, for the sake of fairness, only divisions in which each person gets *at least one slice* of pizza should be considered. How many sequences $a_1, a_2, \ldots, a_6$ of *strictly positive* integers satisfy $\sum_{i=1}^6 a_i = 24$? **ANSWERS:** First each person is given one piece, and then it is stars and bars with $n = 18$ and $k = 6$, so answer is $\binom{23}{5}$.

(c) Each of the six players pulls out a fair twenty-sided die (containing the numbers $\{1, 2, \ldots, 20\}$) and rolls it. (The six rolls are independent of each other.) What is the probability that the sum of the numbers on the dice is exactly 24? **ANSWER:** We realize that in part (b) each person gets a number of pieces of pizza between 1 and 19, so the number of ways to assign each person a die value (with total sum being 24) is exactly the answer in (b). The total number of die roll sequences is 20^6 so answer is $\binom{23}{5}/20^6$.

4. (20 points) An a capella group with 15 members (8 women and 7 men) is organizing a holiday gift exchange. Each member writes his or her name on a piece of paper and puts it in a bowl. Then the pieces of paper are randomly distributed among the 15 people, with all 15! arrangements being equally likely. Each person is assigned to buy a gift for the individual on the paper that he or she chose.

(a) Compute the expected number of people who will be assigned to buy gifts for themselves. **ANSWER:** Let X_i be 1 if ith person gets own name, 0 otherwise. Then $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = n \cdot 1/n = 1$.

(b) Compute the expected number of men who will be assigned to give gifts to women. **ANSWER:** Let X_i be 1 if ith man gives to woman, 0 otherwise. Then $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = 7 \cdot 8/15 = 56/15$.

(c) Compute the probability that *every* man is assigned to give a gift to a woman. **ANSWER:** $\frac{8}{15} \cdot \frac{7}{14} \cdot \frac{6}{13} \cdot \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9} = \frac{8!8!}{15!}$

(d) Compute the probability that *every* individual is part of a cycle of length three (i.e., a group of people A, B, and C where A gives to B, B gives to C, and C gives to A). **ANSWER:** There are $\binom{15}{3,3,3,3,3}$ ways to divide 15 into a pile 1, pile 2, pile 3, pile 4, pile 5 with 3 per pile. If we don't care about ordering the piles, then we have $\binom{15}{3,3,3,3,3}/5!$ ways to divide 15 into the groups of 3. For each such division, there are two directions each cycle can go, so we end up with $2^5\binom{15}{3,3,3,3,3}/5!$, and the probability is $\frac{2^5\binom{15}{3,3,3,3,3}}{5!15!}$

5. (10 points) A standard deck of 52 cards has 13 cards of each suit (diamonds, hearts, clubs, or spades). The deck is randomly divided into 4 bridge hands with 13 cards each (with all divisions being equally likely). What is the probability that *each* of these hands contains cards from only a single suit? (So one hand is only hearts, one hand is only clubs, and so forth.) **ANSWER:** There are $\binom{52}{13,13,13,13}$ ways to give the players their hands, and 4! ways in which each player has a pure-suit hand. So answer is $4!/\binom{52}{13,13,13,13} = \frac{4!(13!)^4}{52!}$.

6. (20 points) Alicia is writing a paper for her history class. Whenever she writes a paper, there is a .7 chance it will be brilliant and a .3 chance it will be mediocre. A professor reading a brilliant paper gives it an A with probability .9. A professor reading a mediocre paper gives it an A with probability .3. Let B be the event that that the paper is brilliant and let A be the event that it gets an A grade, so that our assumptions can be stated as $P(B) = .7$ and $P(A|B) = .9$ and $P(A|B^c) = .3$. Now compute the following:

(a) $P(A)$ (i.e., overall likelihood she gets an A) **ANSWER:** $P(A) = P(BA) + P(B^cA) = P(B)P(A|B) + P(B^c)P(A|B^c) = .7 \cdot .9 + .3 \cdot .3 = .72$

(b) $P(B|A)$ (i.e., likelihood paper is brilliant given it got an A) **ANSWER:** $P(AB)/P(A) = .63/.72 = 7/8$.

(c) $P(B|A^c)$ (i.e., likelihood paper is brilliant given it did not get an A) **ANSWER:** $P(A^cB)/P(A^c) = P(B)P(A^c|B)/P(A^c) = .7 \cdot .1/.28 = 1/4$