Spring 2016 18.600 Final Exam Solutions

1. (10 points) Suppose that $X_1, X_2, \dots$ is an i.i.d. sequence of normal random variables, each of which has mean 1 and variance 1.

(a) Compute the mean and variance of $Y = X_1 - 2X_2 + 3X_3 - 4X_4$. **ANSWER:** By additivity of expectation, mean is $1 - 2 + 3 - 4 = -2$. By additivity of variance for independent random variables (and fact that $\mathrm{Var}(X) = \mathrm{Var}(-X)$), variance is $1 + 4 + 9 + 16 = 30$.

(b) Compute the probability density function for Y. **ANSWER:** Y is normal with mean $\mu = -2$ and variance $\sigma^2 = 30$, so $f_Y(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{30}\cdot\sqrt{2\pi}} e^{-(x+2)^2/60}$.

(c) Compute $P(Y > 0)$ in terms of the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. **ANSWER:** $Y > 0$ if it is $2/\sqrt{30}$ standard deviations above its mean, so $P(Y > 0) = 1 - \Phi(2/\sqrt{30}) = \Phi(-2/\sqrt{30})$.

2. (10 points) Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of i.i.d. normal random variables, this time with mean 0 and variance 1.

(a) Write $Y_n = \sum_{i=1}^{n} X_i$. Is Y_n a martingale? **ANSWER:** Yes, since

$$E[Y_n | Y_0, Y_1, \dots, Y_{n-1}] = E[Y_{n-1} + X_n | Y_0, Y_1, \dots, Y_{n-1}] = E[Y_{n-1} | Y_0, Y_1, \dots, Y_{n-1}] = Y_{n-1}$$

(b) Write $Z_n = Y_n^2 - n$. Compute $E[Z_n - Z_{n-1} | X_1, X_2, \dots, X_{n-1}]$. You can use the following calculation to help you get started:

$$\begin{aligned} Z_n - Z_{n-1} &= (Y_n^2 - n) - (Y_{n-1}^2 - (n-1)) = Y_n^2 - Y_{n-1}^2 - 1 \\ &= (Y_{n-1} + X_n)^2 - Y_{n-1}^2 - 1 \\ &= 2Y_{n-1}X_n + X_n^2 - 1. \end{aligned}$$

ANSWER: By additivity of expectation, this is

$$E[2Y_{n-1}X_n | X_1, X_2, \dots, X_{n-1}] + E[X_n^2 - 1 | X_1, X_2, \dots, X_{n-1}]$$

First term is zero since $E[X_n]$ is zero, and $2Y_{n-1}$ can be treated as a constant for this conditional expectation calculation (since it is known once $X_1, X_2, \dots, X_{n-1}$ is given). Second term is same as $E[X_n^2 - 1]$ (by independence of X_n and $X_1, \dots, X_{n-1}$) which is zero. Overall answer is thus zero.

(c) Is Z_n a martingale? **ANSWER:** By our definition, Z_n is a martingale if and only if $E[Z_n | Z_0, Z_1, Z_2, \dots, Z_{n-1}] = Z_{n-1}$, or equivalently $E[Z_n - Z_{n-1} | Z_0, Z_1, Z_2, \dots, Z_{n-1}] = 0$. We rewrite this as $E[2Y_{n-1}X_n + X_n^2 - 1 | Z_0, Z_1, Z_2, \dots, Z_{n-1}] = 0$. This is true by the analysis used in (b) and the fact that Y_{n-1} and X_n are conditionally independent of each other when $Z_0, Z_1, Z_2, \dots, Z_{n-1}$ is given, with the conditional expectation of X_n being zero.

3. Suppose that $X_1, X_2, \ldots$ is an infinite sequence of i.i.d. Cauchy random variables, so that each has probability density function $\frac{1}{\pi(1+x^2)}$. Let W be an independent normal random variable with mean zero and variance one.

(a) Write $S = X_1 + X_2 + X_3 + X_4 + X_5$. Give an explicit formula for the probability density function f_S. **ANSWER:** We know that $T = S/5$ is a Cauchy random variable. So $f_S(x) = f_{5T}(x) = \frac{1}{5} f_T(x/5) = \frac{1}{5\pi\left(1+(x/5)^2\right)}$.

(b) Write the probability density function for $A = X_1 + W$. (Your answer will involve an integral that you do not have to try to evaluate explicitly.) **ANSWER:**

$$f_{X_1+W}(a) = \int_{-\infty}^{\infty} f_{X_1}(x) f_W(a-x)dx = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \frac{1}{\sqrt{2\pi}} e^{-(a-x)^2/2} dx.$$

(c) Compute (as an explicit rational number) the probability that $X_1 + X_2 > X_3 + 3$. (Hint: remember the spinning flashlight story.) **ANSWER:** Since $-X_3$ is also Cauchy, $B = (X_1 + X_2 - X_3)/3$ is Cauchy, and we are computing $P(B > 1)$, which is $1/4$.

4. Suppose that n people toss their shoes into a bin (two shoes—one left and one right—per person) and then the shoes are randomly shuffled and returned to the n people, with all ways of returning two shoes to each person being equally likely. Let B be the number of people who get one left and one right shoe (regardless of whether they match).

(a) Compute the expectation $E[B]$. **ANSWER:** Let A_i be 1 if ith person gets both a left and a right shoe, zero otherwise. If we imagine shoes handed out one by one, then whatever first person gets as first shoe, that person will have $n/(2n-1)$ chance of getting an opposite type shoe for the second, so $E[A_1] = n/(2n-1)$. Similarly $E[A_i] = n/(2n-1)$ for any i. Thus $E[B] = \sum_{i=1}^n E[A_i] = n^2/(2n-1)$.

(b) Compute $E[B^2]$. **ANSWER:** $E[B^2] = E[\sum_{i=1}^n A_i \sum_{j=1}^n A_j] = \sum_{i=1}^n \sum_{j=1}^n E[A_i A_j]$. There are $(n^2 - n)$ "off diagonal terms" with $i \neq j$ and n diagonal terms with $i = j$. Putting them together gives $n \cdot \frac{n}{2n-1} + (n^2 - n)\frac{n}{2n-1} \cdot \frac{n-1}{2n-3}$.

5. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the triangle $\{(x, y) : x \geq 0, y \geq 0, x + y \leq 1$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$. **ANSWER:** The triangle has area $1/2$, so $f_{X,Y}(x, y)$ is 2 if (x, y) is in the triangle and zero otherwise.

(b) Compute the conditional expectation $E[X|Y]$ as a function of Y. **ANSWER:** Once Y is given, X is conditionally uniform along the segment $[0, 1 - Y]$. So $E[X|Y] = (1 - Y)/2$.

(c) Compute $E[X]$. **ANSWER:**

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)xdxdy = \int_0^1 \int_0^{1-y} 2xdxdy$$

$$= \int_0^1 (1-y)^2 dy = -(1-y)^3/3\Big|_0^1 = 1/3.$$

6. (10 points) Let X and Y be independent uniform random variables on $[0,1]$.

(a) Compute the moment generating function for $Z = X + Y$. **ANSWER:** We know $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} f_X(x)e^{tx}dx = \int_0^1 e^{tx}dx = (e^t - 1)/t$. Thus $M_{X+Y}(t) = M_X(t)M_Y(t) = M_X(t)^2 = (e^t-1)^2/t^2$.

(b) Compute the probablity density function for $W = X^3$. **ANSWER:** For $a \in (0,1)$, $F_W(a) = P(W \le a) = P(X^3 \le a) = P(x \le a^{1/3}) = a^{1/3}$. Thus, $f_W(a) = F_W'(a) = (1/3)a^{-2/3}$.

7. (10 points) Let X, Y, Z be i.i.d. uniform random variables on $[0,5]$.

(a) Set $A = \min\{X,Y,Z\}$. Compute the probability density function f_A. **ANSWER:** For $a \in (0,5)$, $P(A > a) = P(X > a, Y > a, Z > a) = \left(\frac{5-a}{5}\right)^3$. So $F_A(a) = 1 - \left(\frac{5-a}{5}\right)^3$ and $f_A(a) = F_A'(a) = -3(-1/5)\left(\frac{5-a}{5}\right)^2$ for $a \in (0,5)$ and 0 for $a \notin (0,5)$.

(b) Let B be the second largest of the three values in $\{X,Y,Z\}$. Compute $E[B]$. **ANSWER:** $E[B] = 5/2$ by symmetry. (In principle you could also compute this by rescaling to the interval $[0,1]$ and recalling some of our problem set problems on beta random variables.)

(c) Let $C = \max\{X,Y,Z\}$. Compute the probability $P(1 < C < 4)$. **ANSWER:** $(4/5)^3 - (1/5)^3 = 63/125$.

8. (10 points)

(a) Let X be the number of heads that come up when three independent fair coins are tossed. Compute the entropy $H(X)$. **ANSWER:**

$$H(X) = \frac{1}{8}\big(-\log\frac{1}{8}\big) + \frac{3}{8}\big(-\log\frac{3}{8}\big) + \frac{3}{8}\big(-\log\frac{3}{8}\big) + \frac{1}{8}\big(-\log\frac{1}{8}\big).$$

(b) Suppose that X and Y are two (not necessarily independent or identically distributed) random variables, each of which takes values in the set $\{1,2,3,4,5,6\}$. Which of the following is *necessarily* true? (Just circle the corresponding letters.)

(i) $H(X,Y) \ge H(X) + H(Y)$ **ANSWER:** Not necessarily true. For example, if $H(X) > 0$ and $X = Y$ with probability one, then $H(X,Y) = H(X) \le H(X) + H(Y) = 2H(X)$. The properties derived in lecture imply that the inequality actually *does* hold in the other direction, since $H(X,Y) = H(X) + H_X(Y) \le H(X) + H(Y)$.

(ii) $H(X, Y) = H(X) + H(Y)$ **ANSWER:** Not necessarily true, by reasoning above.

(iii) $H(X) \leq H(X, Y)$. **ANSWER:** True, since $H(X, Y) = H(X) + H_X(Y)$ and $H_X(Y) \geq 0$.

(iv) $H(X + Y) \leq \log(11)$. **ANSWER:** True, since $X + Y$ takes one of 11 possibilities, and (as shown on problem set) entropy is at most what if would be if all 11 possibilities were equally likely.

(v) $H(X - Y) \geq 0$. **ANSWER:** True, since entropy of any random variable is non-negative.

9. (10 points) A certain country has three distinct types of leaders: liberal, conservative, and insane. Every four years they elect a new leader.

(i) If the current leader is liberal, there is a 2/3 chance the next leader will be liberal also, a 1/4 chance the next leader will be conservative, and a 1/12 chance the next leader will be insane.

(ii) If the currently leader is conservative, there is a 2/3 chance the next leader will be conservative also, a 1/4 chance the next leader will be liberal, and a 1/12 chance the next leader will be insane.

(iii) If the currently leader is insane, then there is a 1/2 chance the next leader will be conservative and a 1/2 chance the next leader will be liberal. (They never allow two consecutive terms of insanity.)

(a) Use L, C, I to denote the three states. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by L, C, and I. **ANSWER:**

$$\begin{pmatrix} 2/3 & 1/4 & 1/12 \\ 1/4 & 2/3 & 1/12 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(b) If the current leader is insane, what is the probability that, after two elections, the leader will be insane again? **ANSWER:** 1/12

(c) Over the long term, what fraction of the time does the country spend under each of the three types of leaders? **ANSWER:** You can write

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2/3 & 1/4 & 1/12 \\ 1/4 & 2/3 & 1/12 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}$$

and use $a + b + c = 1$ to solve and find $(a, b, c) = (6/13, 6/13, 1/13)$, which are the long term probabilities for states L, C and I respectively.

10. (10 points) A certain town (with constant climate) has had an average of one house fire per year for the past century. At the beginning of one calendar year, Jill moves to town, and during that year there are 6 house fires. A trial is held to determine whether the increase in fires is due to Jill being a witch. During the trial the judge asks a math expert the following (which you should answer):

(a) Suppose that house fire times in this town are a Poisson point process with parameter λ equal to 1 per year. Under this assumption, let p be the probability that there will be exactly 6 house fires during a single given year. What is p? **ANSWER:** $p = e^{-\lambda}\lambda^k/k!$ with $\lambda = 1$ and $k = 6$. Comes to $e^{-1}/6!$.

(b) Under the same assumption, what is the probability that, during the course of a century, there will be *at least* 1 calendar year during which there are exactly 6 house fires? Compute your answer in terms of the p computed in (a). **ANSWER:** $1 - (1-p)^{100}$.

When Jill first moved to town, Nora thought that there was a 1/100 chance that Jill was a witch. She also thought if Jill *wasn't* a witch the number of fires that year would be Poisson with parameter 1, and that if Jill *was* a witch the number would be 6 with probability 1. (Arranging for there to be exactly 6 fires per year is what arsonist witches in this world do.)

(c) *Given* that the number of observed fires during Jill's first year in town is 6, what is Nora's assessment of the *conditional* probability that Jill is a witch? Compute your answer in terms of the p computed in (a). **ANSWER:** Let F be event that there are six fires, W event Jill is a witch. Then

$$P(W|F) = \frac{P(WF)}{P(F)} = \frac{P(W)P(F|W)}{P(W)P(F|W) + P(W^c)P(F|W^c)} = \frac{(1/100)}{(1/100) + (99/100)p}.$$

Remark: In more realistic variants, "Jill" is a polluting industrial facility and "house fires" are health problems allegedly caused by pollution.