18.600 Midterm 1, Spring 2018: Solutions

1. (20 points) Roll six standard six-sided dice independently, and let X be the number of dice that show the number 6.

(a) Compute the expectation $E[X]$. **ANSWER:** The number of heads is binomial with $n = 6$ and $p = 1/6$, so $E[X] = np = 1$.

(b) Compute the variance $\mathrm{Var}(X)$. **ANSWER:** $\mathrm{Var}(X) = npq = 5/6$ (where $q = 1 - p$).

(c) Compute $P(X = 5)$. **ANSWER:**
$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{6}{5}(1/6)^5(5/6) = 6 \cdot 5/6^6 = 5/6^5$

(d) Compute $P(X = 6|X \geq 5)$. **ANSWER:** $P(X = 6) = (1/6)^6$ so

$$P(X = 6|X \geq 5) = \frac{P(X = 6)}{P(X = 6) + P(X = 5)} = \frac{1/6^6}{30/6^6 + 1/6^6} = 1/31$$

2. (10 points) Suppose that E, F and G are events such that

$$P(E) = P(F) = P(G) = .4$$

and

$$P(EF) = P(EG) = P(FG) = .2$$

and

$$P(EFG) = .1.$$

Compute $P(E \cup F \cup G)$. **ANSWER:** Inclusion-exclusion tells us

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$$

$$= 3(.4) - 3(.2) + .1 = 1.2 - .6 + .1 = .7$$

3. (10 points) Compute the following

(a)

$$\lim_{n \to \infty} (1 + 5/n)^n$$

ANSWER: General formula is $e^x = \lim_{n\to\infty}(1 + x/n)^n$. Plug in $x = 5$ and answer is e^5. Alternatively, one can just use definition of e (the special case $x = 1$). Write $n = 5m$ (which implies $5/n = 1/m$) and note that

$$\lim_{n\to\infty}(1 + 5/n)^n = \lim_{m\to\infty}(1 + 1/m)^{5m} = \lim_{m\to\infty}\left((1 + 1/m)^m\right)^5 =$$

$$\left(\lim_{m\to\infty}(1 + 1/m)^m\right)^5 = e^5.$$

(b)

$$\sum_{n=0}^{\infty} 5^n/n!$$

ANSWER: By Taylor expansion $e^x = \sum_{n=0}^{\infty} x^n/n!$. Setting $x = 5$ gives the answer e^5.

4. (20 points) Suppose that 10000 people visit Alice's new restaurant during its first few months of operation. Each person independently chooses to leave a positive Yelp review (e.g., "Great samosas!") with probability 5/10000, a negative Yelp review (e.g., "Rude servers!") with probability 1/10000 or no review at all with probability 9994/10000. Let X be total number of positive reviews received and Y the total number of negative reviews received.

(a) Compute $E(Y)$ and $\text{Var}(Y)$. (Give exact values, not approximations.)
ANSWER: This is binomial with $n = 10000$ and $p = 1/10000$ so $E[Y] = np = 1$ and $\text{Var}(Y) = np(1-p) = (1-p) = 9999/10000$.

(b) Use a Poisson random variable to approximate $P(X = 3)$.
ANSWER: $E[X] = 5$, so X is approximately Poisson with parameter $\lambda = 5$. This suggests

$$P(X = k) \approx e^{-\lambda}\lambda^k/k! = e^{-5}5^3/3! = \frac{125}{6e^5}.$$

(c) Use a Poisson random variable to approximate $P(Y = 0)$.
ANSWER: $e^{-\lambda}\lambda^k/k!$ with $k = 1$ and $\lambda = 1$ is $1/e$. Altenatively, just note directly that $P(Y = 0) = (1 - 1/10000)^{10000} \approx e^{-1} = 1/e$.

5. (10 points) Suppose that a deck of cards contains 120 cards: 30 red cards, 40 black cards, and 50 blue cards. A random collection of 12 cards is chosen (with all possible 12-card subsets being equally likely). What is the probability that this collection contains three red cards, four black cards, and five blue cards? **ANSWER:** Total number of ways to choose 12 cards is $\binom{120}{12}$. The number of ways to choose cards with desired color breakdown is $\binom{30}{3}\binom{40}{4}\binom{50}{5}$. So the ratio is

$$\frac{\binom{30}{3}\binom{40}{4}\binom{50}{5}}{\binom{120}{12}}.$$

6. (15 points) Ten people toss their hats in a bin and have them randomly shuffled and returned, one hat to each person. Let X_i be 1 if ith person

gets own hat back, 0 otherwise. Let $X = \sum_{i=1}^{10} X_i$ be the total number of people who get their own hat back. Compute the following:

(a) The expectation $E[X]$. **ANSWER:** $10 \cdot \frac{1}{10} = 1$

(b) The expectation $E[X_3X_7]$. **ANSWER:** X_3X_7 is 1 on the event that 3rd and 7th people both get own hats, and zero otherwise. So $E[X_3X_7]$ is the probability that both 3 and 7 their own hats. There are 8! permuations in which 3 and 7 get own hats, so answer is $8!/10! = 1/90$.

(c) The expectation $E[X_1^2 + X_2^2 + X_3^2]$. **ANSWER:** Note that $X_j^2 = X_j$ for each j, so this is just $E[X_1 + X_2 + X_3] = 3E[X_1] = 3/10$.

7. (15 points) Bob is at an airport kiosk considering the purchase of a bottle of water with no listed price. Bob is thirsty but is too shy to ask about the price. He happens to know that the price in dollars (denoted X) is an integer between 3 and 7 and he considers each of the values in $\{3, 4, 5, 6, 7\}$ to be equally likely (probability 1/5 for each). According to Bob's probability measure, find the following:

(a) $E[X]$ **ANSWER:** $\frac{1}{5}(3 + 4 + 5 + 6 + 7) = 5$

(b) $\mathrm{Var}[X]$ **ANSWER:** $\mathrm{Var}(X) = E[(X - 5)^2]$. Note that $(X - 5)^2$ is 4 with probability 2/5 and 1 with probability 2/5 and zero otherwise. So $\mathrm{Var}[X] = E[(X - 5)^2] = \frac{2}{5} \cdot 4 + \frac{2}{5} \cdot 1 = 2$.

(c) $\mathrm{Var}[1.05X]$ (That is, the variance of the sales-tax-inclusive price assuming the airport is in a state with five percent sales tax.) **ANSWER:** $\mathrm{Var}(1.05(X)) = 1.05^2\mathrm{Var}(X) = (21/20)^2 \cdot 2 = 441/200$