18.600 Midterm 1, Spring 2016 SOLUTIONS

1. (20 points) Let X_1, X_2, X_3 be independent random variables, each of which is equal to 1 with probability 1/4 and 0 with probability 3/4. Write $Y = X_1 + X_2$ and $Z = X_2 + X_3$. Compute the following:

(a) $E[Y]$ **ANSWER:** $E[X_1] + E[X_2] = 1/4 + 1/4 = 1/2$.

(b) $E[YZ]$ **ANSWER:** $E[X_1X_2 + X_1X_3 + X_2^2 + X_2X_3] =$

$$E[X_1X_2] + E[X_1X_3] + E[X_2^2] + E[X_2X_3] = \frac{1}{16} + \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = 7/16$$

(c) $E[Y^2]$ **ANSWER:** $E[X_1^2] + 2E[X_1X_2] + E[X_2^2] = \frac{1}{4} + \frac{2}{16} + \frac{1}{4} = 5/8$.

(d) $\mathrm{Var}[Y]$ **ANSWER:** $E[Y^2] - E[Y]^2 = 5/8 - 1/4 = 3/8$. Alternate solution: Y is binomial (n,p) so $\mathrm{Var}(y) = npq = 2(1/4)(3/4) = 3/8$.

(e) $\mathrm{Cov}(Y,Z)$ **ANSWER:** $E[YZ] - E[Y]E[Z] = 7/16 - (1/2)^2 = 3/16$. Alternate solution: bilinearity of covariance and fact that independent random variables have covariance zero imply $\mathrm{Cov}(X_1 + X_2, X_2 + X_3) = \mathrm{Cov}(X_2, X_2) = \mathrm{Var}(X_2) = 3/16$.

2. (20 points) At a certain small college, an entering class has 400 first-year college students. Each student tries out for the fencing team with probabiity 1/200, independently of what the other students do. Let N be the total number of first-year students who try out for the fencing team.

(a) Compute the expectation $E[N]$. Give an exact answer, not an approximation. **ANSWER:** $np = 400 \cdot \frac{1}{200} = 2$

(b) Compute the variance $\mathrm{Var}[N]$. Give an exact answer, not an approximation. **ANSWER:** $npq = 2 \cdot \frac{199}{200} = 1.99$

(c) Write down an exact formula for the probability that $N = k$ (for $k \in \{0, 1, \dots, 400\}$). **ANSWER:** $\binom{n}{k}p^k(1-p)^{n-k}$ where $p = 1/200$ and $n = 400$.

(d) Use a Poisson random variable to *approximate* the probability that $N = 3$. **ANSWER:** $e^{-\lambda}\lambda^k/k! = e^{-2}2^3/3!$.

3. (15 points) A deck of card contains 52 distinct card *types*, with exactly one card of each *type*. Suppose that one has two identical decks of cards (so 104 cards total) and that one accidentally loses 10 of these 104 cards

(chosen uniformly from the set of all possible 10-card subsets) so that one now has only 94 cards.

What is the probability that it is possible to form a single complete deck of cards from the 94 cards remaining? In other words, what is the probability that the set of 94 remaining cards includes *at least one* card of each of the 52 types? (Note: this is equivalent to the probability that the set of 10 lost cards includes *at most* one card of each type.)
ANSWER: $2^{10}\binom{52}{10}$ ways to choose the 10 types in the missing set, and which deck the card of each type came from. $\binom{104}{10}$ overall possibilities for missing set. Probability is the ratio $2^{10}\binom{52}{10}/\binom{104}{10}$.
4. (15 points) Seven people toss their hats in a bin and have them randomly shuffled and returned, one hat to each person. Let N be the number of people who get their own hat back. Compute the following:

(a) The expectation $E[N]$. **ANSWER:** By expectation additivity $E[N] = 7 \cdot \frac{1}{7} = 1$.

(b) The probability $P(N = 5)$. **ANSWER:** $\binom{7}{2}$ is number of ways to choose two people to have their hats swapped. So $P(N = 5) = \binom{7}{2}/7!$

(c) The conditional probability $P(N = 7|N \geq 5)$. **ANSWER:** $\frac{1/7!}{\binom{7}{2}/7!+1/7!} = \frac{1}{\binom{7}{2}+1} = 1/22$

5. (10 points) An urn contains 10 black balls and 10 white balls. If a collection of 8 balls is chosen uniformly at random from the urn, what is the probability that 4 of them are black and 4 of them are white?
ANSWER: $\binom{10}{4}\binom{10}{4}/\binom{20}{8}$
6. (20 points) Let X be the number on a standard die roll (assuming values in $\{1, 2, 3, 4, 5, 6\}$ with equal probability). Let Y be the number on an independent roll of the same die. Compute the following expectations:

(a) $E[X]$ **ANSWER:** $7/2$

(b) $E[X^2]$ **ANSWER:** $(1 + 4 + 9 + 16 + 25 + 36)/6 = 91/6$

(c) $E[5X^7 - 5Y^7 + 5]$ **ANSWER:** 5 by additivity of expectation, since $E[5X^7] = E[5Y^7]$.

(d) $E[XY]$ **ANSWER:** $E[X]E[Y] = 49/4$, by independence of X and Y.