18.440 Midterm 1, Spring 2011: 50 minutes, 100 points.
SOLUTIONS

1. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p.

(a) Let X be such that the first heads appears on the Xth toss. In other words, X is the number of tosses required to obtain a heads. Compute (in terms of p) the expectation $E[X]$. **ANSWER: geometric random variable with parameter p has expectation $1/p$.**

(b) Compute (in terms of p) the probability that exactly 5 of the first 10 tosses are heads. **ANSWER: binomial probability $\binom{10}{5}p^5(1-p)^5$**

(c) Compute (in terms of p) the probability that the 5th head appears on the 10th toss. **ANSWER: negative binomial. Need 4 heads in first 9 tosses, 10th toss heads. Probability $\binom{9}{4}p^4(1-p)^5 p$.**

2. (20 points) Jill sends her resume to 1000 companies she finds on monster.com. Each company responds with probability 3/1000 (independently of what all the other companies do). Let R be the number of companies that respond.

(a) Compute $E[R]$. **ANSWER: binomial random variable with $n = 1000$ and $p = 3/1000$. $E[R] = np = 3$.**

(b) Compute $\mathrm{Var}[R]$. **ANSWER: binomial random variable with $n = 1000$ and $p = 3/1000$. $\mathbf{Var}[R] = np(1-p) = 3(1 - 3/1000)$.**

(c) Use a Poisson random variable approximation to estimate the probability $P\{R = 3\}$. **ANSWER: R is approximately Poisson with $\lambda = 3$. So $P\{R = 3\} \approx e^{-\lambda}\lambda^k/k! = e^{-3}3^3/3! = 9e^{-3}/2$.**

3. (10 points) How many four-tuples (x_1, x_2, x_3, x_4) of *non-negative* integers satisfy $x_1 + x_2 + x_3 + x_4 = 10$? **ANSWER: represent partition with stars and bars $**|**||******$. Have $\binom{13}{3}$ ways to do this.**

4. (10 points) Suppose you buy a lottery ticket that gives you a one in a million chance to win a million dollars. Let X be the amount you win. Compute the following:

(a) $E[X]$. **ANSWER: $\frac{1}{10^6}10^6 = 1$.**

(b) $\mathrm{Var}[X]$. **ANSWER: $E[X^2] - E[X]^2 = \frac{1}{10^6}(10^6)^2 - 1^2 = 10^6 - 1$.**

5. (20 points) Suppose that X is continuous random variable with probability density function $f_X(x) = \begin{cases} 2x & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$. Compute the following:

(a) The expectation $E[X]$. **ANSWER:** $\int_{-\infty}^{\infty} f_X(x)xdx = \int_0^1 f_X(x)xdx = \int_0^1 2x^2dx = \frac{2}{3}x^3|_0^1 = 2/3$**.**

(b) The variance $\mathrm{Var}[X]$. **ANSWER:** $E[X^2] = \int_{-\infty}^{\infty} f_X(x)x^2dx = \int_0^1 f_X(x)x^2dx = \int_0^1 2x^3dx = \frac{2}{4}x^4|_0^1 = 1/2$**.** **So variance is** $1/2 - (2/3)^2 = 1/2 - 4/9 = 1/18$**.**

(c) The cumulative distribution function F_X. **ANSWER:**
$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a < 0 \\ a^2 & a \in [0,1] \\ 1 & a > 1 \end{cases}.$$

6. (20 points) A standard deck of 52 cards contains 4 aces. Suppose we choose a random ordering (all 52! permutations being equally likely). Compute the following:

(a) The probability that *all* of the top 4 cards in the deck are aces. **ANSWER:** $4!$ **ways to order aces,** $48!$ **ways to order remainder. Probability** $4!48!/52!$

(b) The probability that *none* of the top 4 cards in the deck is an ace.**ANSWER: choose cards one at a time starting at the top and multiply number of available choices at each stage to get total number. Probability is** $48 \cdot 47 \cdot 46 \cdot 45 \cdot 48!/52!$**.**

(c) The *expected* number of aces among the top 4 cards in the deck. (There is a simple form for the solution.) **ANSWER: have probability** $4/52 = 1/13$ **that top card is an ace. Similarly, probability** $1/13$ **that** j**th card is an ace for each** $j \in \{1,2,3,4\}$**. Additivity of expectation gives answer:** $4/13$**.**