Note: $\omega_E = *\omega_M = \omega$

$$\Rightarrow \begin{cases} d\omega = \Lambda \\ d^*\omega = 0. \end{cases}$$

Extra topics: symplectic form, Hamiltonian vector fields

§3. Integration of forms

Prop^n. $U, V \subseteq \mathbb{R}^n$ open subset, $f: U \to V$ diffeomorphism.

Then for a continuous function $\phi: V \to \mathbb{R}$, we have

$$\int_V \phi(y)\, dy = \int_U (\phi \circ f)(x) |\det Df(x)|\, dx.$$

Goal. Prove this using differential forms.

§3.1, 3.2 Poincaré lemma for compactly supported n-forms.

Def^n. For $\omega \in \Omega^k(\mathbb{R}^n)$, its support is defined to be

$$\operatorname{supp}(\omega) := \overline{\{x \in \mathbb{R}^n \mid \omega_x \neq 0\}}.$$

ω is compactly supported $\iff$ $\operatorname{supp}(\omega)$ is compact.

Notation. $\Omega_c^k(\mathbb{R}^n)$, $\Omega_c^k(U)$.

Def^n. For $U \subseteq \mathbb{R}^n$, if $\omega \in \Omega_c^n(U)$ and write $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$

for $f \in C_c^\infty(U)$, define

$$\int_U \omega = \int_{\mathbb{R}^n} f\, dx.$$

- Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle.

Thm (Poincaré lemma). Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ s.t. $\operatorname{supp}(\omega) \subseteq \operatorname{int}(Q) = (a_1, b_1) \times \cdots \times (a_n, b_n)$.

The TFAE.

(1) $\int \omega = 0$

(2) $\exists \mu \in \Omega_c^n(\mathbb{R}^n)$ w/. $\operatorname{supp}(\mu) \subseteq \operatorname{int}(Q)$ s.t. $d\mu = \omega$.

proof. (2) $\Rightarrow$ (1) This is integration by part.

Write $\mu = \sum_{i=1}^{n} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Then $d\mu = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$.

So, $\int_Q d\mu = \sum_{i=1}^{n} (-1)^{i-1} \int_Q \frac{\partial f_i}{\partial x_i}\, dx$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_{(a_1,b_1)\times\cdots\times\widehat{(a_i,b_i)}\times\cdots\times(a_n,b_n)} \int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i}\, dx_i\, dx_1 \cdots \widehat{dx_i} \cdots dx_n.$$

Note that $\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i}\, dx_i = f_i(b_i) - f_i(a_i) = 0$ because $\mathrm{supp}(\mu) \subseteq \mathrm{int}(Q)$,

so we conclude that $\int \omega = \int_Q d\mu = 0$.

(1) ⇒ (2). We prove it by induction on n.

For $n=1$, consider $\mu(t) = \int_{a_1}^{t} f(\tau)\, d\tau$, for $\omega = f(t)\, dt$.

Idea: μ should be the primitive of ω.

Then: (a) $d\mu = f(t)\,dt$, this is fundamental theorem of calculus.

(b) ~~μ~~ $\mathrm{supp}(\mu)$ is compact and $\subseteq (a_1, b_1)$ ⇒ inspection.

Assuming the assertion holds for some $U \subseteq \mathbb{R}^{n-1}$ (in partical, $(a_1,b_1)\times\cdots\times(a_{n-1},b_{n-1})$)

Consider $U \times (a_n, b_n)$, w/. coordinates $(x_1, \ldots, x_{n-1}, t)$.

Write $\omega = f(x,t)\, dt \wedge dx_1 \wedge \cdots \wedge dx_{n-1} \cancel{\wedge dt}$.

Consider $\theta := \left(\int_{a_n}^{b_n} f(x,t)\, dt\right) dx_1 \wedge \cdots \wedge dx_{n-1}$,

then $\int_U \theta = \int_{U\times(a_n,b_n)} \omega = 0$.

By the induction hypothesis, $\exists\, \alpha \in \Omega_c^{n-2}(U)$ s.t. $d\alpha = \theta$.

~~(Naively: take $dt\wedge\alpha$, then $d(dt\wedge\alpha) = dt\wedge\theta$.~~

Let $\rho: \mathbb{R} \to \mathbb{R}$ be a C^∞ function ~~w.t~~ w/. $\mathrm{supp}(\rho) \subseteq (a_n, b_n)$, $\int \rho = 1$.

Let $k = -\rho(t)\, dt \wedge \alpha \Rightarrow dk = \cancel{\rho(t)\wedge\theta}\ \rho(t)\, dt \wedge \theta$.

so $\omega - dk = \left(f(x,t) - \rho(t)\int_{a_n}^{b_n} f(x,t)\,dt\right) dt \wedge \cancel{\theta}\, dx_1 \wedge \cdots \wedge dx_{n-1}$

Note $\int_{a_n}^{b_n} \left(f(x,t) - \rho(t)\int_{a_n}^{b_n} f(x,t)\,dt\right) dt = 0$, so $\exists\, F(x,t)$, s.t.

$\forall x \in U$, $d_t F(x,t)\,dt = f(x,t) - \rho(t) \int_{a_n}^{b_n} f(x,t)\,dt$.

Therefore, $\omega - dk = \frac{\partial F}{\partial t}(x,t)\, dt \wedge \theta = d\,(\underbrace{F(x,t)\, \cancel{\theta}\, dx_1 \wedge \cdots \wedge dx_{n-1}}_{\gamma})$

$\Rightarrow \omega = d(k + \gamma)$, and $\mathrm{supp}(k+\gamma) \subseteq \mathrm{int}(Q)$. □

Thm. (Poincaré lemma) Let $U \subseteq \mathbb{R}^n$ be a connected open subset. Then for $\omega \in \Omega^n_c(U)$, TFAE.

(1) $\int_{\mathbb{R}^n} \omega = 0$

(2) $\exists\, \mu \in \Omega^{n-1}_c(U)$ s.t. $d\mu = \omega$

proof. (2) $\Rightarrow$ (1) Find a large rectangle Q in $\mathbb{R}^n$ s.t. $\mathrm{supp}(\omega) \subseteq Q$, then we can apply the previous statement.

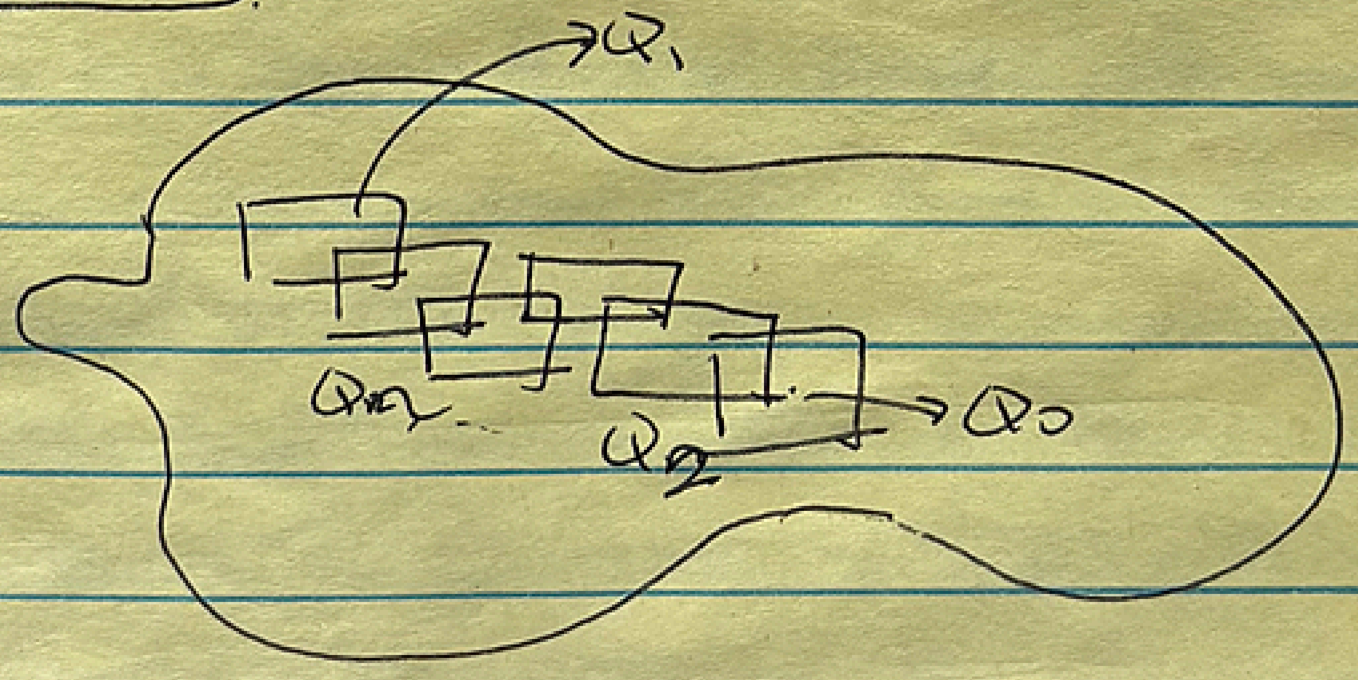
(1) $\Rightarrow$ (2)

Step 1 Claim. Given a rectangle $Q_0 \subseteq \mathbb{R}^n$, let $\omega_0 \in \Omega^n_c(\mathrm{int}(Q_0))$ w. $\int \omega_0 = 1$. Then for any $\alpha \in \Omega^n_c(\mathrm{int}(Q_0))$ s.t. $\int \alpha = c \in \mathbb{R}^n$, $c \cdot \omega_0$ and α are cohomologous: $\exists\, \mu \in \Omega^{n-1}_c(\mathrm{int}(Q_0))$ s.t. $\alpha - c\omega_0 = d\mu$.

proof of the claim observe that $\mathrm{supp}(\alpha - c\omega_0) \subseteq \mathrm{int}(Q_0)$, and $\int \alpha - c\omega_0 = 0 \Rightarrow \mu \in \Omega^{n-1}_c(\mathrm{int}(Q_0))$ $\exists$ by the rectangular Poincaré lemma.

Step 2 claim. For a connected U & two rectangles ~~in~~ $Q_0, Q_1 \subseteq U$, let $\omega_1 \in \Omega^n_c(\mathrm{int}(Q_1))$, $\omega_0 \in \Omega^n_c(\mathrm{int}(Q_0))$ s.t. $\int \omega_0 = 1$. If $\int_{Q_1} \omega_1 = c$, then $\exists\, \mu_1$ s.t. $d\mu_1 = \omega_1 - c \cdot \omega_0$.

proof of the claim.



Find a sequence of rectangles connecting Q_0 & Q_1.

Induction on length.

Key observation: to go from Q_0 to Q_2, we can assume that $\mathrm{supp}(\omega_0) \subseteq \mathrm{int}(Q_0) \cap \mathrm{int}(Q_2)$.

Upshot: we can always find $\tilde{\omega}_i$ w/. $\mathrm{supp}(\tilde{\omega}_i) \subseteq \mathrm{int}(Q'_i) \cap \mathrm{int}(Q'_{i+1})$

which is cohomologous to ω_0.

By connectivity, $\exists\ \widehat{\omega}$ s.t. $\mathrm{supp}(\widehat{\omega}) \subseteq \mathrm{int}(Q_1)$ s.t. $\widehat{\omega} - \omega = d\widehat{\mu}$.

Then we are reduced to Step 1.

<u>Step 3</u>. In general, cover U by open rectangles

$$U = \bigcup_{i=1}^{\infty} \mathrm{int}(Q_i).$$

Find a partition of unity subordinate to this covering,

<u>i.e.</u> $\phi_i : \mathrm{int}(Q_i) \to \mathbb{R}$ w/. cpt support, s.t.

after extending by 0, $\sum \phi_i = 1$ at each point.

As $\mathrm{supp}(\omega)$ is compact, we can find $Q_1, \dots, Q_n$ covering $\mathrm{supp}(\omega)$.

Write $\omega = \sum_{i=1}^{n} \phi_i \omega$, and define $c_i = \int_{Q_i} \phi_i \omega$.

Choose ω_0 w/. $\mathrm{supp}(\omega_0) \subseteq \mathrm{int}(Q_0)$ for some Q_0.

Then by <u>Step 2</u>, $\exists\ \mu_i \in \Omega_c^{n-1}(U)$ s.t. $\phi_i \omega = c_i \cdot \omega_0 + d\mu_i$.

Therefore,
$$\begin{aligned} \omega = \sum_{i=1}^{n} \phi_i \omega &= \sum_{i=1}^{n} c_i \omega_0 + \sum_{i=1}^{n} d\mu_i \\ &= \left(\int_{Q_i} \phi_i \omega\right) \cdot \omega_0 + d\left(\sum_{i=1}^{n} \mu_i\right) \\ &= \left(\int_U \omega\right) \cdot \omega_0 + d\left(\sum_{i=1}^{n} \mu_i\right) = d\left(\sum_{i=1}^{n} \mu_i\right). \end{aligned}$$

Let $\mu = \sum_{i=1}^{n} \mu_i$, then we know that $\omega = d\mu$. $\square$

§3.4 Degree of a differentiable mapping.

<u>Def^n</u>. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^k$ be open subsets. A continuous map $f : U \to V$ is called proper if $\forall K \subseteq V$ compact, $f^{-1}(K)$ is compact.

<u>Def^n</u>. ~~Let $V \subseteq \mathbb{R}^n$ be an open and connected set.~~

<u>Lemma</u>. If $f : U \to V$ is proper and $\omega \in \Omega_c^m(\cancel{\mathbb{R}^k}\ V)$, then $f^*\omega \in \Omega_c^m(\cancel{\mathbb{R}^n}\ U)$.

<u>proof</u>. $\mathrm{supp}(f^*\omega) \subseteq f^{-1}(\mathrm{supp}(\omega))$ ~~because f is continuous~~ is closed.

By properness, $\mathrm{supp}(f^*\omega)$ is compact. $\square$

<u>Def^n</u>. For $n = k$ and $f : U \to V$ proper, let $\omega_0 \in \Omega_c^n(\cancel{\mathbb{R}^n}\ V)$ s.t. $\int_V \omega_0 = 1$.

Define the degree of $\deg(f) := \int_U f^*\omega_0$.

Lemma. $\deg(f)$ is well-defined.

proof. If $\omega_0' \in \Omega_c^n(V)$ is another n-form s.t. $\int_V \omega_0' = 1$,

by the Poincaré lemma, we know that $\exists \mu \in \Omega_c^{n-1}(V)$ s.t. $\omega_0 - \cancel{\omega_0}\,\omega_0' = d\mu$.

Therefore, $\int_U f^*\omega_0 = \int_U f^*\omega_0' + \int_U f^* d\mu$

$$= \int_U f^*\omega_0' + \int_U d f^*\mu = \int_U f^*\omega_0'. \ \square$$

Prop^n. For any $\omega \in \Omega_c^n(V)$, we have

$$\int_U f^*\omega = \deg(f)\cdot \int_V \omega.$$

proof. By the Poincaré lemma, $\exists \mu \in \Omega_c^{n-1}(V)$ s.t.

$$\omega = \left(\int_V \omega\right)\cdot \omega_0 + d\mu.$$

So, $\int_U f^*\omega = \left(\int_V \omega\right)\left(\int_U f^*\omega_0\right) + \int_U d f^*\omega$

$$= \deg(f)\cdot \int_V \omega. \qquad \square$$

Prop^n. For $U, V, W \subseteq \mathbb{R}^n$ open subsets,

$$U \xrightarrow{f} V \xrightarrow{g} W \text{ s.t. } f, g \text{ are } C^\infty \text{ proper maps,}$$

we have $\deg(g\circ f) = \deg(g)\cdot \deg(f)$.

proof. For any $\omega \in \Omega_c^n(W)$, we have

$$\int_U (g\circ f)^*\omega = \int_U f^*(g^*\omega)$$

$$= \deg(f)\cdot \int_V g^*\omega = \deg(f)\cdot\deg(g)\cdot \int_W \omega. \ \square$$

Thm. Let A be an $n\times n$ ~~sin~~ nonsingular matrix and $f_A : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear mapping. Then $\deg(f_A) = 1$ if $\det(A) = 1$, $\deg(f_A) = -1$ if $\det(A) = -1$.

proof. ① Claim. If $A_k \xrightarrow{k\to\infty} A$ $(f_{A_k} \xrightarrow{C^\infty} f_A)$, then $\lim_{k\to\infty} \deg(f_{A_k}) = \deg(f_A)$.

$$\Rightarrow \deg(f_A) = \frac{\int f_A^*\omega_0}{\int \omega_0}$$

$$= \frac{\int \lim_{k\to\infty} f_{A_k}^*\omega_0}{\int \omega_0} = \frac{\lim_{k\to\infty}\int f_{A_k}^*\omega_0}{\int \omega_0} = \lim_{k\to\infty} \deg(f_{A_k}).$$

② Claim. It holds for

$$A = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & \lambda_{k+1}\begin{pmatrix} \cos\alpha_{k+1} & \sin\alpha_{k+1} \\ \sin\alpha_{k+1} & \cos\alpha_{k+1} \end{pmatrix} & \\ & & & & \ddots \\ & & & & & \lambda_m \begin{pmatrix} \cos\alpha_m & -\sin\alpha_m \\ \sin\alpha_m & \cos\alpha_m \end{pmatrix} \end{pmatrix}$$

Choose $\omega_0 = \rho(x_1)\cdots\rho(x_k)\,\rho(x_{k+1}, x_k)\cdots\rho(x_{n-1}, x_n)\,dx_1\cdots dx_n$

w/ $\int \rho(x_i) = 1$, cptly supported $1 \le i \le k$;

$\int \rho(x_{k+1}, x_k) = 1$ ..., cptly supported, rotationally symmetric.

Then by change of variable formula in dim 1, $\deg(A) = \mathrm{sign}(\det(A))$.

③ It holds for any A conjugate to above one by $SO(n)$.

$\Rightarrow$ If $A = O^T A_0 O$, then $\det(O) = 1$.

④ In general, for any A, we can find approximating A, so by ① $\Rightarrow$. $\square$

§3.5 The change of variable formula

$U, V \subseteq \mathbb{R}^n$ open subsets

$f: U \to V$ diffeomorphism (in particular, proper)

Def^n: f is <u>orientation-preserving</u>: iff $\det(Df(x)) > 0 \ \forall x \in U$

<u>orientation-reversing</u>: iff $\det(Df(x)) < 0 \ \forall x \in U$

Thm 1. $\deg(f) = +1$ if f is orientation-preserving; otherwise, $\deg(f) = -1$.

Thm 2. $\phi: V \to \mathbb{R}$ compactly supported continuous function

$\Rightarrow \int_U (\phi \circ f)(x) |\det(Df(x))| = \int_V \phi(y) dy$.

Proof of Thm 1: <u>Step 1</u>. <u>Trivial observation</u>

$Q \subseteq \mathbb{R}^n$ a ball centered at origin, $\omega_0 \in \Omega_c^n(Q)$ s.t. $\int_Q \omega_0 = 1$,

then for the identity map $id: x \mapsto x$, $\int_Q id^* \omega_0 = 1$.

$\Rightarrow \deg(id) = 1$.

<u>Step 2</u>. Suppose f ① $0 \in U$, $0 \in V$

② $f(0) = 0$

③ $Df(0) = Id$.

Then we want to prove that $\deg(f) = 1$ by interpolating between $f(x)$ and Id.

Claim: $\exists \delta > 0$ s.t. $\forall x \in B_\delta(0)$, $|f(x) - x| \le \frac{1}{2} x$.

Indeed, $D(f(x) - x)(0) = 0$, this follows by continuity of $D(f - Id)$ & m.v.t.

In particular, write $g(x) = f(x) - x$, then $g(B_\delta(0)) \subseteq B_{\frac{\delta}{2}}(0)$.

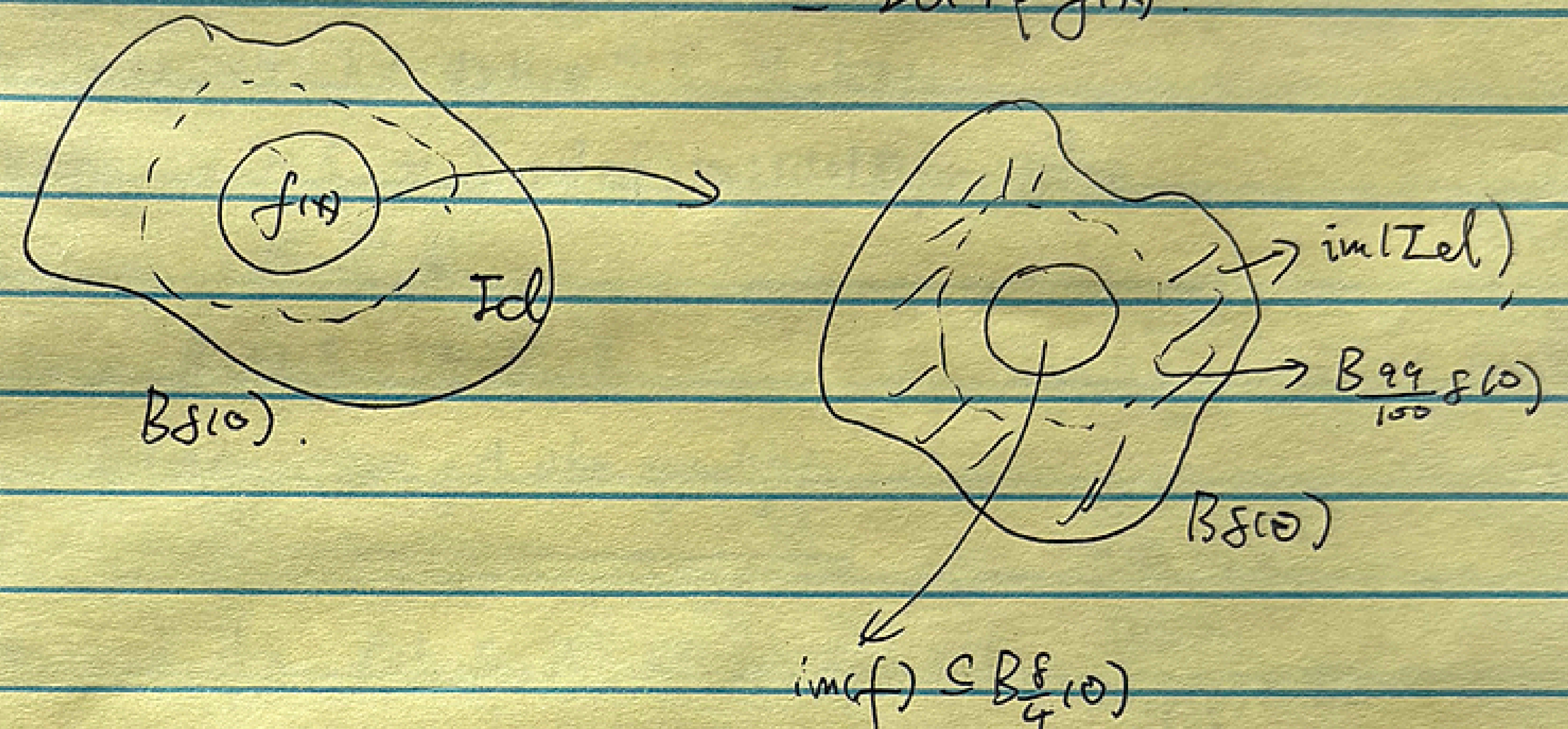
Now: patch x & $f(x)$ together.

Let $\rho: B_\delta(0) \to \mathbb{R}$ be compactly supported s.t.

$\rho = 1$ in $B_{\frac{\delta}{2}}(0)$, and $\rho \geq 0$ in $B_\delta(0)$.

Consider $F(x) = \cancel{\rho(x)\, Id + (1-\rho)\, f(x)}\ \rho \cdot f(x) + (1-\rho)\cdot Id$

$= Id + \rho \cdot g(x)$.



We can do two things

① Consider $\omega_0 \in \Omega_c^n(B_{\frac{\delta}{4}}(0))$ s.t. $\int \omega_0 = 1$.

Then $deg(f) = \dfrac{\int f^*\omega_0}{\int \omega_0} = \int f^*\omega_0 = \int F^*\omega_0$.

② Consider $\widetilde{\omega}_0 \in \Omega_c^n(B_\delta(0) \setminus B_{\frac{99}{100}\delta}(0))$ s.t. $\int \widetilde{\omega}_0 = 1$.

Then $deg(Id) = \int F^*\widetilde{\omega}_0$.

Both ① and ② calculate $deg(F) \Rightarrow deg(f) = deg(Id) = 1$.

<u>Step 3</u>. For a general f

Translate U in $\mathbb{R}^n$ s.t. $0 \in U$;

translate V in $\mathbb{R}^n$ s.t. $f(0) = 0$;

consider $Df(0)^{-1} \circ f$, and use $deg(Df(0)^{-1} \circ f)$

$= deg(Df(0)^{-1}) \cdot deg(f) = deg(f)$. □

<u>proof of Thm 2</u>. For $\phi \in C^\infty(V)$, we ~~get~~ see that

$$f^*(\phi(y)\, dy_1 \wedge \cdots \wedge dy_n) = (\phi \circ f)\, det(f)\, dx_1 \wedge \cdots \wedge dx_n.$$

So $\int_U (\phi \circ f)(x)\, |det(Df(x))|\, dx = \int_V \phi(y)\, dy$.

In general, it follows from: uniformly converge

<u>Lemma</u>. $\phi: V \to \mathbb{R}$ continuous, then $\exists\ \phi_k \xrightarrow{C^0} \phi$ s.t. $\phi_k \in C^\infty(V)\ \forall k$.

Then we have

$$\int_U (\phi_k \circ f)(x)\,|\det(Df(x))|\,dx = \int_V \phi_k(y)\,dy.$$

Use the compact support property,

we can assume that $vol(U), vol(V) \le C$.

Then $\forall\, \varepsilon > 0$, $\exists N$, s.t. $\|\phi - \phi_k\|_{C^0} < \varepsilon$ $\forall\, k \ge N$.

For such $k \ge N$, $L.H.S. - \int_U (\phi \circ f)(x)\,|\det(Df(x))|\,dx$

$$\le C \cdot \varepsilon,$$

$$R.H.S. \le C \cdot \varepsilon,$$

as ε is arbitrary, we get the equality.

To construct such ϕ_k, let $\rho : \mathbb{R}^n \to \mathbb{R}$ be C^∞ s.t. $\int \rho = 1$, $\rho \ge 0$, $supp(\rho) \subseteq B_1(0)$.

Then $\rho_\delta(x) := \rho(\frac{x}{\delta})$ satisfies $\int \rho_\delta = 1$, $\rho_\delta \ge 0$, $supp(\rho_\delta) \subseteq B_\delta(0)$.

Define $\phi_\delta(x) := \int \rho_\delta(\cancel{x-y}\ y-x)\,\phi(y)\,dy$.

Then (a) $\phi_\delta(x)$ is C^∞: commute the limit w/ integration.

(b) $\cancel{\phi_\delta(x) = \int \rho_\delta(y)\phi(x-y)\,dy}$, $\phi(x) = \phi(x) \cdot 1 = \phi(x) \int \rho_\delta(y-x)\,dy$,

$\cancel{\text{so } \phi(x) - \phi_\delta(x)}$ so $\phi_\delta(x) - \phi(x)$

$\cancel{\phi(x) = \phi_\delta(x)}$ $= \int \rho_\delta(y-x)(\phi(x) - \phi(y))\,dy$

$= \int \rho(\frac{y-x}{\delta})(\phi(x) - \phi(y))\,dy.$

Let $z = \frac{y-x}{\delta}$, then $y = x + \delta z$, and the integrand $\neq 0$ only if $|z| \le 1$.

$\Rightarrow \phi_\delta(x) - \phi(x) = \delta \int \rho(z) \cdot (\phi(x) - \phi(x + \delta z))\,dz.$

By making $\delta \ll 1$, we can guarantee that $|\phi(x) - \phi(x + \delta z)| < \varepsilon$ for any $\varepsilon > 0$.

$\Rightarrow \|\phi_\delta(x) - \phi(x)\| \le \varepsilon$ for such δ. Now choose $\varepsilon = \frac{1}{k}$. $\square$