

**Lectures on Categorical Dynamics  
and Symplectic Topology**

**Version 0.2 (last three lectures missing; error in Lecture  
13, marked as such, affects Lectures 13–16)**

Paul Seidel



# Contents

Introduction	5
<b>Part 1. Motivation</b>	7
Lecture 1. Vector bundles on projective space	9
Lecture 2. Equivariant cohomology	17
Lecture 3. Mirror symmetry and circle actions	23
Lecture 4. Derived Picard groups	35
Lecture 5. Flux	41
Lecture 6. Liouville manifolds	51
<b>Part 2. Background</b>	61
Lecture 7. Homological algebra	63
Lecture 8. Hochschild homology	75
Lecture 9. Hochschild cohomology	81
Lecture 10. The Fukaya category of a surface	91
Lecture 11. A four-dimensional example	103
Lecture 12. Symplectic cohomology	117
<b>Part 3. Circle actions</b>	129
Lecture 13. Equivariant modules <b>***WARNING***: CONTAINS AN ERROR (MARKED AS SUCH)</b>	131
Lecture 14. Making objects equivariant <b>***WARNING***: PART OF THIS INHERITS ERRORS FROM THE PREVIOUS LECTURE</b>	145
Lecture 15. Spherical objects and simple singularities <b>***WARNING***: SAME ERROR PROPAGATES</b>	157

Lecture 16. Suspension of Lefschetz fibrations ***WARNING***: SAME ERROR PROPAGATES	165
<b>Part 4. Infinitesimal symmetries</b>	173
Lecture 17. Basic structures	175
Lecture 18. Dilations	185
Lecture 19. Quasi-dilations	193
<b>Part 5. Families of objects</b>	199
Lecture 20. Basic notions	201
Lecture 21. Elliptic curves and mapping tori	203
Lecture 22. Analytic and formal geometry	205
Bibliography	207

## Introduction

These are the notes for an advanced graduate course (given at MIT in Spring 2013). Having been  $\text{\LaTeX}$ 'd, they may look good, but in fact they are in no way comparable to a finished manuscript, insofar as thoroughness and attention to detail are concerned (if you find one of the presumably many errors, please let the author know). Tongue-in-cheek, one could describe these lectures as an antidote to [176]. Whereas that book was focused and self-contained, the discussion here tries to be more open-minded and diverse. This comes at the cost of being much more tentative, and not reaching any major new results.

The subject of the lectures is *symmetries of Fukaya categories*. This means that the technical groundwork is largely of the general categorical kind, more precisely taking place in the framework of  $A_\infty$ -categories. Motivation is drawn from algebraic geometry and mirror symmetry, but the ultimate interest is in applications to symplectic topology. More precisely, we consider:

- *Circle actions*, which more appropriately means actions of the multiplicative group  $G = \mathbb{G}_m$ . This does not mean that we are looking at  $G$ -actions on symplectic manifolds (which would be an entirely different topic)! Instead, let's say that we start with the well-known theory of equivariant coherent sheaves on algebraic varieties with  $G$ -actions. More precisely, the main case of interest is that of a  $G$ -action on a Calabi-Yau variety which rescales the complex symplectic form. The presence of such a symmetry clearly has interesting implications. Having introduced a corresponding abstract algebraic notion of a *dilating  $G$ -action* on an  $A_\infty$ -category, we then apply that to Fukaya categories, following the model of [182]. The main drawback is that, while one can show that some examples of Fukaya categories carry such actions, there is at present no way of constructing these actions geometrically.
- Infinitesimal actions, or *categorical vector fields*, which concretely are just elements of the first Hochschild cohomology. Obviously, a general vector field is a much weaker starting point than a circle action, but one can still extract additional structure from it. On the algebro-geometric side, we would again be interested in vector fields which rescale a complex volume form. The corresponding geometric notion for a symplectic manifold was already introduced in [184] (following ideas of Bezrukavnikov), and called *dilation* (we will find it useful to generalize it a little; similar ideas appear in ongoing work of Abouzaid-Smith on symplectic Khovanov homology).

- *Orbits and vector fields.* Just as in standard ODE theory, one can try to integrate a categorical vector field. Generally speaking, this is an analytic rather than algebraic problem, but there may be orbits of the action which are algebraic (morally, these orbits are curves lying on some “moduli space of objects”, but we consider them more concretely as algebraic families of objects). There is an algebro-geometric model in the form of the derived Picard group of an algebraic variety. However, the main motivation is inherently one of symplectic geometry, where symplectic isotopies which are not Hamiltonian are known to give rise to interesting questions and results (such as the “flux conjecture” [144] or the convergence theory of [70]). The ultimate task, which is far from having been achieved, is to find the correct description of such geometric phenomena in the framework of Fukaya categories.

The word “categorical dynamics” seems to appropriately cover all three parts above, and we have adopted it as part of the title. This is in spite of the fact that it has been previously used in quite a different sense (in [118], which initiated the development of categorical methods in differential geometry, leading to what is now called “synthetic differential geometry” [112]).

The actual structure of the lectures is as follows. The first part (Lectures 1–6) describes the motivations and models for the subsequent developments, drawn both from symplectic topology and other parts of mathematics. Lecture 3, which describes the simplest example of equivariant mirror symmetry, may be the most noteworthy one, since this subject has received considerable interest recently (at a level far deeper than what we are aiming for here, see e.g. [130]). The second part (Lectures 7–12) contains more technical background, much of which is surveyed without going into the details (besides the previously mentioned [176], readers interested in a more thorough exposition may want to look at e.g. [106, 120, 28] for  $A_\infty$ -categories; [71, 21] for Fukaya categories; and [138, 175, 157] for symplectic cohomology). There are a few interesting technical wrinkles in our definition of the Fukaya category (Lecture 10), but otherwise the most significant contribution may be the discussion of local mirror symmetry in Lecture 11 (again, this is limited to the simplest example). The remaining three parts discuss the three topics described above. The first of those (circle action) is essentially algebraic, and even in a limited space one can cover it reasonably well. In contrast, our treatment of the other two topics is probably best thought of as an introduction, which hopefully motivates the reader to dig into the current literature.

*Acknowledgments.* My foremost thanks go to the students who attended the class, and to my collaborators and discussion partners; their specific contributions are mentioned at the start of the relevant lecture. The preparation of these notes was partially supported by a Simons Investigator grant from the Simons Foundation, as well as by NSF grant DMS-1005288.

## Part 1

# Motivation





## LECTURE 1

# Vector bundles on projective space

This lecture is based on Polishchuk's paper [153], whose exposition we simplify by a drastic reduction in generality. For background material on holomorphic vector bundles, see for instance [143, 161]. This is a deep subject, and one to which we cannot really do justice. For us, it serves as a convenient point of entry into several topics which will reappear later: the Mukai pairing; Lefschetz traces; and maybe most significantly, the idea that sufficiently rigid objects can be made equivariant with respect to actions of reductive groups.

*Acknowledgments.* The argument using  $SU(n)$  symmetry presented at the end is due to Lucas Culler; I thank him for allowing me to include it here.

### The Mukai pairing

Let  $V = \mathbb{C}^n$ , and  $M = \mathbb{P}(V)$  its projective space. We are interested in holomorphic vector bundles  $E$  on  $M$  which are *exceptional*, meaning that

$$(1.1) \quad \text{Ext}_M^i(E, E) = H^i(M, E^\vee \otimes E) \cong \begin{cases} \mathbb{C} & i = 0, \\ 0 & i \neq 0. \end{cases}$$

In particular, such vector bundles are indecomposable (by looking at the  $i = 0$  case of the equation above) and rigid, which means that they do not admit nontrivial deformations (taking  $i = 1$ ). Hence, it is not a priori unreasonable to aim for a complete classification. The case of the projective line ( $n = 2$ ) is trivial: any line bundle is exceptional, and by Grothendieck's theorem [87], all higher rank bundles are decomposable, hence certainly not exceptional.

A useful start is to look at the  $K$ -theoretic implications of exceptionality. Namely, let  $K_0(M)$  be the Grothendieck group of holomorphic vector bundles. The *Mukai pairing* is the unsymmetric bilinear form on  $K_0(M)$  defined by

$$(1.2) \quad (E_0, E_1)_{\text{Mukai}} = \sum_i (-1)^i \dim \text{Ext}_M^i(E_0, E_1) = \sum_i (-1)^i \dim H^i(M, E_0^\vee \otimes E_1).$$

The Mukai pairing is quite accessible for general varieties, since it factors through topological  $K$ -theory, and can be computed in terms of ordinary cohomology by the Grothendieck-Riemann-Roch theorem. In our case, algebraic and topological  $K$ -theory actually coincide,  $K_0(M) \cong \mathbb{Z}^n$ . An explicit basis is provided by the Beilinson [26] collection of vector bundles,

$$(1.3) \quad F_1 = \Omega_M^{n-1}(n-1), F_2 = \Omega_M^{n-2}(n-2), \dots, F_n = \mathcal{O}_M,$$

where  $\Omega_M^*$  are the sheaves of holomorphic differential forms, and  $(i)$  is tensoring by the line bundle  $\mathcal{O}_M(i)$ . This collection has particularly nice properties: the only nonzero  $Ext$  spaces between the  $F_i$  are the degree zero morphism spaces

$$(1.4) \quad Hom_M(F_i, F_j) \cong \Lambda^{j-i}(V) \quad \text{for } i \leq j.$$

In particular, each  $F_i$  itself is exceptional. If we choose the  $F_i$  as basis for  $K_0(M)$ , the Mukai pairing is given by the ranks of (1.4), which means by the upper triangular matrix

$$(1.5) \quad \begin{pmatrix} 1 & n & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} \\ 0 & 1 & n & \binom{n}{2} & \cdots & \binom{n}{n-2} \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & & & 0 & 1 & n \\ 0 & \cdots & & & 0 & 1 \end{pmatrix}.$$

By definition, an exceptional vector bundle satisfies  $(E, E)_{Mukai} = 1$ . For instance, this can be used to show that:

LEMMA 1.1 ([61]). *Let  $E$  be an exceptional vector bundle on the projective plane ( $n = 3$ ). Then its rank is not a multiple of 3. Moreover, its rank and degree (defined as the integral of  $c_1(E)$  over a line) are relatively prime.*

PROOF. If the class of  $E$  in  $K_0(M)$  is  $(r_1, r_2, r_3)$ , then

$$(1.6) \quad 1 = (E, E)_{Mukai} = r_1^2 + r_2^2 + r_3^2 + 3(r_1r_2 + r_1r_3 + r_2r_3).$$

We have

$$(1.7) \quad \deg(E) = -r_1 - r_2,$$

$$(1.8) \quad \text{rank}(E) = r_1 + 2r_2 + r_3.$$

If we assume that  $\text{rank}(E) \equiv 0 \pmod{3}$ , then  $r_2 \equiv r_1 + r_3$ , and therefore  $(E, E)_{Mukai} \equiv -(r_1 - r_3)^2$ , which contradicts (1.6).

For the second statement, note that  $\text{rank}(E) \equiv r_2 + r_3 \pmod{\deg(E)}$ . Therefore,

$$(1.9) \quad \begin{aligned} 1 = (E, E)_{Mukai} &= r_1^2 + r_2^2 + r_3^2 + 3(r_1r_2 + r_1r_3 + r_2r_3) \\ &\equiv r_3^2 - r_2^2 \equiv (r_3 - r_2)\text{rank}(E) \pmod{\deg(E)}. \end{aligned}$$

□

REMARK 1.2. *In fact, exceptional vector bundles on the projective plane have been completely classified [61, 83]. While that result requires deeper algebro-geometric tools, Hirzebruch-Riemann-Roch computations such as the one above still play an important role in it.*

Unfortunately, the implications of the single equation  $(E, E)_{Mukai} = 1$  get weaker as the dimension increases. As an illustration, let's look again at the question of ranks, assuming

for simplicity that  $n$  is a prime number. In that case the  $K$ -theory class  $(r_1, \dots, r_n)$  of an exceptional bundle  $E$  satisfies

$$(1.10) \quad r_1^2 + \dots + r_n^2 \equiv 1 \pmod{n},$$

since (1.5) is diagonal mod  $n$ . Similarly, since

$$(1.11) \quad \text{rank}(F_{i-1}) + \text{rank}(F_i) = \binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i} \equiv 0 \pmod{n},$$

we have

$$(1.12) \quad \text{rank}(E) \equiv r_1 - r_2 + r_3 - \dots \pmod{n}.$$

Unfortunately, (1.10) does not seem to constrain (1.12) in general. It turns out that one can introduce an equivariant version of the Mukai pairing, for which the analogue of (1.10) is much more effective, in particular yielding strong restrictions on the ranks of exceptional bundles for prime  $n$  (Corollary 1.7).

### Cyclotomic integers

As preparation, we need some elementary number-theoretic results. For some integer  $p \geq 2$ , let  $\mathbb{Z}[\zeta] \subset \mathbb{C}$  be the subring generated by  $\zeta = e^{2\pi i/p}$ .

LEMMA 1.3. [153, Proposition 2.6] *Take  $z_1, \dots, z_r \in \mathbb{Z}[\zeta]$ , such that  $|z_1|^2 + \dots + |z_r|^2 = 1$ . Then all but one of the  $z_i$  must be zero, and the remaining one is a root of unity.*  $\square$

If  $p$  is prime,  $\mathbb{Z}[\zeta]$  is isomorphic to the abstractly defined ring  $\mathbb{Z}[t]/(1+t+\dots+t^{p-1})$  (where  $t$  is a variable), by mapping  $t$  to  $\zeta$ . In particular:

LEMMA 1.4. *If  $p$  is prime, the only roots of unity contained in  $\mathbb{Z}[\zeta]$  are  $\pm\zeta^k$ .*  $\square$

LEMMA 1.5. *If  $p$  is prime, there is a unique ring homomorphism  $\mathbb{Z}[\zeta] \rightarrow \mathbb{Z}/p$  which takes  $\zeta \mapsto 1$ . It fits into a commutative diagram*

$$(1.13) \quad \begin{array}{ccc} \mathbb{Z}[t]/(t^p - 1) & \xrightarrow{t \mapsto 1} & \mathbb{Z} \\ \downarrow t \mapsto \zeta & & \downarrow \text{reduction mod } p \\ \mathbb{Z}[\zeta] & \dashrightarrow & \mathbb{Z}/p. \end{array}$$

$\square$

### Lefschetz traces

Take a finite subgroup  $G \subset GL(V)$ . Using equivariant vector bundles for the obvious action of  $G$  on  $M$ , one defines the equivariant  $K$ -theory  $K_0^G(M)$ . This is naturally a module over the virtual representation ring  $K_0(\mathbb{C}[G])$ . Beilinson's original argument goes through in this context, and shows that the sheaves from (1.3), with their natural equivariant structure, are basis elements for  $K_0^G(M) \cong K_0(\mathbb{C}[G])^n$ . There is also an equivariant version of the Mukai

pairing, denoted by  $(\cdot, \cdot)_{Mukai}^G$ , where one considers the  $Ext$  groups in (1.2) as representations of  $G$ , and which therefore takes values in  $K_0(\mathbb{C}[G])$ . Note that this equivariant pairing is hermitian in the following generalized sense: if  $\rho$  is a representation of  $G$ , then the equivariant Mukai pairing satisfies

$$(1.14) \quad \begin{aligned} (E_0 \otimes \rho, E_1)_{Mukai}^G &= (E_0, E_1)_{Mukai}^G \otimes \rho^\vee, \\ (E_0, E_1 \otimes \rho)_{Mukai}^G &= (E_0, E_1)_{Mukai}^G \otimes \rho. \end{aligned}$$

From now on, we take  $G$  to be a cyclic group of prime order  $p$ , generated by some element  $g \in GL(V)$ . Then  $K_0(\mathbb{C}[G]) \cong \mathbb{Z}[t]/(1-t^p)$ , where  $t = [\chi]$  is the class of the one-dimensional representation in which  $g$  acts by the root of unity  $\zeta = e^{2\pi i/p}$ . One can use the map  $K_0(\mathbb{C}[G]) \rightarrow \mathbb{Z}[\zeta]$ ,  $t \mapsto \zeta$  (more geometrically, this is given by the trace of the action of  $\zeta$  on representations) to define a simplified version of the Grothendieck group

$$(1.15) \quad K_0^G(M) \otimes_{K_0(\mathbb{C}[G])} \mathbb{Z}[\zeta] \cong \mathbb{Z}[\zeta]^n.$$

By Lemma 1.5, this sits in a commutative diagram

$$(1.16) \quad \begin{array}{ccc} K_0^G(M) & \xrightarrow{\text{forgetful map}} & K_0(M) \\ \downarrow & & \downarrow \text{reduction mod } p \\ K_0^G(M) \otimes_{K_0(\mathbb{C}[G])} \mathbb{Z}[\zeta] & \dashrightarrow & K_0(M) \otimes \mathbb{Z}/p. \end{array}$$

This means that, in this situation, the class of an equivariant sheaf in (1.15) recovers its non-equivariant  $K$ -theory class mod  $p$ . There is a corresponding simplification of the equivariant Mukai pairing, which can be concretely written as the Lefschetz trace of the action of  $g$ :

$$(1.17) \quad (E_0, E_1)_{Mukai}^g = \sum_i (-1)^i \text{Tr}(g : Ext_M^i(E_0, E_1) \rightarrow Ext_M^i(E_0, E_1)) \in \mathbb{Z}[\zeta].$$

As a consequence of (1.14), this pairing is hermitian in the ordinary sense, with respect to the structure of (1.15) as a module over  $\mathbb{Z}[\zeta] \subset \mathbb{C}$ .

We now make our particular choice of group  $G \subset GL(V)$ . Namely, we assume from now on that  $n = \dim(M) + 1$  is prime, and choose the subgroup of size  $p = n$  generated by  $g = \text{diag}(1, \zeta, \dots, \zeta^{n-1})$ . The linear algebra computation

$$(1.18) \quad \sum_{k=0}^n (-1)^k x^k \text{Tr}(g : \Lambda^k V \rightarrow \Lambda^k V) = \det(id - xg : V \rightarrow V) = \prod_{j=0}^{n-1} (1 - x\zeta^j) = 1 - x^n$$

( $x$  a formal variable) shows that the trace of  $g$  on all the nontrivial representations in (1.4) is zero. This means that in the basis given by the  $F_i$ , the pairing (1.17) becomes the standard hermitian form  $(z_1, \dots, z_n) \mapsto |z_1|^2 + \dots + |z_n|^2$ .

**PROPOSITION 1.6.** *Suppose that  $n$  is prime. Let  $E$  be an exceptional holomorphic vector bundle on  $M$  which is  $G$ -equivariant. Then its class in  $K_0(M)$  is congruent mod  $n$  to one of the classes of the  $F_i$  or its opposite.*

PROOF. The action of  $G$  on  $\text{Hom}_M(E, E)$  is trivial, since it must preserve the identity endomorphism. Hence  $(E, E)_{\text{Mukai}}^g = 1$ . Let  $(z_1, \dots, z_n)$  be the class of  $E$  in (1.15). Our previous argument shows that  $|z_1|^2 + \dots + |z_n|^2 = 1$ . In view of Lemmas 1.3 and 1.4, this means that  $(z_1, \dots, z_n) = (0, \dots, \pm \zeta^k, \dots, 0)$ . The result then follows by applying (1.16).  $\square$

COROLLARY 1.7. *In the situation of Proposition 1.6,  $\text{rank}(E) \equiv \pm 1 \pmod{n}$ .*

PROOF. This follows directly from (1.12) and Proposition 1.6.  $\square$

### Making vector bundles equivariant

It may appear that the considerations above do not address the original problem. Generally speaking, there is no reason why an exceptional vector bundle on some variety should be equivariant under a discrete group of symmetries of that variety. However, the situation for continuous symmetries (connected Lie groups or algebraic groups) is quite different. The rigidity of exceptional vector bundles provides an infinitesimal version of equivariance, and this can be integrated under suitable assumptions on the group. In particular:

PROPOSITION 1.8. *Let  $M$  be a compact complex manifold carrying an action of the circle  $S^1$ . Let  $E$  be a holomorphic vector bundle over  $M$  such that  $\text{Ext}_M^1(E, E) = 0$  and  $\text{Hom}_M^0(E, E) \cong \mathbb{C}$ . Then  $E$  can be made  $S^1$ -equivariant.*

PROOF. We first proceed infinitesimally, meaning that we use only the holomorphic vector field  $Z$  which generates our group action. Given that, for any holomorphic vector bundle  $E$  we have a well-defined class

$$(1.19) \quad \text{Def}(E) \in \text{Ext}_M^1(E, E) = H^1(M, \text{End}(E)),$$

which expresses the infinitesimal deformation of  $E$  obtained by moving it in  $Z$ -direction. Generally, this can be obtained by taking the Atiyah class [19]

$$(1.20) \quad \text{At}(E) \in \text{Ext}^1(E, \Omega_M^1 \otimes E)$$

and pairing that with  $Z$ . To get a concrete representative, choose a lift of  $Z$  to a  $C^\infty$  vector field  $\tilde{Z}$  on the total space of  $E$ , which is fibrewise linear (this is like choosing a  $C^\infty$  connection, but where only differentiation in direction of  $Z$  is allowed). In a local holomorphic trivialization of  $E$ ,

$$(1.21) \quad \tilde{Z}_{x,\xi} = (B_x \xi, Z_x),$$

where  $B$  is a matrix-valued  $C^\infty$ -function of the base coordinates  $x$ . Changing the trivialization by  $\Phi$  transforms  $B$  into  $\Phi B \Phi^{-1} + (Z \cdot \Phi) \Phi^{-1}$ . Hence, the expressions  $\bar{\partial} B$  glue together to give a globally well-defined section of  $\Omega^{0,1}(M, \text{End}(E))$ , which we denote by  $\text{def}(E, \tilde{Z})$ . Its Dolbeault cohomology class is independent of the choice of lift: changing  $\tilde{Z}$  by adding  $C \in C^\infty(M, \text{End}(E))$  just means adding  $\bar{\partial} C$  to  $\text{def}(E, \tilde{Z})$ . The cohomology class defined in this way is the Dolbeault version of (1.19).

Under our assumptions, that Dolbeault cohomology class is necessarily zero. Hence, by the same argument as before, one can choose  $\tilde{Z}$  so that  $\text{def}(E, \tilde{Z}) = 0$ . Then, integrating  $\tilde{Z}$  yields a family of automorphisms compatible with the holomorphic structure, which lift the flow of  $Z$ . Now supposing that  $Z$  generates a circle action, look at the flow of  $\tilde{Z}$  going once around the circle, which gives an automorphism of  $E$  (trivial over the base). By the second part of our assumptions, this automorphism is multiplication with some nonzero complex number. By subtracting the logarithm of that number (times the identity) from  $\tilde{Z}$ , we can modify its flow so that the automorphism becomes the identity. This defines the desired  $S^1$ -action on  $E$ .  $\square$

Proposition 1.8 is folklore. More precisely, what may be more familiar is the algebro-geometric version, in which  $S^1$  is replaced by  $\mathbb{C}^*$ , and compact complex manifolds by (possibly singular) proper schemes. In that form, the statement appears in the literature about exceptional vector bundles, and a purely algebro-geometric proof is given in the Appendix to [29] (written by Vologodsky, and unfortunately absent from the published version of the paper). Returning to the original application to projective space  $M = \mathbb{P}(V)$ , one can make any exceptional vector bundle equivariant with respect to the group of matrices  $\text{diag}(1, t, t^2, \dots, t^{n-1}) \in GL(V)$ , where  $t \in S^1$ , and then restrict to the subgroup generated by  $t = \zeta$ . Hence:

**COROLLARY 1.9** ([152, Theorem 1.2 and Corollary 1.3]). *The conclusions of Proposition 1.6 and Corollary 1.7 hold for all exceptional vector bundles.*

There are two directions in which one can go from here. One is the generalisation to chain complexes of vector bundles, which means exceptional objects of the bounded derived category. The analogue of Corollary 1.9 still holds in that context, see again [153]. The strategy of proof remains the same, but a highly nontrivial generalisation of Proposition 1.8 is necessary (we will return to this issue later). The other possible development is to make exceptional vector bundles equivariant under the entire torus  $(\mathbb{C}^*)^{n-1}$  of automorphisms of  $M$ , thus putting them into the general framework of toric vector bundles. For results that can be obtained in this way, see for instance [99].

### A variant argument

As it turns out, one can bypass the number-theoretic considerations above by making more use of symmetry and representation theory. Namely, let  $G = SU(V)$  be the special unitary group, and  $\mathcal{R}^{fin}(G)$  the category of finite-dimensional complex representations. Recall (see for instance [40, p. 265]) that

$$(1.22) \quad K_0(\mathcal{R}^{fin}(G)) = \mathbb{Z}[\lambda_1, \dots, \lambda_{n-1}],$$

where  $\lambda_i = \Lambda^i(V)$  are the exterior powers of the fundamental representation. Dualization defines an involution on this ring, explicitly given by  $\lambda_i^\vee = \lambda_{n-i}$ . We denote by  $I \subset K_0(\mathcal{R}^{fin}(G))$  the ideal generated by  $(\lambda_1, \dots, \lambda_{n-1})$ .

The equivariant  $K$ -theory of projective space with respect to the obvious  $G$ -action is again  $K_0^G(M) \cong K_0(\mathcal{R}^{fin}(G))^n$ , with Mukai pairing

$$(1.23) \quad (F_i, F_j)^G = \begin{cases} 0 & i > j, \\ 1 & i = j, \\ \lambda_{j-i} & i < j. \end{cases}$$

Take a  $G$ -equivariant exceptional vector bundle  $E$ , and write its  $K$ -theory class as  $(z_1 = x_1 + y_1, \dots, z_n = x_n + y_n)$ , where  $x_i \in \mathbb{Z} \subset K_0(\mathcal{R}^{fin}(G))$  is a multiple of the identity, and  $y_i \in I$ . Then

$$(1.24) \quad 1 = (E, E)^G = \sum_i z_i^\vee z_i + \sum_{i < j} z_i^\vee z_j \lambda_{j-i} \equiv x_i^2 \pmod{I}.$$

Hence, exactly one of the  $x_i$  is  $\pm 1$ , and the others are 0.

Now suppose as before that  $n$  is prime. Under the forgetful map (which counts the virtual dimension)  $\mathcal{R}^{fin}(G) \rightarrow \mathbb{Z}$ , any element of  $I$  goes to a number in  $n\mathbb{Z}$ . Therefore, under the corresponding map  $K_0^G(M) \rightarrow K_0(M)$ , the element  $(z_1, \dots, z_n)$  goes to  $(x_1, \dots, x_n) \pmod{n}$ . This proves the analogue of Proposition 1.6 for bundles which are equivariant with respect to our group  $G$ . On the other hand, a more complicated version of the argument from Proposition 1.8 (or alternatively, an appeal to [152, Lemma 2.1]) shows that any exceptional vector bundle can be equivariant.

REMARK 1.10. *From this perspective, the role of the previously chosen cyclic subgroup can be explained as follows. The character yields a map from  $K_0(\mathcal{R}^{fin}(G))$  to the ring of class functions on  $G$ . As (1.18) shows, the element  $g = \text{diag}(1, \zeta, \dots, \zeta^{n-1})$  is chosen so that the ideal  $I$  maps to that of functions vanishing at  $g$ .*

REMARK 1.11. *In the case where  $n = p^k$  is a power of a prime  $p$ , the argument above gives a congruence mod  $p$ . That statement also already appears in [152], with a proof that generalizes the first argument (based on equivariance for cyclic groups) described here.*





## LECTURE 2

# Equivariant cohomology

In Lecture 1, we encountered equivariant  $K$ -theory as a recipient for information about equivariant vector bundles. In that particular case (of projective space), it was easy to describe the relevant Grothendieck group; but such a situation is rather untypical, and in general one often prefers to pass from  $K$ -theory to some form of cohomology.

We start our discussion by recalling the standard topological notion of  $G$ -equivariant cohomology [46] (for simplicity, restricting to the case of  $G = S^1$  acting smoothly on a manifold). In complex geometry, there is also an analogue based on Hodge rather than de Rham cohomology [193, 122]. Interestingly, part of that theory makes sense infinitesimally, which means for arbitrary holomorphic vector fields. This is related to the insight, going back at least to [34, 45], that the presence of holomorphic vector fields has cohomological implications. Finally, we look at another related algebro-geometric construction, which is closer to noncommutative geometry.

*Acknowledgments.* The last-mentioned construction was explained to me by David Nadler.

## De Rham cohomology

Let  $M$  be a manifold with an action of  $G = S^1$ , generated by a vector field  $Z$ . Equivariant cohomology  $H_G^*(M)$  (with complex coefficients) can be defined through the Cartan model, which means as the cohomology of the complex

$$(2.1) \quad (\Omega_G^*(M), d_Z) \stackrel{\text{def}}{=} (\Omega^*(M)^G[[u]], d - u i_Z).$$

Here,  $\Omega^*(M)^G$  is the graded space of invariant (complex-valued) differential forms, and  $u$  is a formal variable of degree 2. The fact that the differential  $d_Z$  squares to zero is a consequence of the Cartan formula

$$(2.2) \quad d i_Z + i_Z d = L_Z.$$

By averaging differential forms, one shows that  $(\Omega^*(M)^G, d) \hookrightarrow (\Omega^*(M), d)$  is a quasi-isomorphism. Hence, the  $u$ -adic filtration of (2.1) gives rise to a spectral sequence

$$(2.3) \quad H^*(M)[[u]] \implies H_G^*(M).$$

REMARK 2.1. *It is worth while to reflect a little on the homological algebra behind (2.1). The operation  $i_Z$  makes  $(\Omega^*(M)^G, d)$  into a dg (differential graded) module over the graded commutative algebra  $\mathbb{C}[\theta]$ , where  $|\theta| = -1$  (and  $\theta^2 = 0$ , by the assumption of commutativity).*

Koszul duality [25, 82] yields a full and faithful functor from the derived category of dg modules for  $\mathbb{C}[\theta]$  to that for  $\mathbb{C}[[u]]$ , given explicitly by

$$(2.4) \quad (Q, d_Q) \mapsto (Q[[u]], d - u\theta).$$

Clearly, for  $(Q, d_Q) = (\Omega^*(M)^G, d)$  this recovers (2.1). This viewpoint can be computationally useful, as explained in [82]. The structure of  $(\Omega^*(M)^G, d)$  as a  $\mathbb{C}[\theta]$ -module transfers to give  $H^*(M)$  the structure of a (strictly unital)  $A_\infty$ -module over  $\mathbb{C}[\theta]$ , which is concretely expressed by a sequence of operations

$$(2.5) \quad H^*(M) \longrightarrow H^*(M)[-1 - 2d].$$

These operations determine the differentials in (2.3) (the operations themselves are not unique, but the partial information about them contained in the spectral sequence is).

If  $Q$  is a free dg module, say  $Q = W \otimes \Lambda$  for some graded vector space  $W$ , with vanishing differential, then  $(Q[[u]], d - u\theta)$  is quasi-isomorphic to  $W$  with trivial  $u$ -action. In particular, this becomes acyclic after taking the tensor product with  $\mathbb{C}((u))$ . To formalize this observation, take the derived category of dg modules over  $\Lambda$ , and divide out by free modules. The quotient category still comes with a functor to the derived category of dg modules over  $\mathbb{C}((u))$ . This is the algebraic mechanism behind localization theorems [33] (in the simplest situation of the circle acting on itself by left multiplication,  $\Omega^*(G)^G \cong \Lambda[-1]$  is obviously free).

### Hodge cohomology

Now suppose that  $M$  is a complex manifold. Changing notation, we write  $A^{*,*}$  for the bigraded space of  $C^\infty$  differential forms, and reserve  $\Omega^*$  for holomorphic differential forms. Let  $Z$  be a holomorphic vector field, which as before generates a circle action. Split  $i_Z = \iota_Z + \bar{\iota}_Z$  into its components of bidegree  $(0, -1)$  and  $(-1, 0)$ , respectively, and write (2.2) accordingly as

$$(2.6) \quad \begin{aligned} \iota_Z \partial + \partial \iota_Z &= 0, \\ \bar{\iota}_Z \partial + \partial \bar{\iota}_Z + \bar{\partial} \iota_Z + \iota_Z \bar{\partial} &= L_Z, \\ \bar{\iota}_Z \bar{\partial} + \bar{\partial} \bar{\iota}_Z &= 0. \end{aligned}$$

Take  $A_G^{*,*}(M) = A^{*,*}(M)^G[[u]]$  with the same differential  $d_Z$  as in (2.1). One can introduce a decreasing filtration of that space, in which  $A^{p,*}(M)^G u^r$  appears at level  $p+r$ . The outcome is the equivariant analogue of the Hodge-de Rham spectral sequence. Its starting page is the cohomology with respect to part of  $d_Z$ , namely

$$(2.7) \quad \bar{\partial}_Z = \bar{\partial} - u\bar{\iota}_Z.$$

The cohomology of  $(A_G^{*,*}(M), \bar{\partial}_Z)$  is called the *equivariant Hodge cohomology* of  $M$  (this theory was considered in [193, Section 7] and [122, Section 5]). We denote it by  $H_{Hodge, G}^{*,*}(M)$ , where the bigrading is such that  $p+q$  is the total degree, and classes in  $H_{Hodge, G}^{p,*}(M)$  are represented by cochains in  $\bigoplus_r A^{p-r,*} u^r$ ; hence, the action of  $u$  has bidegree  $(1, 1)$ .

There is a larger cohomology group which is also of interest. Namely, by looking at (2.6) one sees that  $\bar{\partial}_Z$  squares to zero even on forms which are not necessarily invariant. Let's call the cohomology of  $(A^{*,*}(M)[[u]], \bar{\partial}_Z)$  the *holomorphic equivariant cohomology* (this theory appears in [45, 123]). We denote it by  $H_{hol,Z}^{p,q}(M)$ , and equip it with the same bigrading as before. Holomorphic equivariant cohomology carries an action of  $G$ , and

$$(2.8) \quad H_{hol,Z}^{*,*}(M)^G \cong H_{Hodge,G}^{*,*}(M).$$

This is shown by an averaging process. For instance, if  $\alpha \in A^{*,*}(M)^G[[u]]$  can be written as  $\alpha = \bar{\partial}_Z \beta$  for some  $\beta \in A^{*,*}(M)[[u]]$ , then one can make  $\beta$  invariant by averaging its pullbacks over  $G$ . Note that while the  $G$ -action on  $H_{hol,Z}^{*,*}(M)$  is nontrivial in general, it always becomes trivial after multiplying with  $u$ . On the infinitesimal level, this is because

$$(2.9) \quad uL_Z = \bar{\partial}_Z h + h\bar{\partial}_Z, \quad h = uL_Z - \partial.$$

For another approach to holomorphic equivariant cohomology, consider the Koszul type complex

$$(2.10) \quad I_Z = \left\{ \cdots \rightarrow \Omega_M^2[[u]] \xrightarrow{-u i_Z} \Omega_M^1[[u]] \xrightarrow{-u i_Z} \mathcal{O}_M[[u]] \rightarrow 0 \right\},$$

where  $\Omega_M^p$  is the coherent analytic sheaf of holomorphic  $p$ -forms (the way in which we have written (2.10) is familiar but potentially confusing: in constructing the complex, we still place  $\Omega_M^p u^r$  in total degree  $p + 2r$ ). By taking the Dolbeault resolution of each such sheaf and arranging them into a double complex, one sees that  $(A_G^{*,*}(M), \bar{\partial}_Z)$  is a Cartan-Eilenberg resolution of that complex. Hence, holomorphic equivariant cohomology computes the hypercohomology of (2.10) [193, Lemma 7.9]. We write this as

$$(2.11) \quad H_{hol,Z}^{*,*}(M) \cong H^{*,*}(M, I_Z)$$

(hypercohomology is again bigraded since the complex (2.10) splits as a direct sum of suitable pieces). By combining this with (2.8), one can bring the equivariant Hodge-de Rham spectral sequence into a form which no longer involves choices of resolutions:

$$(2.12) \quad H^{*,*}(M, I_Z)^G \implies H_G^*(M).$$

EXAMPLE 2.2. *Take  $M = \mathbb{C}$  with the obvious circle action by rotation. Since  $M$  is Stein, we can compute hypercohomology through global sections of (2.10), which yields*

$$(2.13) \quad H_{hol,Z}^{p,q}(M) \cong \begin{cases} \mathcal{O}(M), & p = q = 0 \\ \mathcal{O}(M)/z\mathcal{O}(M) \cong \mathbb{C} & p = q > 0, \\ 0 & \text{otherwise.} \end{cases}$$

*The  $G$ -invariant part consists of constant functions, and (2.12) degenerates.*

EXAMPLE 2.3. *Take  $M = \mathbb{C}P^1$ , again with the obvious circle action. In this case, the computation of the hypercohomology of  $I_Z$  reduces to the long exact sequence*

$$(2.14) \quad \cdots \rightarrow H^*(M, \mathcal{O}_M)[[u]] \rightarrow H^{*,*}(M, I_Z) \rightarrow H^*(M, \Omega_M^1)[[u]] \xrightarrow{-u i_Z} \cdots$$

Therefore

$$(2.15) \quad H_{hol,Z}^{p,q}(M) \cong \begin{cases} \mathbb{C} & p = q = 0, \\ \mathbb{C}^2 & p = q > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The  $G$ -action on  $H^1(M, \Omega_M^1)$  is trivial (by Serre duality, for instance), and hence the whole of (2.15) carries a trivial  $G$ -action. The spectral sequence (2.12) again degenerates.

REMARK 2.4. One can put some of the features of Example 2.3 in a wider context. Namely, suppose that  $M$  is a smooth projective variety. Degeneration of the classical Hodge-de Rham spectral sequence implies that the  $G$ -action on  $H^*(M, \Omega_M^*)$  is trivial. By another spectral sequence argument, the same holds for  $H_{hol,Z}^{*,*}(M)$ , which is therefore isomorphic to  $H_{Hodge,G}^{*,*}(M)$ . Now suppose in addition that the  $G$ -action is linearizable (this means that it lifts to a circle action on an ample line bundle, which is the analogue of the Hamiltonian condition in symplectic topology). Then the spectral sequence (2.12) always degenerates (see [122, Theorem 5.1] or [193, Theorem 7.3]).

To conclude this discussion, let's look at Chern classes living in Hodge cohomology. The standard construction of such classes for a holomorphic vector bundle  $E$  goes as follows [19, Section 5]. One starts with the Atiyah class (1.20), then takes its exponential with respect to the algebra structure on  $\Omega^* \otimes \text{End}(E)$ , which yields

$$(2.16) \quad \exp(\text{At}(E)) \in \bigoplus_p H^p(M, \Omega^p \otimes \text{End}(E)),$$

and then takes the trace  $\text{End}(E) \rightarrow \mathcal{O}_M$ , which yields a form of the Chern character of  $E$ . Now suppose that  $E$  is  $G$ -equivariant. In that case, an argument similar to the one used to prove Proposition 1.8 shows that the Atiyah class has a natural lift

$$(2.17) \quad \text{At}_Z(E) \in H^{1,1}(M, I_Z \otimes \text{End}(E)).$$

Since  $I_Z$  is a sheaf of commutative differential graded algebras, one can proceed as before and get a form of the equivariant Chern character taking values in  $\prod_p H_{hol,Z}^{p,p}(M)$ . In fact, this is  $G$ -invariant, hence belongs to the equivariant Hodge cohomology.

## Vector fields

Let's temporarily go back to de Rham theory. In principle, the condition for a differential form  $\alpha$  to be  $G$ -invariant can be written as  $L_Z \alpha = 0$ , hence (2.1) depends only on  $Z$ . Nevertheless, in order for the outcome to be well-behaved, the fact that  $Z$  integrates to a circle action (or at least, is part of the action of a compact Lie group) is important. This is clear from the appearance of an averaging process in the argument leading to (2.3). The same is true for equivariant Hodge cohomology.

On the other hand (as already suggested by the notation),  $H_{hol,Z}^{*,*}(M)$  makes sense for an arbitrary holomorphic vector field  $Z$ . In fact, as [45] shows, one can use the local-to-global

spectral sequence associated to (2.10) to derive localisation-type results for any  $Z$ . As another aspect of the same idea, closer in spirit to [34], one can define (2.17) (and hence equivariant Chern classes in holomorphic equivariant cohomology) for vector bundles  $E$  which are infinitesimally equivariant, meaning that (1.19) vanishes.

EXAMPLE 2.5. *Take  $Z = 0$ , in which case  $H_{hol,Z}^{*,*}(M)$  is ordinary Hodge cohomology tensored with  $\mathbb{C}[[u]]$ . To make a vector bundle infinitesimally equivariant, one just equips it with an arbitrary holomorphic automorphism  $\tilde{Z}$ . The equivariant version of the Atiyah class is the ordinary Atiyah class plus  $u\tilde{Z}$ . Hence, the equivariant analogue of the first Chern class is ordinary first Chern class in  $H^1(M, \Omega_M^1)$  together with  $u\text{Tr}(\tilde{Z}) \in uH^0(M, \mathcal{O}_M)$ .*

### Hochschild homology

From now on, let  $M$  be a smooth projective variety, with a linearizable action of the multiplicative group  $G = \mathbb{G}_m = \mathbb{C}^*$ , again generated by a vector field  $Z$ . The graph of that action is a smooth subvariety

$$(2.18) \quad \Gamma = \{(g, g(x), x)\} \subset G \times M \times M,$$

which is itself invariant under the  $G$ -action  $h \cdot (g, y, x) = (g, hg(y), h(y))$ . Let  $p_G : G \times M \times M \rightarrow G$  be the projection, and  $\Delta \subset M \times M$  the diagonal. Consider the “derived intersection”

$$(2.19) \quad Rp_{G,*}(\mathcal{O}_{G \times \Delta} \overset{L}{\otimes} \mathcal{O}_\Gamma) \in D^b \text{Coh}_G(G).$$

Here,  $G$  acts on itself trivially (for a general group  $G$ , it would act by conjugation), and  $D^b \text{Coh}_G(G)$  is the bounded derived category of equivariant coherent sheaves for that action. In fact,  $G$  is affine, so (2.19) is just a bounded complex of finitely generated  $\mathbb{C}[G]$ -modules, which come with another action of  $G$  (preserving the module structure). We define the *equivariant Hochschild homology*  $HH_{*,G}(M)$  to be the  $G$ -invariant part of the cohomology of (2.19).

REMARK 2.6. *Consider the analogous construction when  $G$  is a finite group. Then the cohomology of (2.19) is*

$$(2.20) \quad \bigoplus_{g \in G} H^*(X \times X, \mathcal{O}_\Delta \overset{L}{\otimes} \mathcal{O}_{\Gamma_g}),$$

where  $\Gamma_g \subset X \times X$  is the graph of the action of  $g$ . For  $g$  the identity, one gets the Hochschild homology

$$(2.21) \quad H^*(X \times X, \mathcal{O}_\Delta \overset{L}{\otimes} \mathcal{O}_\Delta) \cong \bigoplus_{q-p=*} H^q(X, \Omega_X^p).$$

The other summands in (2.20) can similarly be thought of as the Hochschild homology groups of  $X$  with coefficients in the sheaf  $\mathcal{O}_{\Gamma_g}$ . The  $G$ -invariant part of (2.20) is the Hodge analogue of the orbifold cohomology of  $X/G$ . This is quite a well-studied theory (maybe even more so in its Hochschild cohomology version), see e.g. [137].

CONJECTURE 2.7. *Take a formal disc around the identity element in  $G$ , parametrized as  $t = e^u$ . Then, the restriction of (2.19) to that formal disc is quasi-isomorphic to the complex (2.10), in a way which is equivariant for the action of  $G$  on both sides.*

Note that the variable  $u$  switched degrees from 2 (in the discussion of holomorphic Hodge cohomology) to 0 (in the present context). With this taken into account, the cohomology level implication of Conjecture 2.7 says that the cohomology of the restriction of (2.19) to the formal disc is the graded  $\mathbb{C}[[u]]$ -module

$$(2.22) \quad \prod_{p,q} H_{hol,G}^{p,q}(M)[p-q].$$

To make this a little more plausible, it is useful to look at the local situation first, so let's switch to  $M$  being an affine variety. In that case, the restriction of (2.19) to a formal disc around the origin in  $G$  is computed by an appropriate deformation of the (algebraic) Hochschild complex for the ring of regular functions  $\mathcal{O}(M)$ , of the form

$$(2.23) \quad \dots \longrightarrow \mathcal{O}(M)^{\otimes 2}[[u]] \xrightarrow{f_2 \otimes f_1 \mapsto f_2 \cdot f_1 - e^{uL_Z}(f_1) \cdot f_2} \mathcal{O}(M)[[u]] \longrightarrow 0.$$

One would then hope to use a suitable version of the Hochschild-Kostant-Rosenberg map to show that this is quasi-isomorphic to the algebro-geometric version of (2.10). Indeed, in the first nontrivial degree one can easily find a map satisfying the desired chain equation:

$$(2.24) \quad \begin{aligned} \mathcal{O}(M)^{\otimes 2}[[u]] &\longrightarrow \Omega^1(M)[[u]], \\ f_2 \otimes f_1 &\mapsto f_2 \epsilon(uL_Z)(df_1), \end{aligned}$$

where  $\epsilon(x) = (e^x - 1)/x$ . I have not tried to extend this to higher degrees, nor to seriously attack the next step in the proof, which would be to globalize the argument as in [74]. Alternatively, one could think of Conjecture 2.7 as belonging to a general circle of ideas involving the relation between commutative and noncommutative geometry, and try to apply the deep general comparison results that have been proved in that context.

## LECTURE 3

# Mirror symmetry and circle actions

We begin by reviewing some aspects of mirror symmetry (by now, textbook material [55, 91]), illustrated by the example of elliptic curves. After that, we consider equivariant mirror symmetry, following [39] but again restricting it to the most basic example. The equivariant picture is not well understood, and our discussion is heuristic, with the aim of seeing what kind of mathematics (both existing and yet to be constructed) can play a role in it.

*Acknowledgments.* I would like to thank Roman Bezrukavnikov, Davesh Maulik, Michael McBreen, Andrei Okounkov, and Rahul Pandharipande for explaining the philosophy behind [39] to me (repeatedly, until it finally started to percolate into my head).

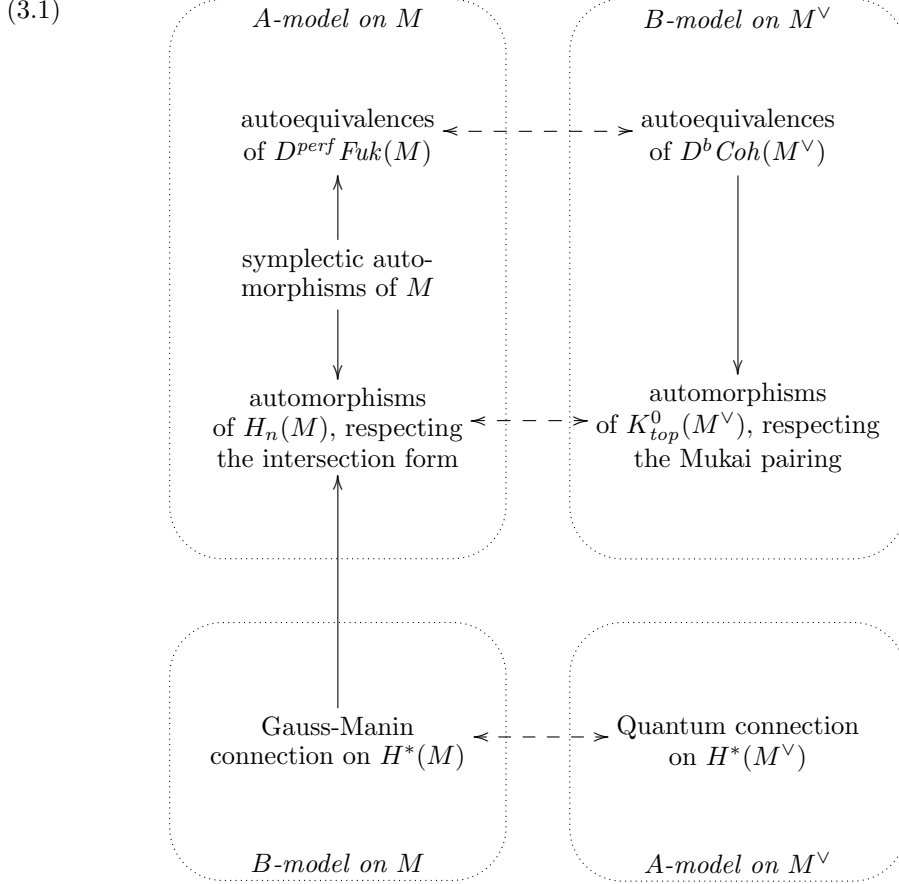
### A superficial introduction

Any mathematically correct formulation of the statements of mirror symmetry is bound to be quite complicated, even if one limits oneself to the most familiar context of compact Calabi-Yau manifolds. A reasonable approximation would be to say that the objects involved are smooth projective varieties  $M$  with trivial canonical bundles  $K_M \cong \mathcal{O}_M$ , equipped with a complexified Kähler class, which means a class  $[\omega_M] + iB_M \in H^2(M; \mathbb{C}/2\pi i\mathbb{Z})$  whose real part lies in the Kähler cone (hence can be represented by a Kähler form  $\omega_M$ ). One considers such manifolds from two different points of view:

- The *A-model* involves symplectic topology, which means that it remains unchanged under deformations of the complex structure on  $M$ , as long as we keep the complexified Kähler class constant.
- The *B-model* involves algebraic geometry, which means that it remains unchanged under deformations of the complexified Kähler class.

Mirror symmetry relates the *A-model* on  $M$  to the *B-model* on a different variety  $M^\vee$  of the same kind, and vice versa. While this symmetry manifests itself in many ways, we are

specifically interested in monodromy phenomena, as summarized here:



The notation in (3.1) is not entirely mathematical: roughly speaking, solid arrows go from more sophisticated to simpler structures, and dashed arrows indicate the relationships established by mirror symmetry. Instead of continuing with the discussion in general terms, we prefer to work through an example.

### The Hesse family of elliptic curves

Consider the elliptic curve  $M \subset \mathbb{C}P^2$  with equation

$$(3.2) \quad f(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3 - 3zx_0x_1x_2 = 0.$$

This depends on a parameter  $z \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , with singularities at  $z = \infty$  and  $z \in \sqrt[3]{1}$ . In some (not particularly distinguished) basis of  $H_1(M) \cong \mathbb{Z}^2$ , the monodromies around the roots of unity, and their product which is the inverse monodromy around infinity, are:

$$(3.3) \quad \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}.$$



The Gauss-Manin connection is the natural connection on the bundle with fibres  $H^*(M; \mathbb{C})$  induced by local trivializations of this family of elliptic curves. Besides its abstract isomorphism type, which is determined by the monodromy representation (3.3), mirror symmetry is also concerned with its interplay with the Hodge filtration, which in our case consists of a single one-dimensional subspace

$$(3.4) \quad F_{Hodge}^1 \subset H^1(M; \mathbb{C}).$$

To express this concretely, one chooses a local covariantly constant trivialization  $H^1(M; \mathbb{C}) \cong \mathbb{C}^2$ , writes elements of (3.4) as pairs  $(\phi_0(z), \phi_1(z))$ , and then considers the quotient  $\phi_1(z)/\phi_0(z)$  as a locally defined function on our parameter space. This still depends on the choice of trivialization, which is usually fixed by monodromy considerations around a specified “large complex structure limit point”, in our case  $z = \infty$ .

To make a direct computation, we choose the complex volume form

$$(3.5) \quad \eta_M = \text{res}_M \left( \frac{1}{f} (x_0 dx_1 \wedge dx_2 - x_1 dx_0 \wedge dx_2 + x_2 dx_0 \wedge dx_1) \right)$$

which is a nowhere vanishing section of the family of spaces (3.4). In a local trivialization of the Gauss-Manin connection,  $[\eta_M]$  becomes a solution of the Picard-Fuchs equation (this is classical, but see e.g. [151, Example 6.5.1])

$$(3.6) \quad ((z^3 - 1)\partial_z^2 + 3z^2\partial_z + z)\phi = 0.$$

REMARK 3.1. *This form of the Picard-Fuchs equation may be slightly unfamiliar. If we change variables to  $\tilde{z} = z^{-1}$ , and simultaneously replace  $\eta_M$  by  $\tilde{\eta}_M = z\eta_M$ , which extends smoothly over  $z = \infty$ , the equation turns into a form which is more common in the mirror symmetry literature:*

$$(3.7) \quad ((\tilde{z}\partial_{\tilde{z}})^2 - \tilde{z}^3(\tilde{z}\partial_{\tilde{z}} + 1)(\tilde{z}\partial_{\tilde{z}} + 2))\phi = 0.$$

More precisely, we can integrate  $[\eta_M]$  against each locally constant class in  $H_1(M; \mathbb{C})$ , and that yields a solution of the Picard-Fuchs equation. Hence, the previously explained recipe is to take two linearly independent solutions of that equation, and consider their quotient. While the Picard-Fuchs equation itself depends on the choice of holomorphic volume form, that quotient will be a well-defined function locally on our moduli space.

Let’s stay with  $M$  for the moment, and consider its  $A$ -model, for which we need to fix a complexified Kähler class  $[\omega_M] + iB_M$ . Parallel transport realizes the monodromies (3.3) as symplectic automorphisms (volume-preserving diffeomorphisms, in this dimension) for  $\omega_M$ , unique up to Hamiltonian isotopy. Concretely, each matrix in the left hand side of (3.3) corresponds to a positive Dehn twist along three parallel curves; on the right hand side we have a negative Dehn twist along three parallel curves, combined with a downwards shift in the grading by 2. For general reasons, these symplectic automorphisms induce auto-equivalences of the Fukaya category, unique up to isomorphism and shifts (in fact, there is a preferred lift to the graded symplectomorphism group  $Symp_{gr}(M)$  [169], which removes the shift ambiguity). That takes care of the top left box in (3.1).

REMARK 3.2. *We have been implicitly assuming that the Fukaya category can be defined over  $\mathbb{C}$ , by integrating  $[\omega_M] + iB_M$  over holomorphic curves, as commonly used in the physics literature. While this is possible in this particular case [154], the resulting construction is disconnected from general symplectic topology, where one uses a formal parameter to rescale  $\omega_M$  (the “large volume limit”). Since we use statements from the general theory, such as the fact about symplectic automorphisms inducing auto-equivalences, our discussion has certainly not been rigorous at this point.*

The Gauss-Manin connection is the connection on the bundle with fibres  $H^*(M; \mathbb{C})$  induced by parallel transport. The abstract isomorphism type of that connection is encoded in its monodromy (3.3).

The mirror  $M^\vee$  is again an elliptic curve. Identify  $H^{even}(M^\vee; \mathbb{Z}) \cong \mathbb{Z}^2$  by choosing generators  $\psi_0 = 1$  and  $\psi_1 = [point]$ . In particular, one can then write the complexified Kähler class as

$$(3.8) \quad [\omega_{M^\vee}] + iB_{M^\vee} = -\log(q)\psi_1$$

where  $q$  is a complex number with  $0 < |q| < 1$ . The *quantum connection* or *A-model connection* [55, Section 8.5.2] on the trivial bundle with fibres  $H^{even}(M^\vee; \mathbb{C})$  is given by

$$(3.9) \quad \nabla_q \phi = \partial_q \phi - \frac{1}{2\pi i q} \psi_1 \phi,$$

where  $\psi_1$  acts by the cup product. This is the counterpart of the Gauss-Manin connection. The counterpart of the Hodge filtration is the filtration by degrees, which in this case consists of the subspace  $F_{deg}^1 = H^0(M^\vee; \mathbb{C})$ . To investigate the interplay of connection and filtration, we proceed formally as before. Namely, take two covariantly constant sections of (3.9), say  $\psi_0 + (\log(q)/2\pi i)\psi_1$  and  $\psi_1/2\pi i$ ; express  $\psi_0 \in F_{deg}^1$  as a linear combination of those two sections; and consider the quotient of the resulting coefficients, which is the function  $\log(q)$ .

The *mirror map* transforms the complex structure parameter  $z$  for  $M$  into the complexified Kähler parameter  $q$  for  $M^\vee$ , in such a way that  $z = \infty$  corresponds to  $q = 0$ . This must transform  $\log(q)$  into the quotient of solutions of (3.6), and one uses that relation to reconstruct the mirror map. The explicit solution [111] has the form

$$(3.10) \quad q = \frac{1}{27}z^{-3} + \frac{5}{243}z^{-6} + \dots$$

Note that the monodromy maps around roots of unity on the left hand side of (3.3) do not have counterparts for (3.9), because the mirror map converges only for  $|z| > 1$ . In other words, if one starts from (3.9), then those monodromies only appear after changing coordinates to  $z$  and analytic continuation in that variable.

So far, we have not mentioned the *B-model* aspect of  $M^\vee$ . Suppose that we fix a generic value of  $[\omega_M] + iB_M$ , corresponding to a generic choice of complex structure of  $M^\vee$ . The group of autoequivalences of the derived category  $D^b Coh(M^\vee)$  is isomorphic to [146]

$$(3.11) \quad (M^\vee \times M^\vee) \rtimes SL_2(\mathbb{Z}).$$

Here, we assume a choice of base point on  $M^\vee$ . Then, the first copy of  $M^\vee$  is given by translations, and the second by tensoring with degree zero line bundles. The final part

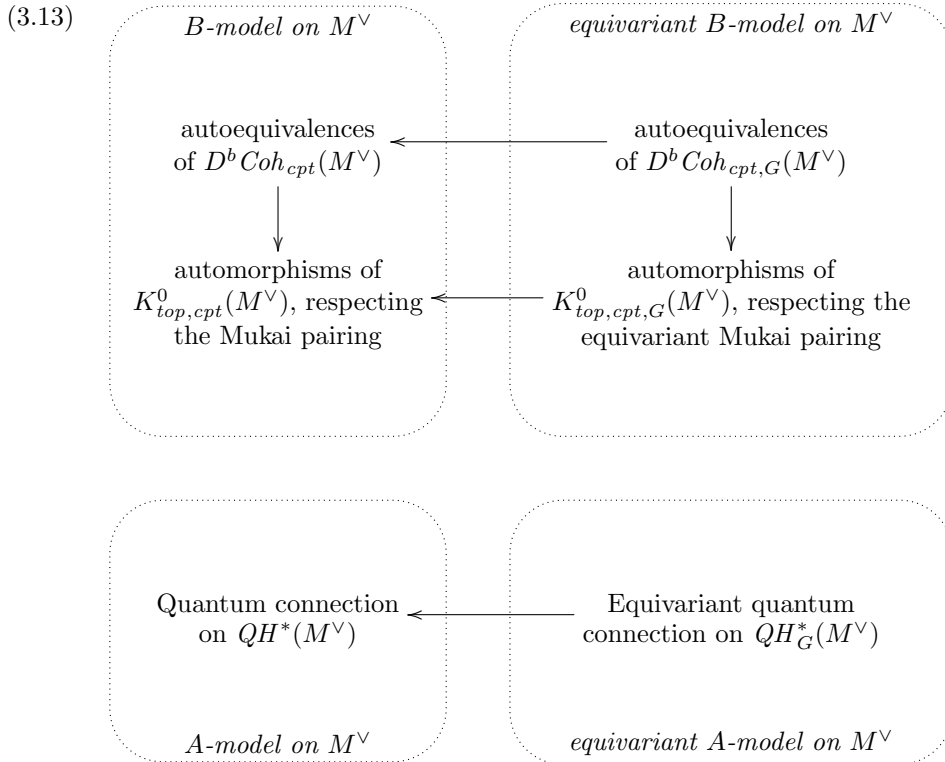
$SL_2(\mathbb{Z})$  acts on  $K_{top}^0(M^\vee) \cong \mathbb{Z}^2$  by the obvious representation. In particular, one can construct mirrors of the monodromy maps appearing in (3.3). For instance, the monodromy around  $z = \infty$  corresponds to the autoequivalence

$$(3.12) \quad D^b \text{Coh}(M^\vee) \rightarrow D^b \text{Coh}(M^\vee) \quad E \mapsto E \otimes L[2],$$

where  $L$  is a line bundle of degree 3 (this is an instance of a general result, see e.g. [43]). There are many interesting examples of such “mirror monodromy” computations in the literature, starting with [18, 93, 185].

### Circle actions

We will be interested in applying mirror symmetry to situations where one side, namely  $M^\vee$ , carries an (algebraic) action of  $G = S^1$ , and such that one has a complex volume form  $\eta_{M^\vee}$  on which the action has positive weight. This can happen only for noncompact  $M^\vee$ , which means that we are in a modified version of our original framework, usually called “local mirror symmetry”. For the  $B$ -model, we will use the category of compactly supported coherent sheaves  $\text{Coh}_{cpt}(M^\vee)$ , and the corresponding version of topological  $K$ -theory,  $K_{top,cpt}^0(M^\vee)$ . Then, the situation on this side of the mirror roughly looks as follows:



The equivariant Mukai pairing, which we already encountered in Lecture 1, is a virtual representation of  $G$ . Concretely, if one identifies the Grothendieck ring of such representations

with  $\mathbb{Z}[t, t^{-1}]$ , then it can be written as

$$(3.14) \quad (E_0, E_1)_{Mukai}^G : t \mapsto \sum_i (-1)^i \text{Tr}(t : \text{Hom}^i(E_0, E_1) \rightarrow \text{Hom}^i(E_0, E_1)).$$

The ordinary Mukai pairing on a Calabi-Yau variety is graded symmetric, by Serre duality. In the current situation, one has to take into account the fact that the canonical bundle is not equivariantly trivial. The resulting formula is

$$(3.15) \quad (E_1, E_0)_{Mukai}^G = (-1)^n t^d \left( (E_0, E_1)_{Mukai}^G \right)_{t \rightarrow t^{-1}},$$

where  $n$  is the dimension of  $M^\vee$ , and  $d$  is the weight of the circle action on  $\eta_{M^\vee}$ . The same holds for the extension of the Mukai pairing to  $K_{top, cpt, G}^0(M^\vee)$ . Hence, what one would want on the mirror side is:

**DESIDERATUM 3.3.** *A  $\mathbb{Z}[t, t^{-1}]$ -module together with a map from its  $t = 1$  reduction to  $H_n(M)$ . This module should carry a bilinear pairing satisfying the analogue of (3.15). Moreover, the specialization of that pairing to  $t = 1$  should recover the ordinary intersection pairing on  $H_n(M)$ .*

Similarly, looking at the equivariant Gromov-Witten theory of  $M^\vee$  leads one to suspect the existence of the following:

**DESIDERATUM 3.4.** *A graded filtered  $\mathbb{Z}[[u]]$ -module together with a map from its  $u = 0$  reduction to  $H^*(M)$ , taking the filtration to the Hodge filtration. As the complex structure of  $M$  varies, these modules should carry a flat connection whose  $u = 0$  specialization recovers the Gauss-Manin connection.*

At present, these requirements are no more than intelligent guesses. In the first case, we can at least construct some approximations to the desired object, but in the second case nothing seems to be known. Rather than continuing the discussion in the abstract, we now turn to an example.

### The equivariant derived category

Let  $N$  be the total space of the canonical bundle over  $C = \mathbb{C}P^1$  ( $N$  is also the cotangent bundle  $T^*C$ , or the minimal resolution of the singularity  $\mathbb{C}^2 / \pm 1$ ). Let  $D^b \text{Coh}_C(N)$  be the derived category of coherent sheaves with (reduced) support along the zero-section  $C \subset N$ . This category can be described in more elementary terms by using the McKay correspondence [100], but we will not need that here. The relevant Grothendieck group (algebraic and topological  $K$ -theories coincide in this case) is

$$(3.16) \quad K_C^0(N) \cong K_{top, cpt}^0(N) \cong \mathbb{Z}^2,$$

with basis given by the classes of  $\mathcal{O}_C(-1)[1]$  and  $\mathcal{O}_C$ . In this basis, the Mukai pairing is

$$(3.17) \quad \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The category  $D^b \text{Coh}_C(N)$  carries an action of the affine braid group

$$(3.18) \quad Br_2^{\text{aff}} = \langle F_0, F_1 \mid F_0 F_1 F_0 F_1 = F_1 F_0 F_1 F_0 \rangle$$

(this can be viewed as a toy example for the categorical braid group actions arising from representation theory). Concretely, consider the autoequivalences

$$(3.19) \quad T_d = T_{\mathcal{O}_C(d)}, \quad \text{the twist functor [185] along the spherical object } \mathcal{O}_C(d) \ d \in \mathbb{Z},$$

$$(3.20) \quad \Theta = \mathcal{O}_N(F) \otimes -, \quad \text{tensoring with the line bundle } \mathcal{O}_N(F), \ F = \text{fibre of } N \rightarrow C.$$

Since  $\Theta(\mathcal{O}_C(d)) \cong \mathcal{O}_C(d+1)$ , we have  $\Theta T_d \Theta^{-1} \cong T_{d+1}$ . Similarly, since  $T_d(\mathcal{O}_C(d+1)) \cong \mathcal{O}_C(d-1)[1]$ , we have  $T_d T_{d+1} T_d^{-1} \cong T_{d-1}$ . Taking the two together shows that

$$(3.21) \quad \Theta T_d \Theta T_d \cong \Theta T_d T_{d+1} \Theta \cong \Theta T_{d-1} T_d \Theta \cong T_d \Theta T_d \Theta.$$

In particular, setting  $F_0 = \Theta$  and  $F_1 = T_0$  satisfies the relation (3.18) up to isomorphism of functors. The resulting representation on (3.16) is given by

$$(3.22) \quad F_0 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad F_1 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

In fact, with some more work one can prove a stronger relation

$$(3.23) \quad F_0 F_1 F_0 F_1 \cong Id,$$

which means that the action descends to the quotient  $Br_2^{\text{aff}} \rightarrow \mathbb{Z}/2 * \mathbb{Z}$  (with generators  $(F_0 F_1, F_1)$  for the free product).

Let  $G = S^1$  act on  $N$  by rotating the fibres of  $N \rightarrow C$  with weight 2. Let  $D^b \text{Coh}_{C,G}(N)$  be the equivariant analogue of the previous category. The equivariant Grothendieck group is

$$(3.24) \quad K_{C,G}^0(N) \cong K_{\text{top}, \text{cpt}, G}^0(N) \cong \mathbb{Z}[t, t^{-1}]^2.$$

$G$  acts trivially on  $\text{Hom}_N(\mathcal{O}_C, \mathcal{O}_C)$ , but with weight  $-2$  on  $\text{Ext}_N^2(\mathcal{O}_C, \mathcal{O}_C)$  (as one can see by applying Serre duality). Hence, the equivariant version of the Mukai pairing has the form

$$(3.25) \quad \begin{pmatrix} 1 + t^{-2} & -2t^{-1} \\ -2t^{-1} & 1 + t^{-2} \end{pmatrix}.$$

(We have twisted the  $G$ -action on  $\mathcal{O}_C(-1)$  by a character, in order to make the matrix look more symmetric). The affine braid group action lifts to the equivariant case, and the associated automorphisms of (3.24) are now given by

$$(3.26) \quad F_0 \mapsto \begin{pmatrix} 0 & t^{-2} \\ -1 & 2t^{-1} \end{pmatrix}, \quad F_1 \mapsto \begin{pmatrix} 1 & 0 \\ 2t^{-1} & -t^{-2} \end{pmatrix}.$$

In particular, unlike the non-equivariant case,

$$(3.27) \quad F_1^2 \mapsto \begin{pmatrix} 1 & 0 \\ 2t^{-1} - 2t^{-3} & t^{-4} \end{pmatrix}$$

acts highly nontrivially. The equivariant analogue of (3.23) says that  $F_0 F_1 F_0 F_1$  acts by changing the equivariant structure of any object by tensoring with the one-dimensional representation of weight  $-4$ . One can of course adjust to make this product equal to the identity instead, and then the resulting action again descends to  $\mathbb{Z}/2 * \mathbb{Z}$ .

### Equivariant Gromov-Witten theory

As recalled in Lecture 2, equivariant cohomology groups for  $G = S^1$  are modules over  $H_G^*(point) = H^*(BG) \cong \mathbb{C}[[u]]$ . If  $V$  is the one-dimensional representation of  $G$  with weight  $d$ , then its equivariant Euler class in  $H_G^2(point)$  is  $du$ . Any oriented  $G$ -manifold  $N$  carries an equivariant fundamental class, which we think of dually as a pushforward operation on compactly supported equivariant cohomology:

$$(3.28) \quad \int_N^G : H_{cpt,G}^*(N) \longrightarrow H_G^*(point)[-dim(N)].$$

Now suppose that  $N$  is a smooth quasiprojective variety (on which  $G$  acts compatibly with that structure), with the property that all non-constant closed curves are contained in a compact subset of  $N$ . The genus zero equivariant Gromov-Witten invariants [110, 39] are symmetric multilinear maps, defined for nonzero  $\beta \in H_2(N; \mathbb{Z})$  and any  $n \geq 0$ ,

$$(3.29) \quad \langle \cdots \rangle_{0,n,\beta}^G : H_G^*(N)^{\otimes n} \longrightarrow H_G^*(point).$$

These satisfy an equivariant analogue of the divisor axiom, which comes in two parts: first,

$$(3.30) \quad \langle 1, \dots \rangle_{0,n,\beta}^G = 0;$$

and secondly, for  $x \in H_G^2(N)$  we have [77, Equation (6)]

$$(3.31) \quad \langle x, \cdots \rangle_{0,n,\beta}^G = x(\beta) \langle \cdots \rangle_{0,n-1,\beta}^G,$$

where  $x(\beta)$  is defined by first mapping  $x$  to ordinary (non-equivariant) cohomology. Specializing to  $n = 3$ , one defines the small quantum product on  $QH_G^*(N) = H_G^*(N)[[q]]$  by requiring that

$$(3.32) \quad \int_N^G (x_1 *_G x_2) x_3 = \int_N^G x_1 x_2 x_3 + \sum_{\beta} q^{\deg(\beta)} \langle x_1, x_2, x_3 \rangle_{0,3,\beta}^G.$$

Here,  $x_3 \in H_{cpt,G}^*(N)$  is a compactly supported equivariant cohomology class, and on the right hand side one uses its image in  $H_G^*(N)$ . The degree  $\deg(\beta)$  is taken with respect to some choice of integral Kähler class  $[\omega_N]$ . Suppose that the symplectic form  $\omega_N$  is  $G$ -invariant, and that the group action is Hamiltonian, with moment map  $\mu_N$ . This yields a class  $[\omega_N, \mu_N] \in H_G^2(N)$ . One defines the equivariant form of the quantum connection by

$$(3.33) \quad \nabla_q^G x = \partial_q x - \frac{1}{2\pi i q} [\omega_N, \mu_N] *_G x.$$

Let's apply this to the same example  $N = T^*C$  as before. Equip it with a  $G$ -invariant Kähler form  $\omega_N$ , normalized in such a way that the integral over  $C$  is 1. Choose the moment map  $\mu_N$  so that  $\mu_N|_C = 0$ . We write  $\psi_1$  both for the equivariant cohomology class  $[\omega_N, \mu_N]$  and its unique lift to  $H_{cpt,G}^2(N)$ . Then

$$(3.34) \quad \begin{aligned} \psi_1^2 &= 0 \in H_G^4(N), \\ \int_N^G \psi_1^2 &= -1/2. \end{aligned}$$

The relevant equivariant Gromov-Witten invariants are

$$(3.35) \quad \langle \psi_1, \psi_1, \psi_1 \rangle_{0,3,d[C]}^G = d^3 \langle \rangle_{0,0,d[C]}^G = d^3 \cdot 2ud^{-3} = 2u.$$

In the trivial case  $d = 1$ , the relevant moduli space of stable maps is a single point, consisting of the inclusion  $C \hookrightarrow N$ . Since  $TN|_C = TC \oplus K_C$ , we have

$$(3.36) \quad H^1(C, TN|_C) = H^1(C, K_C) \cong \mathbb{C},$$

and  $G$  acts with weight 2 on that space. Hence, the equivariant Euler class is  $2u$ . In higher degrees, one uses an equivariant version of the Aspinwall-Morrison formula [129]. Taking  $\psi_1$  and the unit class  $\psi_0$  as a basis, one finds that the equivariant quantum connection satisfies

$$(3.37) \quad \begin{aligned} \nabla_q^G \psi_0 &= -\frac{1}{2\pi i q} \psi_1, \\ \nabla_q^G \psi_1 &= -\frac{1}{2\pi i} \frac{4u}{1-q} \psi_1. \end{aligned}$$

Setting  $u = 0$  kills all the invariants (3.35) (a general fact about complex symplectic manifolds, but particularly obvious in this case for dimension reasons), hence the ordinary non-equivariant quantum connection has the same essentially trivial form as for elliptic curves. In contrast, the equivariant version (3.37) shows a new pole at  $q = 1$ , with associated monodromy matrix

$$(3.38) \quad \exp \begin{pmatrix} 0 & 0 \\ 0 & -4u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (e^u)^{-4} \end{pmatrix}.$$

The notable fact here is that after setting  $t = e^u$ , (3.38) becomes conjugate to (3.27) over  $\mathbb{C}[[u]]$  (the matrix (3.27) is diagonalizable away from  $t = \sqrt{-1}$ , and we are considering its Taylor expansion around  $t = 1$ ).

### The mirror

There is no way to apply the Strominger-Yau-Zaslow (SYZ) construction of mirrors directly to  $N$  (any Lagrangian torus in it is necessarily nullhomologous, hence can't be Special Lagrangian). The most common approach, going back to [84], is to introduce a rational section of the canonical bundle which is nowhere zero and has poles along a suitably chosen divisor, and then take  $M^\vee$  to be the complement of that divisor. One constructs  $M$  by applying the SYZ process to  $M^\vee$ , possibly with instanton corrections [20, Section 5]. Concretely, in our case the divisor is a copy of  $\mathbb{C}^*$  inside  $N = T^*C$  which is disjoint from the zero-section (the graph of a one-form on  $C$  with two poles). Corresponding to our fixed choice of complex structure on  $M^\vee$ , we have a specific choice of Kähler class on  $M$ , which turns out to be zero. Concretely, one can write the mirror as an affine algebraic surface

$$(3.39) \quad \begin{aligned} M &= \{x_1 x_2 + W(y) = 0\} \subset \mathbb{C}^2 \times \mathbb{C}^*, \\ W(y) &= zy^{-1} - 2 + y \end{aligned}$$

where  $z$  is a parameter. We equip  $M$  with the exact symplectic form  $\omega_M = d\theta_M = -dd^c h$ , where the Kähler potential is  $\frac{1}{4}\|x\|^2 + \frac{1}{2}(\log\|y\|)^2$  (it turns out that  $M$  is diffeomorphic to  $M^\vee$ , but that is a coincidence).

Projection to  $y$  shows  $M$  to be a family of conics which degenerate over

$$(3.40) \quad y_{\pm} = 1 \pm \sqrt{1 - z}.$$

The singular values are  $z = 0$  (where  $y_-$  becomes zero, creating a singularity at infinity);  $z = 1$ , where  $M$  develops a node; and  $z = \infty$ , which one would define by appropriately rescaling the  $y$ -coordinate. If we keep away from those values, one has

$$(3.41) \quad H_2(M) \cong H_1(\mathbb{C}^*; \{y_{\pm}\})$$

(more geometrically, embedded paths in  $\mathbb{C}^*$  joining  $y_-$  to  $y_+$  give rise to Lagrangian spheres in  $M$ , following the procedure in [109]). In a suitable basis, the monodromy maps around  $z = 0, 1$  exactly match the matrices (3.22) (in that order), and the inverse monodromy around  $z = \infty$  is therefore given by

$$(3.42) \quad \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The family (3.39) is part of a larger one, where the points  $y_{\pm}$  are allowed to vary arbitrarily in  $\mathbb{C}^*$ . The base of the smooth part of that family is parametrized by the configuration space  $Conf_2(\mathbb{C}^*)$ , and this is compatible with the  $\mathbb{C}^*$ -action by rotation on  $Conf_2(\mathbb{C}^*)$ . The quotient  $Conf_2(\mathbb{C}^*)/\mathbb{C}^*$  is itself a copy of  $\mathbb{C}^*$  with one  $\mathbb{Z}/2$ -orbifold point. Hence, the monodromy yields a homomorphism

$$(3.43) \quad \mathbb{Z}^2 * \mathbb{Z} \cong \pi_1^{orb}(Conf_2(\mathbb{C}^*)/\mathbb{C}^*) \longrightarrow \pi_0(Symp_{ex,gr}(M))$$

into the ‘‘mapping class group’’ of exact (preserving  $\theta_M$  up to exact one-forms) and graded symplectic automorphisms. In particular, if  $\phi_0, \phi_1$  are the monodromy maps of (3.39) around  $z = 0, 1$ , then we get an isotopy  $\phi_0\phi_1\phi_0\phi_1 \simeq Id$ , corresponding to our previous discussion of autoequivalences of the derived category.

Take the natural complex volume form on  $M$ ,

$$(3.44) \quad \eta_M = \text{res}_M \frac{dx_1 \wedge dx_2 \wedge y^{-1} dy}{x_1 x_2 + W(y)},$$

Under the cohomology isomorphism corresponding to (3.41),  $\eta_M$  maps to the class of the one-form  $dy/y$ . This means that the periods (solutions of the Picard-Fuchs equation) can be computed in entirely elementary terms,

$$(3.45) \quad \int_{2\text{-cycle}} \eta_M = \int_{y_-}^{y_+} \frac{dy}{y} = \log(y_+/y_-).$$

To match this with the quantum connection, which means the  $u = 0$  specialization of (3.37), one can at least tentatively take the mirror map to be

$$(3.46) \quad q = \frac{y_+}{y_-} \implies z = 1 - \left( \frac{q-1}{q+1} \right)^2$$

In particular,  $z = 1$  corresponds to  $q = 1$ , but the map  $q \rightarrow z$  is double branched at that point, and since the monodromy of  $H_2(M)$  around  $z = 1$  has order two, its pullback by that map is trivial.



Passing from  $N$  to  $M^\vee$  is unfortunately not compatible with the  $G$ -action. One could get a mirror of  $N$  itself by equipping  $M$  with a suitable regular function (which yields a Landau-Ginzburg model) [20]. If we allow ourselves to bypass this issue, Desideratum 3.3 takes on the following form. By choosing two non-isotopic paths in  $\mathbb{C}^*$  that go from  $y_-$  to  $y_+$ , and only intersect at the endpoints, one gets two Lagrangian spheres  $S_0, S_1 \subset M$  whose intersection numbers are

$$(3.47) \quad S_0 \cdot S_0 = S_1 \cdot S_1 = -2, \quad S_0 \cdot S_1 = S_1 \cdot S_0 = 2,$$

matching (3.17) (up to an overall  $-1$  sign that is due to orientation conventions). One would then want to define  $t$ -intersection numbers which yield the refinement corresponding to (3.25):

$$(3.48) \quad S_0 \cdot_t S_0 = S_1 \cdot_t S_1 = -1 - t^{-2}, \quad S_0 \cdot_t S_1 = S_1 \cdot_t S_0 = 2t^{-1}.$$

Moreover, the action of the monodromy maps  $\phi_0, \phi_1$  should reflect (3.26). Now  $\phi_1$  is a four-dimensional Dehn twist, hence  $\phi_1^2$  is isotopic to the identity as a diffeomorphism, but not symplectically so [168]. That makes it conceivable, at least in principle, that one could reproduce phenomena like (3.27) in symplectic topology. Indeed, a construction of improved intersection numbers with the desired properties has been given in [184], and we will show later that this construction can be applied to  $M$ .



## LECTURE 4

### Derived Picard groups

Derived categories of coherent sheaves, and their automorphisms, have already appeared in Lecture 3. We return to the same topic, and more specifically consider the “continuous” part of the autoequivalence group, which is what appears in the following theorem of Rouquier:

**THEOREM 4.1** ([159, Theorem 4.18]). *The (connected algebraic) group*

$$(4.1) \quad DPic_0(M) = Pic_0(M) \times Aut_0(M)$$

*attached to a smooth projective variety  $M$ , is a derived invariant.*

This reflects the infinitesimal (Lie algebra) equality

$$(4.2) \quad H^1(M, \mathcal{O}_M) \oplus H^0(M, TM) \cong HH^1(M, M),$$

where  $HH^1(M, M)$  is the first Hochschild cohomology group, which is easily seen to be derived invariant. However, the global result (4.1) is stronger: besides its own algebraic group structure,  $DPic_0(M)$  can be equipped with additional data which are also derived invariant (sheaves of twisted Hochschild cohomology groups). For subgroups of the actual geometric automorphism group  $Aut(M)$ , we had already considered part of this data in Lecture 2, and interpreted it as a version of equivariant cohomology. The more general construction, and its interaction with more classical notions of algebraic geometry, is the object of ongoing study [156, 155, 125].

In principle, this approach is not limited to algebraic geometry, but can be applied to other categories which admit a meaningful notion of “family of objects parametrized by an algebraic variety”. See for instance [107] for the general framework. As another concrete instance, there a parallel statement to Theorem 4.1 for modules over finite-dimensional algebras (besides [159], see [94, 208]).

#### Fourier-Mukai transforms

We consider only smooth projective varieties over  $\mathbb{C}$ . Given  $P_2 \in Ob D^b Coh(M_1 \times M_0)$  and  $P_1 \in Ob D^b Coh(M_2 \times M_1)$ , we define their *convolution*  $P_2 * P_1 \in Ob D^b Coh(M_2 \times M_0)$  by the formula

$$(4.3) \quad P_2 * P_1 \stackrel{\text{def}}{=} p_{02,*}(p_{12}^*P_2 \otimes p_{01}^*P_1).$$

The tensor product takes place on  $M_2 \times M_1 \times M_0$ , and the  $p_{ij}$  are the projections from there to  $M_i \times M_j$ . The pushforwards and tensor product in (4.3) are taken in the derived sense (for

the pullbacks this is not an issue since they are already exact); this notational convention will continue throughout the rest of our discussion. Convolution is an exact functor in either argument. The structure sheaves of diagonals are the two-sided identity elements:

$$(4.4) \quad P * \mathcal{O}_{\Delta_{M_0}} \cong \mathcal{O}_{\Delta_{M_1}} * P \cong P \quad \text{for all } P \in \text{Ob } D^b \text{Coh}(M_1 \times M_0).$$

One can also think of  $P \in \text{Ob } D^b \text{Coh}(M_1 \times M_0)$  as a “kernel” which defines a functor, the *Fourier-Mukai transform*

$$(4.5) \quad \begin{aligned} \Phi_P : D^b \text{Coh}(M_0) &\longrightarrow D^b \text{Coh}(M_1), \\ \Phi_P(X) &= p_{1,*}(P \otimes p_0^* X). \end{aligned}$$

Here  $p_0, p_1$  are again the projections from  $M_1 \times M_0$  to the two factors. Of course, this is itself technically a special case of convolution, but a case which intuitively (and historically) plays a particularly important role. One has

$$(4.6) \quad \Phi_{P_2 * P_1} \cong \Phi_{P_2} \circ \Phi_{P_1}$$

and

$$(4.7) \quad \Phi_{\mathcal{O}_{\Delta_M}} \cong \text{Id}.$$

It is also interesting to see the formulae for left and right adjoint functors in this context (see [95, Proposition 5.9] or [136]), which involve Grothendieck-Serre duality. Namely, for  $P \in \text{Ob } D^b \text{Coh}(M_1 \times M_0)$  one has

$$(4.8) \quad \begin{aligned} \text{Hom}_{D^b \text{Coh}(M_1)}(\Phi_P(X_0), X_1) &= \text{Hom}_{D^b \text{Coh}(M_1)}(p_{1,*}(P \otimes p_0^* X_0), X_1) \\ &\cong \text{Hom}_{D^b \text{Coh}(M_1 \times M_0)}(P \otimes p_0^* X_0, p_1^* X_1 \otimes K_{M_0}[n]) \\ &\cong \text{Hom}_{D^b \text{Coh}(M_1 \times M_0)}(p_0^* X_0, (P^\vee \otimes K_{M_0}[n]) \otimes p_1^* X_1) \\ &\cong \text{Hom}_{D^b \text{Coh}(M_0)}(X_0, p_{0,*}(P^\vee \otimes K_{M_0}[n] \otimes p_1^* X_1)) \\ &= \text{Hom}_{D^b \text{Coh}(M_0)}(X_0, \Phi_{P^{\text{right}}}(X_1)), \end{aligned}$$

where  $n = \dim(M)$ ;  $K_{M_0}$  is the canonical bundle (pulled back to  $M_0 \times M_1$ , even though the notation does not reflect that); and

$$(4.9) \quad P^{\text{right}} = P^\vee \otimes K_{M_0}[n].$$

The left adjoint is similarly given by

$$(4.10) \quad P^{\text{left}} = P^\vee \otimes K_{M_1}[n].$$

**REMARK 4.2.** *It is much harder to say how much  $\Phi_P$  knows about the kernel  $P$ . If  $\Phi_P$  is the identity functor, then  $P \cong \mathcal{O}_{\Delta_M}$ . From that, one concludes for any  $P$  which has an inverse under convolution, the functor  $\Phi_P$  determines  $P$  up to isomorphism (the same is true for fully faithful  $\Phi_P$  [145], but it fails for general Fourier-Mukai transforms: see [44], which follows an earlier negative answer to a related question [95, Example 5.15]). Finally, any exact equivalence  $D^b \text{Coh}(M_0) \rightarrow D^b \text{Coh}(M_1)$  is isomorphic to a Fourier-Mukai transform [145] (this is an open question for general exact functors). One can doubt whether such questions are even natural: they are the consequence of working in the world of classical triangulated categories, inside which functor categories do not have particularly good properties.*

Following standard usage, let's say that  $M_0$  and  $M_1$  are derived equivalent if  $D^b \text{Coh}(M_0)$  is equivalent to  $D^b \text{Coh}(M_1)$  as a triangulated category. Again by [145], there are

$$(4.11) \quad \begin{aligned} P_1 &\in D^b \text{Coh}(M_1 \times M_0), & P_2 &\in D^b \text{Coh}(M_0 \times M_1), \\ P_2 * P_1 &\cong \mathcal{O}_{\Delta_{M_0}}, & P_1 * P_2 &\cong \mathcal{O}_{\Delta_{M_1}}. \end{aligned}$$

We are interested in the implication of this for the categories  $D^b \text{Coh}(M_k \times M_k)$  with their tensor (convolution) structure.

LEMMA 4.3. *Suppose that  $M_0$  and  $M_1$  are derived equivalent. Then there is an exact equivalence*

$$(4.12) \quad D^b \text{Coh}(M_1 \times M_1) \cong D^b \text{Coh}(M_0 \times M_0)$$

*which is compatible with convolution; maps the structure sheaves of the diagonals to each other, and the same more generally for the objects  $\mathcal{O}_{\Delta_{M_i}} \otimes K_{M_i}^d$  ( $d \in \mathbb{Z}$ ).*  $\square$

PROOF. If the derived equivalence is as in (4.11), then (4.12) is given by

$$(4.13) \quad P \longmapsto P_2 * P * P_1.$$

Only the last property, involving the canonical bundle, is worth commenting on. Because of convolution, it is sufficient to prove the case  $d = 1$ . For that, one can argue abstractly in terms of Serre functors, or slightly more concretely as follows. Since  $\Phi_{P_1}$  is an equivalence, its left and right adjoints must coincide, which in view of (4.10), (4.9) (and the fact that an equivalence determines its kernel up to isomorphism) means that  $P_1^\vee \otimes K_{M_0} \cong P_0^\vee \otimes K_{M_1}$ . After dualizing and then tensoring with  $K_{M_0 \times M_1}$ , this takes on the form

$$(4.14) \quad P_1 \otimes K_{M_0} \cong P_0 \otimes K_{M_1}.$$

But then

$$(4.15) \quad \begin{aligned} P_2 * (\mathcal{O}_{\Delta_{M_1}} \otimes K_{M_1}) * P_1 &\cong P_2 * (K_{M_1} \otimes P_1) = P_2 * (K_{M_0} \otimes P_1) \\ &\cong P_2 * P_1 * (\mathcal{O}_{\Delta_{M_0}} \otimes K_{M_0}) \cong \mathcal{O}_{\Delta_{M_0}} \otimes K_{M_0}. \end{aligned}$$

This argument is not particularly elegant, since it passes from kernels to functors and back, but that allowed us to use the familiar language of adjoint functors.  $\square$

### The derived Picard group

DEFINITION 4.4. *Consider the semigroup of isomorphism classes of objects in  $D^b \text{Coh}(M \times M)$ , with the multiplication given by (4.3). The derived Picard group  $DPic(M)$  is the subset of invertible elements inside that semigroup.*

By the previously discussed general results, this is really the ‘‘automorphism group’’ of  $D^b \text{Coh}(M)$ , which means the group of exact self-equivalences up to isomorphism. In particular, it is itself a derived invariant of  $M$ . The group  $DPic(M)$  can be described explicitly in some cases (see (3.11), and more generally [32, 146]), but in general it can be rather large and hard to approach. On the other hand, there are some subgroups that can be easily understood, and one of those is specifically of interest to us. Let  $Pic(M)$  be the Picard

group, and  $Aut(M)$  the automorphism group. The latter acts on the former, and we can form the semidirect product  $Pic(M) \rtimes Aut(M)$ . Any pair  $([L], f)$  in this semidirect product determines a Fourier-Mukai kernel

$$(4.16) \quad P_{L,f} = p_1^* L \otimes \mathcal{O}_{\Gamma_f},$$

where  $\Gamma_f \subset M \times M$  is the graph. This defines an injective homomorphism  $Pic(M) \rtimes Aut(M) \rightarrow DPic(M)$ . Here is a characterization of (4.16) (roughly following [159, Section 4.4.1]):

LEMMA 4.5. *Let  $P$  be a coherent sheaf on  $M \times M$  with the following properties:*

- *The projections  $p_0, p_1 : Supp(P) \rightarrow M$  are finite morphisms;*
- *$p_{0,*}(P)$  and  $p_{1,*}(P)$  are line bundles.*

*Then  $P \cong P_{L,f}$  for some  $([L], f)$ .* □

Now restrict to the subgroup  $Pic_0(M) \subset Pic(M)$  of topologically trivial line bundles (which is an abelian variety), and similarly to the neutral connected component  $Aut_0(M) \subset Aut(M)$ , which is an algebraic group. The action of  $Aut_0(M)$  on  $Pic_0(M)$  is trivial, hence with the definition (4.1) one has

$$(4.17) \quad DPic_0(M) \hookrightarrow Pic(M) \rtimes Aut(M) \hookrightarrow DPic(M).$$

### The parametrized version

Let  $S$  be an auxiliary smooth quasiprojective variety. A family of Fourier-Mukai kernels parametrized by  $S$  is an object  $\underline{P} \in Ob D^b Coh(S \times M \times M)$ . When it comes to defining the notion of isomorphism between two families, we find it convenient to work locally over  $S$ . This means that two families over the same base are called isomorphic if there is a cover of  $S$  by Zariski-open subsets, such that the restrictions to each of those subsets are isomorphic (with no coherence condition imposed on the isomorphisms). There is a version of Lemma 4.5 for families, leading to the following:

LEMMA 4.6. *Let  $\underline{P}$  be a family of Fourier-Mukai transforms parametrized by  $S$ . Suppose that there is a point  $s \in S$  such that  $\underline{P}_s \cong P_{L,f}$  for some  $([L], f)$ . Then, there is a Zariski-open subset  $U \subset S$  containing  $s$ , an map  $\underline{f} : U \times M \rightarrow M$  which is fiberwise an automorphism, and a line bundle  $\underline{L}$  on  $U \times M$ , such that*

$$(4.18) \quad \underline{P}|_U \cong P_{\underline{L}, \underline{f}}$$

*is given by the family version of (4.16).* □

The same thing holds if one considers only pairs  $([L], f)$  lying in  $DPic_0(M)$ , and we will use this to characterize that group by a universal property. Take a connected algebraic group  $G$ . A *weak derived action* of  $G$  on  $M$  is given by a family of Fourier-Mukai kernels  $\underline{P}$  parametrized by  $G$  itself, with the following two properties:

- Restriction to  $e \in G$  yields  $\underline{P}_e \cong \mathcal{O}_{\Delta_M}$ ;
- There is an isomorphism of families over  $G \times G$  (in the sense described above), which fibrewise yields

$$(4.19) \quad \underline{P}_{g_2 g_1} \cong \underline{P}_{g_2} * \underline{P}_{g_1}.$$

To be more precise, let  $m : G \times G \rightarrow G$  be the multiplication, and  $q_0, q_1 : G \times G \rightarrow G$  the projections. What we then mean by (4.19) is that Zariski-locally over  $G \times G$ ,

$$(4.20) \quad m^* \underline{P} \cong q_1^* \underline{P} * q_0^* \underline{P},$$

where  $*$  is convolution fibered over  $G \times G$ . Now, given such an action, Lemma 4.6 applies in a Zariski neighbourhood of the identity element in  $G$ . One can use (4.20) to translate that Zariski neighbourhood. The outcome is that we have a cover of  $G$  by open subsets  $U_i$  on which isomorphisms of type (4.18) hold, say

$$(4.21) \quad \underline{P}|_{U_i} \cong \underline{P}_{\underline{L}_i, \underline{f}_i}.$$

Moreover, each  $\underline{L}_i$  is topologically trivial, and each  $\underline{f}_i$  lies fibrewise in  $Aut_0(M)$ . These isomorphisms glue together to define a morphism of algebraic varieties

$$(4.22) \quad G \longrightarrow DPic_0(M).$$

Again appealing to (4.20), one shows that this is also a group homomorphism. On the other hand, there is clearly a weak derived action of  $DPic_0(M)$  itself on  $M$ , constructed using the universal line bundle on  $Pic_0(M) \times M$  and the evaluation map  $Aut_0(M) \times M \rightarrow M$ . Hence:

LEMMA 4.7. *Weak derived actions of  $G$  on  $M$  (up to isomorphism) correspond to homomorphisms of algebraic groups,  $G \rightarrow DPic_0(M)$ .*  $\square$

Our notion of action is easily seen to be derived invariant, by a parametrized version of Lemma 4.3. Therefore, Lemma 4.7 implies Theorem 4.1.

### Twisted Hochschild cohomology

The derived invariance of the group  $DPic_0(M)$  comes with a corresponding property for the kernels  $P_{L,f}$  associated to points  $([L], f)$ . By combining this with Lemma 4.3, one concludes that the following spaces are also derived invariant:

$$(4.23) \quad \begin{aligned} & Hom_{D^b Coh(M \times M)}^*(\mathcal{O}_{\Delta_M} \otimes K_M^d, P_{L,f}) \cong \\ & \cong Hom_{D^b Coh(M \times M)}^*(\mathcal{O}_{\Delta_M}, P_{L,f} \otimes K_M^{-d}) \\ & \cong H^*(M \times M, \mathcal{O}_{\Delta_M} \otimes P_{L,f} \otimes K_M^{-d-1})[-n] \\ & \cong H^*(M, \delta^* P_{L,f} \otimes K_M^{-d-1})[-n], \end{aligned}$$

where  $\delta : M \rightarrow M \times M$  is the diagonal embedding;  $n = \dim(M)$  as before; and  $d$  is any integer. The transformations between the various expressions (4.23) used Serre duality, as well as the fact that the derived dual of  $\mathcal{O}_{\Delta_M}$  is  $\mathcal{O}_{\Delta_M} \otimes K_M^{-1}[-n]$ .

Here are some illustrative examples:

- If  $f = id_M$ , we can use the isomorphism  $\delta^*\mathcal{O}_{\Delta_M} \cong \bigoplus_i \Omega_M^i[i]$  (a sheafified version of Hochschild-Kostant-Rosenberg, proved in [207, Equation 0.5], see also [15]) to rewrite (4.23) as

$$(4.24) \quad \bigoplus_i H^*(M, L \otimes \Omega_M^i[i] \otimes K_M^{-d-1})[-n] \cong \bigoplus_j H^*(M, L \otimes \Lambda^j TM[-j] \otimes K_M^{-d}).$$

For  $d = 0$  and trivial  $L$ , (4.24) is the Hochschild cohomology of  $M$ . For any  $d$  and  $L$ , the degree zero part of (4.24) is  $H^0(M, L \otimes K_M^{-d})$ . For further discussion, see [125].

- For  $d = -1$ , trivial  $L$ , and any  $f$ , (4.23) is the cohomology of the derived intersection  $\mathcal{O}_{\Delta_M} \otimes \mathcal{O}_{\Gamma_f}$ , which is a form of Hochschild homology and has already appeared previously in (2.19).
- If  $f$  has only nondegenerate fixed points,  $\delta^*P_{L,f} \cong \bigoplus_{x \in \text{Fix}(f)} \mathcal{O}_x$  is a direct sum of skyscraper sheaves at those fixed points. Hence (4.23) is concentrated in degree  $n$ , and isomorphic as a vector space to  $\mathbb{C}^{|\text{Fix}(f)|}$ , irrespective of the choice of  $L$  and  $d$ .

For fixed  $d$ , the spaces (4.23) are naturally fibres of an object of  $D^b \text{Coh}(DPic_0(M))$ . Since the only automorphisms of  $P_{L,f}$  are multiplication with scalars, this object itself is derived invariant only up to tensoring with a line bundle on  $DPic_0(M)$  (which is clearly the best possible result, since the construction of the universal line bundle on  $Pic_0(M)$  suffers from the same ambiguity).



## LECTURE 5

### Flux

Flux is a classical notion in symplectic topology, expressed as an isomorphism

$$(5.1) \quad \text{Symp}_0(M)/\text{Ham}(M) \cong H^1(M; \mathbb{R})/\Gamma.$$

One can study flux using fixed point Floer cohomology, in a way which parallels the algebro-geometric object (4.1), or alternatively using Lagrangian Floer cohomology. In fact, those invariants yield useful information even in situations where  $\Gamma$  is trivial. We will illustrate this by looking at symplectic mapping tori and some related manifolds.

#### Symplectic isotopies

We begin by recalling some standard terminology. Let  $(M, \omega_M)$  be a closed symplectic manifold. We write  $\text{Symp}(M)$  for the symplectic automorphism group, and  $\text{Symp}_0(M)$  for the connected component of the identity. Formally, these are infinite-dimensional Lie groups whose Lie algebra, the space of symplectic vector fields, can be identified with the space of closed one-forms:

$$(5.2) \quad L\text{Symp}(M) \cong \ker(d : \Omega^1(M) \rightarrow \Omega^2(M)),$$

where a vector field  $X$  corresponds to the one-form  $-i_X\omega_M$ . Vector fields that correspond to exact one-forms are called Hamiltonian. The subspace of such vector fields is invariant under the adjoint action, hence gives rise to a foliation of  $\text{Symp}(M)$  of codimension  $\dim H^1(M; \mathbb{R})$ . Isotopies which are tangent to that foliation are called Hamiltonian. We denote by  $\text{Ham}(M)$  the (normal) subgroup of elements of  $\text{Symp}_0(M)$  which can be reached from the identity by a Hamiltonian isotopy.

Let  $\Phi = (\phi_t)_{0 \leq t \leq 1}$  be any symplectic isotopy, which means a smooth path in  $\text{Symp}(M)$ . To this corresponds a family  $(\delta_t)$  of closed one-forms. The cohomology class

$$(5.3) \quad \text{Flux}(\Phi) = \int_0^1 [\delta_t] dt \in H^1(M; \mathbb{R})$$

is called the flux of the isotopy. A Hamiltonian isotopy has zero flux; and conversely, an isotopy with zero flux can be deformed (keeping endpoints fixed) to a Hamiltonian isotopy (for this and related elementary background, see [133, Chapter 10]). The *flux subgroup*

$$(5.4) \quad \Gamma \subset H^1(M; \mathbb{R})$$

consists of all the fluxes associated to loops in  $Symp(M)$  (one does not lose anything by restricting to loops in  $Symp_0(M)$ , since  $\text{Flux}(\psi \circ \Phi) = \text{Flux}(\Phi)$  for any symplectomorphism  $\psi$ ). Equivalently,  $\text{Flux}(\cdot)$  is the image of  $[\omega_M]$  under

$$(5.5) \quad H^2(M; \mathbb{R}) \xrightarrow{\text{evaluation}} H^2(\text{Symp}(M) \times M; \mathbb{R}) \xrightarrow{\text{K\"unneth}} \text{Hom}(\pi_1(\text{Symp}(M)), H^1(M; \mathbb{R})).$$

An elementary reformulation of (5.5) may be useful. Take some loop  $\Phi = (\phi_t)$ ,  $t \in S^1$ , in  $\text{Symp}(M)$ . Given a loop  $\lambda : S^1 \rightarrow M$ , one defines a map

$$(5.6) \quad T^2 = S^1 \times S^1 \longrightarrow M, \quad (s, t) \longmapsto \phi_t(\lambda(s)).$$

The integral of  $\omega_M$  over this torus equals  $\langle \text{Flux}(\Phi), [\lambda] \rangle$ .

EXAMPLE 5.1. *If  $[\omega_M]$  is integral,  $\Gamma$  is a subgroup of  $H^1(M; \mathbb{Z})$ .*

EXAMPLE 5.2. *The flux subgroup is trivial for surfaces of genus  $\geq 2$ , since in that case any map  $T^2 \rightarrow M$  has degree zero.*

EXAMPLE 5.3 (Taken from [131]). *More generally, suppose that  $[\omega_M]$  is a multiple of  $c_1(M)$ . The pullback  $\lambda^*TM$  is a symplectic vector bundle over the circle, hence trivial, and any choice of trivialization induces a trivialization of the pullback of  $TM$  under (5.6). Hence, the integral of  $c_1(M)$  over that torus is zero, which means that  $\Gamma$  must again be trivial.*

EXAMPLE 5.4 (Taken from [103]). *Similar topological arguments prove the following result: if some Chern number of  $M$  is nonzero and the map*

$$(5.7) \quad \smile [\omega_M^{n-1}] : H^1(M; \mathbb{R}) \longrightarrow H^{2n-1}(M; \mathbb{R}),$$

*for  $2n = \dim(M)$ , is an isomorphism (the latter condition always holds for K\"ahler manifolds), then  $\Gamma = 0$ .*

Here is another reformulation. For any class  $d \in H^1(M; \mathbb{R})$ , choose a family  $(\delta_t)_{0 \leq t \leq 1}$  of closed one-forms whose cohomology classes integrate to  $d$ . This induces a symplectic isotopy (starting at the identity), whose endpoint  $\phi^{(d)}$  is independent of the choice of one-forms up to Hamiltonian isotopy. Then,  $d$  belongs to the flux subgroup if and only if  $\phi^{(d)}$  is Hamiltonian isotopic to the identity. This explains the isomorphism (5.1).

EXAMPLE 5.5. *Take a symplectic vector space  $(V, \omega_V)$  and a lattice  $G \subset V$ . Form the symplectic torus  $M = V/G$ , so that  $H_1(M; \mathbb{Z}) = G$ . Take a loop  $\Phi$  of symplectic automorphisms, which is such that the orbit of any point is in the class corresponding to  $g_1 \in G$ . Take a loop  $\lambda \in G$  whose homology class corresponds to  $g_2 \in G$ . Then by definition of the symplectic form,*

$$(5.8) \quad \langle \text{Flux}(\Phi), [\lambda] \rangle = \omega_V(g_2, g_1)$$

*(ultimately, a formula of this kind can exist because  $[\omega_M]$  lies in the subring generated by one-dimensional cohomology classes). In particular, if we use  $\omega_V$  to identify  $H_1(M; \mathbb{R}) = V \cong V^\vee = H^1(M; \mathbb{R})$ , then the flux is precisely  $g_1$ . Hence, with respect to this identification,*

$$(5.9) \quad \Gamma \subset G.$$

*On the other hand, every class in  $H^1(M; \mathbb{R})$  can be represented uniquely by a constant one-form, which corresponds to a constant symplectic vector field, in other words to an element*

$v \in V$ . The flow of that vector field is translation  $x \mapsto x + tv$ . In particular, if  $v \in G$ , the time-one map is the identity, so  $G \subset \Gamma$ . This shows that (5.9) is an equality.

Alternatively, one could have obtained (5.9) as follows. If  $v \notin G$ , the time-one map is fixed point free, hence can't be Hamiltonian isotopic to the identity by the (solution of the) Arnol'd conjecture for tori. This is our first indication of a link between flux and deeper issues in symplectic topology. Another such indication is provided by the proof of the flux conjecture:

**THEOREM 5.6** (Ono [144]). *For any closed symplectic manifold  $M$ ,  $\Gamma \subset H^1(M; \mathbb{R})$  is a discrete subgroup.*

### Fixed point Floer cohomology

To describe the formal structure of “closed string” or fixed point Floer cohomology, we need to introduce the one-variable Novikov field  $\mathbb{A}$ , which is the field of formal series

$$(5.10) \quad f(q) = a_0 q^{r_0} + a_1 q^{r_1} + \cdots$$

where the coefficients  $a_k$  lie in some auxiliary field  $\mathbb{K}$  (an arbitrary field of characteristic 0), and the exponents  $r_k$  are real numbers, satisfying  $\lim_k r_k = \infty$ . Write  $\mathbb{A}^{\geq 0} \subset \mathbb{A}$  for the subring of series (5.10) in which only nonnegative powers of  $q$  appear, and  $\mathbb{A}^{> 0}$  for the ideal in which only strictly positive powers appear.

The *fixed point Floer cohomology* group  $HF^*(\phi)$  is a  $\mathbb{Z}/2$ -graded vector space over  $\mathbb{A}$  associated to a symplectic automorphism  $\phi \in \text{Symp}(M)$ . More generally, one can introduce twisted versions  $HF^*(\phi; \xi)$ , where  $\xi$  is a bundle over  $M$  with fibre  $\mathbb{A}$  and (discrete) structure group  $\mathbb{A}^{\geq 0, \times}$  (the group of invertible elements in  $\mathbb{A}^{\geq 0}$ ). Fixed point Floer cohomology is invariant under Hamiltonian isotopies of  $\phi$  (in the twisted case, carrying  $\xi$  along with the isotopy). It is also conjugation invariant. Another fundamental property is that

$$(5.11) \quad HF^*(id; \xi) \cong H^*(M; \xi),$$

where the right hand side is ordinary cohomology with twisted coefficients (and the grading has been reduced to  $\mathbb{Z}/2$ ). In fact, those two properties (for trivial  $\xi$ ) together yield the proof of the Arnol'd conjecture.

**REMARK 5.7.** *These statements are not quite the most general ones. Let's associate to  $\phi$  the loop space*

$$(5.12) \quad \mathcal{L}_\phi M = \{x : \mathbb{R} \rightarrow M : x(t) = \phi(x(t+1))\}.$$

*Formally, fixed point Floer cohomology looks like a Morse-Novikov theory on this space. It can therefore be twisted by a  $\mathbb{A}^{\geq 0, \times}$ -bundle on  $\mathcal{L}_\phi M$ . In the description above, we have only used bundles pulled back by the evaluation map  $\mathcal{L}_\phi \rightarrow M$ .*

*On a related note,  $HF^*(\phi)$  remains invariant under all isotopies whose flux lies in the image of  $id - \phi^* : H^1(M; \mathbb{R}) \rightarrow H^1(M; \mathbb{R})$ . This is again quite transparent from a loop space perspective, or alternatively one can derive it from the invariance under Hamiltonian isotopies and conjugation.*

For our application, the relevant groups are  $HF^*(\phi^{(d)}; \xi)$ . These depend on  $d \in H^1(M; \mathbb{R})$  and the isomorphism type of  $\xi$ , which one can combine into a single expression

$$(5.13) \quad q^d[\xi] \in H^1(M; \Lambda^\times).$$

From this point of view, what Floer cohomology yields is a family of  $\mathbb{Z}/2$ -graded vector spaces parametrized by  $H^1(M; \Lambda^\times)$ , and invariant under a suitably defined action of  $\Gamma$ . More geometrically, (5.13) is represented by a bundle whose holonomy around a loop  $\lambda$  in  $M$  is  $q^{d(\lambda)}$  times the corresponding holonomy of  $\xi$ . Let's denote that bundle by  $q^d\xi$ , in a slight abuse of notation. The main result from [144], which underlies the proof of the flux conjecture, is a generalization of (5.11):

$$(5.14) \quad HF^*(\phi^{(d)}; \xi) \cong H^*(M; q^d\xi) \quad \text{for small } d$$

(there are some special classes of symplectic manifolds for which (5.14) continues to hold for all  $d \in H^1(M; \mathbb{R})$ , see [119], but this is definitely not true in general, as one can already see from Example 5.5). It would be good to go further in this direction, and to have theorems describing the local structure of the stratification of  $H^1(M; \Lambda^\times)/\Gamma$  given by the ranks of Floer cohomology groups (in analogy with the algebro-geometric situation, where the spaces (4.23) give rise to an algebraic stratification of the derived Picard group).

EXAMPLE 5.8. *Take an arbitrary  $\psi \in \text{Symp}(M)$ . Its symplectic mapping torus  $E = E_\psi$  is the total space of the fibration over the two-torus with fibre  $M$  and monodromy  $\psi$  in one direction. Explicitly,*

$$(5.15) \quad \begin{aligned} E &= \mathbb{R}^2 \times M / (p, q, x) \sim (p-1, q, x) \sim (p, q-1, \psi(x)), \\ \omega_E &= dp \wedge dq + \omega_M. \end{aligned}$$

*If we take  $d$  to be the pullback of  $(1, 0) \in H^1(T^2; \mathbb{R})$  under the projection  $E \rightarrow T^2$ , and represent it by the one-form  $dp$ , the associated symplectic vector field is  $X = \partial_q$ , whose time-one map is  $\phi^{(d)}(p, q, x) = (p, q+1, x) = (p, q, \psi(x))$ . One expects the fixed point Floer cohomology of  $\phi^{(d)}$  to reflect that of  $\psi$ , and indeed*

$$(5.16) \quad HF^*(\phi^{(d)}) \cong H^*(T^2; \Lambda) \otimes HF^*(\psi).$$

*To be precise, this is easy to show in situations where the construction of fixed point Floer cohomology can be carried out with smooth moduli spaces of holomorphic curves, for instance if  $c_1(M) = 0$ . The general case, which uses virtual perturbations, should be similar, but would require more attention to the details (hence is strictly speaking conjectural). See [167, Section 4] for further discussion of (5.16) and its generalization to other cohomology classes  $d$ .*

*Suppose that  $H^1(M; \mathbb{R}) = 0$ , that  $\psi_* = \text{id}$  on cohomology, and that  $HF^*(\psi)$  has total rank different from that of  $H^*(M)$ . Then (5.16) implies that  $d$  does not lie in the flux subgroup of  $E$ ; hence, that group does not coincide with  $H^1(E; \mathbb{Z})$ , which distinguishes  $E$  from the trivial mapping torus  $T^2 \times M$ . If one assumes that  $\psi$  is fragile [177, Section 1], then  $E$  will be symplectically deformation equivalent (and in particular diffeomorphic) to  $T^2 \times M$ . This is a good source of nonstandard (deformation equivalent, but not symplectomorphic) symplectic structures.*

We want to mention one more general property:

PROPOSITION 5.9 (Arnol'd). *For any symplectic automorphism  $\phi$ , the total rank of  $HF^*(\phi^k)$  grows at most exponentially in  $k$ .*

PROPOSITION 5.10 (Arnol'd). *Equip the vector space  $H^1(M; \mathbb{R})$  with some Euclidean metric. There are constants  $C, D$  such that for all  $d \in H^1(M; \mathbb{R})$ ,*

$$(5.17) \quad \text{rank } HF^*(\phi^{(d)}) \leq D^{\|d\|} C.$$

This is not unexpected, and indeed, the proofs (which we will outline later on) are elementary. Proposition 5.10 is noteworthy because it provides a (weak) global bound, which complements local statements such as (5.14). Finally, note that even though we have not mentioned the twistings  $\xi$  in the two Propositions above, that's just because the inequalities hold for any choice of those (with the same constants).

### Lagrangian Floer cohomology

The “open string” or Lagrangian intersection version of Floer cohomology is trickier to use, because of the complicated structure of the auxiliary choices that enter into its definition. Let's first consider the “topologically unobstructed” case, which does not suffer from such complications. Inside a fixed  $M$ , consider Lagrangian submanifolds  $L \subset M$  which are oriented, *Spin*, and such that

$$(5.18) \quad \langle [\omega_M], [u] \rangle = 0 \quad \text{for each map } u : (D, \partial D) \longrightarrow (M, L).$$

Floer cohomology associates to each pair  $(L_0, L_1)$  of such submanifolds a  $\mathbb{Z}/2$ -graded vector space  $HF^*(L_0, L_1)$  over  $\mathbb{A}$ . There are also twisted versions, in which each  $L_i$  carries a bundle  $\xi_{L_i}$  of the same kind as before. Floer cohomology is invariant under Hamiltonian isotopies of either of the two submanifolds involved. The application to flux goes via the implication

$$(5.19) \quad d \in \Gamma \implies HF^*(\phi^{(d)}(L_0), L_1) \cong HF^*(L_0, L_1) \text{ for all } L_0, L_1.$$

We also would like to mention the analogue of Proposition 5.9 in this context:

PROPOSITION 5.11 (Arnol'd). *Let  $\phi$  be a symplectic automorphism of  $M$ . Then there is a constant  $D > 0$  depending only on  $\phi$ , such that the following holds. For any pair  $(L_0, L_1)$  of Lagrangian submanifolds, there is a constant  $C$  such that*

$$(5.20) \quad \text{rank } HF^*(\phi^k(L_0), L_1) \leq D^k C \quad \text{for all } k > 0.$$

As before, the same bound holds for all choices of bundles on  $L_0, L_1$ .

EXAMPLE 5.12. *Let  $M$  be a closed surface of genus  $> 0$ , and  $E$  the symplectic mapping torus (5.15) of some  $\psi \in \text{Symp}(M)$ . To any non-contractible simple closed curves  $\alpha \subset M$ , one can associate a Lagrangian torus*

$$(5.21) \quad L_\alpha = S^1 \times \{0\} \times \alpha \subset E.$$

*This is incompressible, meaning that its fundamental group injects into  $\pi_1(E)$ , and we also have  $\pi_2(E) = 0$ , which implies that (5.18) holds for  $L_\alpha$ . The (much more straightforward)*

counterpart of (5.16) says that

$$(5.22) \quad HF^*(L_{\alpha_0}, L_{\alpha_1}) \cong H^*(S^1; \mathbb{A}) \otimes HF^*(\alpha_0, \alpha_1).$$

Lagrangian intersection Floer cohomology in  $M$  essentially reduces to the (nontrivial, but purely topological) theory of geometric intersection numbers [68]. Suppose from now on that  $\psi$  has infinite order in the mapping class group. Then there are curves  $\alpha_0, \alpha_1$  such that

$$(5.23) \quad \overline{\lim}_{k \rightarrow \infty} \text{rank } HF^*(\psi^k(\alpha_0), \alpha_1) = \infty.$$

By combining this with the same elementary considerations as in Example 5.8, one sees that the image of  $(k, 0)$  under the pullback  $H^1(T^2; \mathbb{R}) \rightarrow H^1(E; \mathbb{R})$  can not lie in  $\Gamma$ , for any  $k \in \mathbb{R} \setminus \{0\}$ .

We should stress that this is only a toy model. The same conclusion about the flux subgroup can be derived using fixed point Floer cohomology, since that has been fully computed for surface diffeomorphisms [54]. It is even possible that it might be within reach of elementary topological arguments involving  $\pi_1(E)$ , in the manner of [103, Theorem A.2] or Example 5.5.

To illustrate the issues with Lagrangian Floer cohomology in general, let's consider the case of Lagrangian surfaces  $L$  (oriented, with a *Spin* structure) in a closed symplectic four-manifold  $M$ . More specifically, we are interested in the groups  $HF^*(\phi(L), L)$ , for some  $\phi \in \text{Symp}(M)$ . Such groups can always be defined, but they are not unique. To make the additional choices more explicit, let's take a compatible almost compatible structure  $J$  which is generic in the following sense:

- There are no non-constant  $J$ -holomorphic spheres of Chern number  $< 1$ . Those of Chern number 1 are regular, and intersect  $L$  transversally.
- There are no non-constant  $J$ -holomorphic discs with boundary on  $L$  of Maslov number  $< 2$ . Those of Maslov number 2 are regular.

Given such a  $J$ , we can make an additional choice of a bundle  $\xi_L$  as before. For  $\phi(L)$ , we then choose  $\phi_*(J)$  and  $\phi_*(\xi_L)$ . The resulting Floer cohomology is well-defined by a cancellation mechanism first encountered in [142]. The problematic part is that passing from one such  $J$  to a different one, while preserving Floer cohomology, requires a highly nontrivial identification of the spaces of  $\xi_L$  on both sides (and more generally, one may not want to base one's definition on choosing such a  $J$  at all, but instead on a more abstract obstruction theory). Understanding this requires the full formalism of [71]. Still, the outcome is that if the auxiliary choices are suitably correlated,  $HF^*(\phi(L), L)$  becomes invariant under Hamiltonian deformations of  $\phi$  or  $L$ . Moreover, the analogue of Proposition 5.11 holds.

EXAMPLE 5.13. *The symplectic mapping torus construction has a relative version, which has been extensively explored since it gives rise to knotted symplectic submanifolds [69]. The simplest class of examples goes as follows. Let  $\text{Conf}_m(S^2)$  be the configuration space of unordered  $m$ -tuples of points on  $S^2$ , for some  $m \geq 3$ . Its fundamental group, the spherical*

$m$ -stranded braid group, sits in a short exact sequence

$$(5.24) \quad 1 \rightarrow \mathbb{Z}/2 \longrightarrow \pi_1(\text{Conf}_m(S^2)) \longrightarrow \pi_0(\text{Diff}^+(S^2, \{m \text{ points}\})) \rightarrow 1.$$

Fix a loop  $\beta$  in configuration space, representing some conjugacy class  $[\beta]$  in the spherical braid group. We can associate to it the symplectic surface

$$(5.25) \quad S = S_\beta = \{(p, q, x) : x \in \beta(q)\} \subset E = T^2 \times S^2.$$

Take a simple closed non-contractible curve  $\alpha \subset S^2 \setminus \beta(0)$ , and consider the Lagrangian torus  $L_\alpha \subset E \setminus S$  defined in (5.21). The analogue of (5.22) is

$$(5.26) \quad HF_{E \setminus S}^*(L_{\alpha_0}, L_{\alpha_1}) \cong H^*(S^1) \otimes HF_{S^2 \setminus \beta(0)}^*(\alpha_0, \alpha_1),$$

where we have included the symplectic manifolds under consideration in the notation, in order to avoid misunderstandings. Defining the Floer cohomology groups in (5.26) is straightforward, since (5.18) is satisfied (even though  $E \setminus S$  is noncompact, that does not cause any technical issues, since one can work in  $E$  but use holomorphic curves which avoid  $S$ ). Let  $d = (1, 0) \in H^1(E; \mathbb{R})$ . For a suitable choice of one-forms, the associated symplectic isotopy will preserve  $S$ , and then

$$(5.27) \quad \phi^{(d)}(L_\alpha) = L_{\psi(\alpha)}$$

where  $\psi$  is (a symplectic representative of) the image of  $\beta$  under (5.24). In particular, for any  $k \in \mathbb{Z}$  and any  $\alpha$  one knows from (5.26) that

$$(5.28) \quad HF_{E \setminus S}^*(\phi^{kd}(L_\alpha), L_\alpha) \cong H^*(S^1) \otimes HF_{S^2 \setminus \beta(0)}^*(\psi^k(\alpha), \alpha).$$

From now on, we consider only pure braids  $\beta$  (so  $S$  is the disjoint union of  $m$  tori, each of which can be identified with  $T^2$  by projection), and also we want to assume that our braids are framed, which amounts to a trivialization of the normal bundle to (each connected component of)  $S$ .

Take another closed four-manifold  $\tilde{E}$  containing a symplectic torus  $\tilde{S} \cong T^2$  with trivialized normal bundle. Suppose that the following two conditions are satisfied:

- One has  $\pi_1(\tilde{E} \setminus \tilde{S}) \cong \mathbb{Z}$ , in such a way that the meridian of  $\tilde{S}$  maps to 0, and the two longitudes to 1 and 0, respectively.
- For a generic compatible almost complex structure on  $\tilde{E}$  with respect to which  $\tilde{S}$  is holomorphic, there is no holomorphic sphere which passes through  $\tilde{S}$ .

Concrete examples of such manifolds are easy to construct, see [183]. Let's perform the fibre connected sum [80]

$$(5.29) \quad M = M_\beta = E \# \overbrace{(\tilde{E} \cup \cdots \cup \tilde{E})}^{m \text{ copies}},$$

where we glue each connected component of  $S$  to a copy of  $\tilde{S}$ . The outcome is a symplectic four-manifold with  $\pi_1(M) \cong \mathbb{Z}$ . All the tori  $L_\alpha$  will carry over to  $M$  (in a way which is unique up to Hamiltonian isotopy).

By construction,  $M$  carries a unique cohomology class whose restriction to  $E \setminus S_\beta$  equals that of  $d$ . For simplicity, let's denote that class again by  $d$ . A (technically highly nontrivial) degeneration argument in the spirit of [96] shows that the cohomology groups  $HF^*(\phi^{(kd)}(L_\alpha), L_\alpha)$  in  $M$  are the same as in  $E \setminus S$ . In particular, by the analogue of Proposition 5.11, we get a bound

$$(5.30) \quad \text{rank } HF^*(\phi^{(kd)}(L_\alpha), L_\alpha) \leq D^k C \quad \text{for } k > 0,$$

where  $C$  depends on the choice of  $\alpha$ , but  $D$  depends only on  $\phi^{(d)}$  up to Hamiltonian isotopy. As we will now see, this allows one to distinguish between the manifolds  $M_\beta$ , but in a way which is not effective.

Namely, by a suitable choice of  $\beta$ , one can achieve that there are  $\alpha$  such that the ranks of the Floer cohomology groups in (5.28) grow exponentially in  $k$  (one can see this easily using Thurston's classification of surface diffeomorphisms [68]). After that, the exponential growth rate can be made arbitrarily large, simply by replacing the chosen  $\beta$  with some power. By comparison with (5.30), one concludes that there are infinitely many distinct (not pairwise symplectically isomorphic)  $M_\beta$ . This is closely related to the results in [183] (those were proved using fixed point Floer cohomology, which is a little harder to compute but yields more clear-cut statements).

### Growth of intersections

Our final task is to explain the proof of Propositions 5.9, 5.10, and 5.11. The argument is taken from [17], and applies in a variety of geometric situations; we will begin by considering diffeomorphisms, and add the symplectic structures later.

Let  $M$  be a closed Riemannian manifold. Under any diffeomorphism  $\phi \in \text{Diff}(M)$ , the Riemannian volume of submanifolds grows at most by a constant, more precisely

$$(5.31) \quad \text{vol}(\phi(L)) \leq \|D\phi\|^{\dim(L)} \text{vol}(L).$$

Similarly, if  $(\phi_t)$  is the flow of a vector field  $X$ , one has

$$(5.32) \quad \frac{d}{dt} \text{vol}(\phi_t(L)) \leq \dim(L) \|\nabla X\| \text{vol}(\phi_t(L)),$$

and hence for any  $t \geq 0$ ,

$$(5.33) \quad \text{vol}(\phi_t(L)) \leq e^{t \dim(L) \|\nabla X\|} \text{vol}(L).$$

Take two submanifolds  $L_0, L_1 \subset M$  of complementary dimension. Let  $\text{int}_U(L_0, L_1)$  be the number of *intersection points not removable by a small perturbation*. Here,  $U$  is a neighbourhood of the point corresponding to  $L_1$  inside the space of all submanifolds, with the  $C^\infty$ -topology (by taking  $U$  small, one can ensure that all  $\tilde{L}_1$  in  $U$  are isotopic to  $L_1$ ). The definition is

$$(5.34) \quad \text{int}_U(L_0, L_1) \stackrel{\text{def}}{=} \min_{\tilde{L}_1} (|L_0 \cap \tilde{L}_1|),$$



where the minimum is taken all  $\tilde{L}_1$  in  $U$  which intersect  $L_0$  transversally. We will later modify this definition further to fit the situation in symplectic geometry.

Arnol'd's insight is that (5.34) is governed by the volume, in the following sense.

LEMMA 5.14 ([17, Lemma 1]). *For any  $L_1$  and  $U$  there is a constant  $C = C(L_1, U)$  such that*

$$(5.35) \quad \text{int}_U(L_0, L_1) \leq C \text{vol}(L_0) \quad \text{for all } L_0.$$

PROOF. This is essentially a probabilistic argument. Among a suitable family of small perturbations of  $L_1$ , we prove that there must be at least one which intersects  $L_0$  transversally and in not more than  $C \text{vol}(L_0)$  points.

**Claim:** *There is a map*

$$(5.36) \quad \iota : P \times L_1 \longrightarrow M,$$

where  $P$  is a small closed ball around the origin in  $\mathbb{R}^N$  for some large  $N$ , with the following properties. First,  $\iota|_{\{p\} \times L_1}$  is an embedding for any  $p$ , specializing to the given inclusion for  $p = 0$ . We denote the images of these embeddings by  $\tilde{L}_{1,p}$ , and also assume that they all lie in the neighbourhood  $U$ . The final requirement is that  $\iota$  should be a submersion.

To verify this claim, one first constructs a map satisfying the first requirement, and such that  $D\iota$  is onto at every point of the form  $(0, x)$ . Then, making  $P$  smaller yields the desired submersion property. In fact, let's agree that we shrink it by slightly more than strictly necessary.

**Claim:** *Fix some Riemannian metric on  $P \times L_1$ . Then there is a constant  $C > 0$  such that for any submanifold  $L_0 \subset M$ ,*

$$(5.37) \quad \text{vol}(\iota^{-1}(L_0)) \leq C \text{vol}(L_0).$$

If this is true for some metric on  $P \times L_1$ , then it is true for any other metric. Hence, we may assume without loss of generality that the metric is chosen in such a way that  $\iota$  is a Riemannian submersion. Then,

$$(5.38) \quad \text{vol}(\iota^{-1}(L_0)) = \int_{L_0} \text{vol}(\iota^{-1}(x)) |dx|.$$

Note that outside a subset of Lebesgue measure zero,  $\iota^{-1}(x)$  intersects  $\partial P \times L_1$  transversally, hence is a compact manifold with boundary, with well-defined volume. Take some  $x_0 \in L_0$ . There is a ball  $\tilde{P}$  slightly larger than  $P$ , and an extension  $\tilde{\iota}$  of  $\iota$  to  $\tilde{P} \times L_1$ , such that  $\tilde{\iota}^{-1}(x_0)$  intersects  $\partial \tilde{P} \times L_1$  transversally. Then, the same holds for all  $x$  sufficiently close to  $x_0$ . Hence,  $\text{vol}(\tilde{\iota}^{-1}(x))$  is locally a continuous function near  $x = x_0$ . This yields a local bound for  $\text{vol}(\iota^{-1}(x))$  in the same region, and since  $L_0$  is compact, we get an overall bound  $C$  valid for all  $x$ .

**Claim:** *We have*

$$(5.39) \quad \int_P |L_0 \cap \tilde{L}_{1,p}| |dp| \leq \text{vol}(\iota^{-1}(L_0)),$$

where: the integral on the left is with respect to the standard flat metric on  $P$ , normalized so that  $\text{vol}(P) = 1$ ; and the volume on the right side is with respect to the metric on the submanifold  $\iota^{-1}(L_0)$  induced by the product of our metric on  $P$  and an arbitrary Riemannian metric on  $L_1$ .

The intersection  $L_0 \cap \tilde{L}_{1,p}$  is transverse (hence in particular finite) for all  $p$  outside a closed subset of measure zero. Let  $P^{reg}$  be its complement. Then, the projection

$$(5.40) \quad \iota^{-1}(L_0) \cap (P^{reg} \times L_1) \longrightarrow P^{reg}$$

is a finite covering. Moreover, the pullback of  $|dp|$  yields a density which is pointwise less or equal than that used to compute the volume of  $\iota^{-1}(L_0)$ .

After combining the inequalities above into

$$(5.41) \quad \int_P |L_0 \cap \tilde{L}_{1,p}| |dp| \leq C \text{vol}(L_0),$$

one sees that there must be some subset of positive measure inside  $P$  on which  $|L_0 \cap \tilde{L}_{1,p}| \leq C \text{vol}(L_0)$ . Inside that subset, one can find a  $p$  where the intersection is transverse.  $\square$

Let's assume from now on that  $M$  is symplectic (not necessarily in a way which is compatible with the metric) of dimension  $2n$ ; that our submanifolds are Lagrangian; and that the topology of the space of Lagrangian submanifolds is defined in such a way that only Hamiltonian isotopies are continuous (this affects the definition of neighbourhoods  $U$ ). In the proof of Lemma 5.14, we can construct  $\iota$  in such a way that all the  $\tilde{L}_{1,p}$  are Lagrangian, and depend in a Hamiltonian way on  $p$ . Then, that Lemma still holds with our modified definition of  $\text{int}_U(L_0, L_1)$ . In view of this and (5.31), we have:

**PROPOSITION 5.15.** *Take  $\phi \in \text{Symp}(M)$ . Then for any Lagrangian submanifolds  $(L_0, L_1)$ , and any  $U$ , there is a constant  $C$  such that*

$$(5.42) \quad \text{int}_U(\phi^k(L_0), L_1) \leq \|D\phi\|^{nk} C \quad \text{for all } k > 0.$$

$\square$

The number on the left hand side of (5.42) is an upper bound for the total rank of Lagrangian Floer cohomology (whenever that is defined). Hence, Proposition 5.11 follows immediately, and so does its analogues for other situations where Lagrangian Floer cohomology is well-defined. Next, recall that any symplectic fixed point problem can be converted to a Lagrangian intersection problem by looking at the graph and the diagonal. Applying that to Proposition 5.15 immediately yields Proposition 5.9. Proposition 5.11 is obtained in a similar way, but using (5.32).

## LECTURE 6

# Liouville manifolds

This lecture has three aims. The first one is to review some elementary notions of symplectic topology, with an emphasis on explicit constructions of open symplectic manifolds. The original sources are [66, 205]; expository accounts can be found in [148] (specifically for the low-dimensional case) and [53] (a comprehensive reference including many new developments). The second aim is to show how a version of Floer cohomology (symplectic cohomology) can be applied to these manifolds. Among the existing literature on this topic, we follow [134] most closely; but the state of the art is probably better represented by [36]. The final aim is to explain some possible limitations of the use of symplectic cohomology as an invariant.

*Acknowledgments.* The discussion of Weinstein handle attachment (and in particular Example 6.5) reflects discussions with Mohammed Abouzaid.

### Symplectic manifolds with Liouville flows

We will consider exact symplectic manifolds, meaning symplectic manifolds  $(M, \omega_M)$  which additionally come with a distinguished one-form primitive  $\theta_M$ ,  $d\theta_M = \omega_M$ . The vector field  $Z_M$  defined by  $i_{Z_M}\omega_M = \theta_M$  is a *Liouville vector field*, which expands the symplectic form:

$$(6.1) \quad L_{Z_M}\omega_M = d(i_{Z_M}\omega_M) = \omega_M.$$

We will always require that  $Z_M$  should be transverse to  $\partial M$ , which implies that  $\theta_M|_{\partial M}$  is a contact one-form. Within this fairly general class of manifolds, we will more specifically consider the following cases:

- A *Liouville cobordism* is a compact  $M$  with the properties mentioned above. We then write  $\partial_- M$  and  $\partial_+ M$  for the (open and closed) subsets of the boundary where  $Z_M$  points inwards and outwards, respectively.

A Liouville cobordism is called *trivial* if every flow line of  $Z_M$  is a compact interval (with endpoints on  $\partial_{\pm} M$ ). In particular,  $M$  is then diffeomorphic to  $\partial_- M \times [0, 1]$ .

- A *Liouville domain* is the special case of a Liouville cobordism where  $\partial_- M = \emptyset$ .

- A *Liouville manifold* (really, this should be called a finite type complete Liouville manifold) is an open manifold  $M$  without boundary, with the following property. There is an exhausting (proper and bounded below) function  $h \in C^\infty(M, \mathbb{R})$ , such that  $dh(Z_M) > 0$  outside a compact subset. Moreover, we require that the flow of  $Z_M$  exists for all positive times (the corresponding statement for negative times follows from the previous requirement).

EXAMPLE 6.1. *Let  $M$  be a complex manifold which is Stein. One can equip  $M$  with a strictly plurisubharmonic exhausting function  $h$ . Then the regular sublevel sets of  $h$  are Liouville domains for  $\theta_M = -d^c h$  and its exterior derivative  $\omega_M = -d(d^c h)$ . The Liouville vector field is the gradient of  $h$  with respect to the Kähler metric associated to  $\omega_M$ .*

EXAMPLE 6.2. *Let  $M \subset \mathbb{C}^N$  be a smooth affine algebraic variety. Then, taking  $h(x) = \frac{1}{4}\|x\|^2$  and proceeding as before equips  $M$  with the structure of a Liouville manifold (one has  $dh(\nabla h) = \|dh\|^2 \neq 0$  outside a compact subset, because  $\|dh\|^2$  is a real polynomial). By construction,  $\omega_M$  is the restriction of the standard (constant) symplectic form on  $\mathbb{C}^N$  to  $M$ .*

EXAMPLE 6.3. *If  $L$  is a closed manifold, then  $M = T^*L$  with its canonical one-form is a Liouville manifold.*

Compact manifolds with boundary lend themselves well to gluing processes. Most obviously, if  $M_1$  and  $M_2$  are Liouville cobordisms such that  $\partial_+ M_1$  is diffeomorphic to  $\partial_- M_2$  in a way which is compatible with the respective one-forms, then

$$(6.2) \quad M = M_1 \cup_{\partial_+ M_1 \sim \partial_- M_2} M_2$$

naturally becomes a Liouville cobordism (the Liouville vector fields give canonical collar neighbourhoods, which one uses to define the differentiable structure on  $M$ ). More generally, suppose that  $\partial_+ M_1$  and  $\partial_- M_2$  are isomorphic as (oriented) contact manifolds. In that case, the gluing process (6.2) needs to be modified as follows:

$$(6.3) \quad M = M_1 \cup_{\partial_+ M_1 \sim \partial_- C} C \cup_{\partial_+ C \sim \partial_- M_2} M_2.$$

The intermediate piece  $C$  is a trivial cobordism, whose negative boundary is diffeomorphic to  $\partial_+ M_1$  compatibly with the one-forms; and whose positive boundary is diffeomorphic to  $\partial_- M_2$ , in such a way that  $\theta_C|_{\partial_+ C}$  corresponds to  $e^c \theta_{M_2}|_{\partial_- M_2}$  for some constant  $c$ . This of course means that one needs to rescale  $\omega_{M_2}$  by the same factor  $e^c$  when forming (6.3).

The outcome of (6.3) is no longer strictly unique (because of the choice of  $c$ ), which brings us to the overdue question of what the appropriate notion of isomorphism is for Liouville domains. The most narrow notion would be this:

- Let  $M_1, M_2$  be Liouville domains. A symplectic isomorphism  $\phi : M_1 \rightarrow M_2$  is called *exact rel boundary* if  $\phi^* \theta_{M_2} = \theta_{M_1} + dk$  for some function  $k$  which vanishes near the boundary.

However, this notion, as well as that of plain symplectic isomorphism, is too rigid for most purposes; the reason being that it preserves quantitative invariants such as volume and

capacities, which we do not want to consider. We therefore want to allow the two following operations as well:

- Attaching a trivial cobordism to the boundary;
- Deformation (of  $\theta_M$  and  $\omega_M$ , within the class of Liouville domains).

In fact, it is enough to allow one of those operations, and then the other can be expressed as a combination of that and symplectic isomorphisms which are exact rel boundary. Let's call the resulting equivalence relation *Liouville isomorphism*.

The situation is a little simpler for Liouville manifolds, for which there can be no quantitative invariants (since  $(M, \theta_M, \omega_M)$  is Liouville isomorphic to  $(M, e^c \theta_M, e^c \omega_M)$  for any  $c \in \mathbb{R}$ , using the flow of  $Z_M$ ). In this context, one defines a Liouville isomorphism to be a diffeomorphism such that  $\phi^* \theta_{M_2} = \theta_{M_1} + dk$  for some compactly supported function  $k$ . Deformation invariance is built in by a Moser-style argument. More precisely:

LEMMA 6.4. *Fix an open manifold  $M$  without boundary. On it, take a family of exact symplectic structures  $\omega_t = d\theta_t$ , with associated vector fields  $Z_t$ , and a family of exhausting functions  $h_t$ , both depending smooth on  $t \in [0, 1]$ . Suppose that for each  $t$ , the flow of  $Z_t$  is defined for all positive times. Suppose also that for all  $(t, x)$  outside a compact subset of  $[0, 1] \times M$ , we have  $dh_t(Z_t) > 0$ . Then  $(M, \omega_0, \theta_0)$  is Liouville isomorphic to  $(M, \omega_1, \theta_1)$ .  $\square$*

Still remaining on the same subject, note that one can pass from Liouville domains to Liouville manifolds by attaching an infinite cone:

$$(6.4) \quad M^{complete} = M \cup_{\partial M} ([0, \infty) \times \partial M),$$

where the conical part carries the one-form  $e^r(\theta_M|_{\partial M})$  ( $r$  is the coordinate on  $[0, \infty)$ ) and its exterior derivative. There is also a truncation construction in the opposite direction, and the two establish a bijection on the level of Liouville isomorphism types.

Finally, given that the symplectic form is exact, it makes sense to impose corresponding exactness conditions on Lagrangian submanifolds. For a closed Lagrangian submanifold, exactness means that  $[\theta_M|_L] \in H^1(L; \mathbb{R})$  is zero. Here are two generalizations:

- Suppose that  $M$  is a Liouville domain, and  $L \subset M$  a Lagrangian submanifold with  $\partial L = L \cap \partial M$ , which is of Legendrian type near the boundary. The last-mentioned condition means that  $\theta_M|_L$  vanishes in a neighbourhood of  $\partial L$ ; equivalently,  $\partial L \subset \partial M$  is Legendrian, and the Liouville flow is tangent to  $L$  near the boundary. Then, we say that  $L$  is exact rel boundary if  $[\theta_M|_L] \in H^1(L, \partial L; \mathbb{R})$  vanishes.
- Let  $M$  be a Liouville manifold, and  $L \subset M$  a properly embedded Lagrangian submanifold which is of Legendrian type at infinity, meaning that  $\theta_M|_L$  vanishes outside a compact subset. Then, we say that  $L$  is exact rel infinity if  $[\theta_M|_L] \in H_{cpt}^1(L; \mathbb{R})$  vanishes.

### Weinstein handle attachment

Suppose that we are given a Liouville domain  $M_1$  of dimension  $2n$ , and a submanifold  $K_1 \subset \partial M_1$  which is Legendrian for our contact structure, meaning that  $\dim(K_1) = n - 1$  and  $\theta_{M_1}|_{K_1} = 0$ . Suppose also that we have a compact manifold  $L_2$  and a diffeomorphism

$$(6.5) \quad \partial L_2 \cong K_1 \subset \partial M_1.$$

Recall that the normal bundle of  $K_1 \subset \partial M_1$  is  $\mathbb{R} \oplus T^*K_1 \cong T^*L_2|_{\partial L_2}$ . One can therefore attach a ball cotangent bundle  $D^*L_2$  to  $M_1$  along a tubular neighbourhood of  $K_1$ . Strictly speaking, the outcome has concave corners, but one can smooth them to get a manifold with boundary. In fact, a slightly fattened version of the same construction can be thought of as attaching a Liouville cobordism  $M_2$  (which homotopy retracts to  $\partial M_1 \cup_{K_1 \sim \partial L_2} L_2$ ). Let's denote the result of this attachment process by  $M$ .

EXAMPLE 6.5. *Suppose that  $M_1$  itself contains a Lagrangian submanifold  $L_1$  which is of Legendrian type near the boundary, exact rel boundary, and such that  $\partial L_1 = K_1$ . Then  $L = L_1 \cup_{\partial L_1 \sim \partial L_2} L_2$  is a closed exact Lagrangian submanifold in  $M$ .*

EXAMPLE 6.6. *The most important special case is where we start with a Legendrian  $K_1 \subset \partial M_1$  diffeomorphic to a sphere. In order to identify that sphere with the boundary of the standard ball  $L_2 = D^n$ , we suppose that a distinguished element  $\pi_0(\text{Diff}(S^{n-1}, K_1)/O(n))$  has been chosen (for  $n \leq 4$  this quotient is trivial). One says that  $M$  is obtained by attaching a critical Weinstein handle to  $M_1$  along  $K_1$ .*

More generally, suppose that we have  $K_1 \subset \partial M_1$  which is contact isotropic, meaning that  $\theta_{M_1}|_{K_1} = 0$ , but whose dimension can be arbitrary ( $\leq n - 1$  because of the isotropy condition). Such a manifold carries a canonical symplectic vector bundle of rank  $2(n - 1 - \dim(K_1))$ , namely

$$(6.6) \quad TK_1^\perp / TK_1 \longrightarrow K_1,$$

where the orthogonal complement is with respect to the symplectic form on the contact hyperplane bundle  $\ker(\theta_{M_1}|_{\partial M_1}) \subset T(\partial M_1)$ . Take a compact manifold  $L_2$  together with a symplectic vector bundle  $\nu$ . Suppose that there is a diffeomorphism  $\partial L_2 \cong K_1$ , covered by a bundle isomorphism  $\nu|_{\partial L_2} \cong TK_1^\perp / TK_1$ . Then we can again carry out a suitable attachment process, using the disc bundle of  $T^*L_2 \oplus \nu$ .

EXAMPLE 6.7. *If we again suppose that  $K_1$  is a sphere, and  $L_2$  a ball, the additional bundle data is a trivialization of (6.6). For  $\dim(K_1) < n - 1$ , one calls the resulting process attaching a subcritical Weinstein handle. In the lowest-dimensional case, one starts with two points in  $\partial M_1$  and attaches a 1-handle (in that case, no additional choices are required, since the linear symplectic group is connected).*

### Symplectic cohomology

Fix a coefficient field  $\mathbb{K}$ , which can be arbitrary. Symplectic cohomology associates to a Liouville domain or manifold  $M$  a  $\mathbb{Z}/2$ -graded  $\mathbb{K}$ -vector space  $SH^*(M)$ , of at most countable

dimension. This comes with a rich structure of operations, of which the only one relevant for now is the structure of a graded commutative ring with unit. Symplectic cohomology is invariant under Liouville isomorphism.

EXAMPLE 6.8. *The symplectic cohomology of  $\mathbb{C}^n$  is zero.*

EXAMPLE 6.9. *Suppose that  $M = T^*L$ , where  $L$  is closed and oriented. If  $\text{char}(\mathbb{K}) \neq 2$ , assume additionally that  $L$  is Spin. Then*

$$(6.7) \quad SH^*(M) \cong H_{n-*}(\mathcal{L}L; \mathbb{K}),$$

where  $\mathcal{L}L$  is the free loop space [202, 163, 1] (the isomorphism depends on the Spin structure: changing this structure by a class  $\alpha \in H^1(L; \mathbb{Z}/2)$  yields a change in (6.7) which is  $\pm \text{Id}$  on each component on the loop space, depending on how  $\alpha$  evaluates on loops). Importantly, (6.7) carries the ring structure on symplectic cohomology to the string product [2].

EXAMPLE 6.10. *The product of two Liouville manifolds is naturally again a Liouville manifold (the same is true for Liouville domains after rounding off corners). One then has a Künneth formula [139], which is an isomorphism of  $\mathbb{Z}/2$ -graded rings*

$$(6.8) \quad SH^*(M_1 \times M_2) \cong SH^*(M_1) \otimes SH^*(M_2).$$

Also important is the *Viterbo functoriality* property of symplectic cohomology [203]. Namely, suppose that  $M$  is obtained by attaching a Liouville cobordism to  $M_1$ , as in (6.2) or (6.3). One then has a canonical homomorphism (of unital rings)

$$(6.9) \quad SH^*(M) \longrightarrow SH^*(M_1).$$

In particular, if  $SH^*(M)$  vanishes, so does  $SH^*(M_1)$ , because the unit element is zero. The maps (6.9) are functorial with respect to repeated attachment.

REMARK 6.11. *One use of Viterbo functoriality is to show that symplectic cohomology is invariant under symplectic isomorphisms  $\phi : M_1 \rightarrow M_2$  which are exact, meaning that  $\phi^*\theta_{M_1} = \theta_{M_2} + dk$  for some function  $k$ . As explained in [36, Lemma 1.1], any symplectic isomorphism between Liouville manifolds can be deformed to an exact one. Hence, symplectic cohomology is actually a symplectic invariant of Liouville manifolds in the standard (undecorated) sense of the word.*

The following theorems allow one to change the topology of a Liouville domain without affecting its symplectic cohomology:

THEOREM 6.12 (Cieliebak [52]). *If  $M$  is obtained from  $M_1$  by attaching a subcritical Weinstein handle, the map (6.9) is an isomorphism.*

THEOREM 6.13 (Bourgeois-Ekholm-Eliashberg). *Consider four-dimensional Liouville domains, and use coefficients in a field with  $\text{char}(\mathbb{K}) = 0$ . Suppose that  $M$  is obtained from  $M_1$  by attaching a Weinstein handle along a stabilized Legendrian knot. Then (6.9) is an isomorphism.*

The last result may require some explanation. Stabilization is a local modification of a Legendrian knot, which depends on a choice of orientation [67, Section 2.7]. For Legendrian

knots in  $\mathbb{R}^3$ , one sees easily that the Chekanov homology [51] of a stabilized knot vanishes. When translated into pseudo-holomorphic curve technology, that argument depends on the existence of a low-energy once-punctured holomorphic disc, hence carries over to  $\partial M_1$ . Theorem 6.13 is then a consequence of [36, Theorem 5.6 and Section 7.2]. Strictly speaking, one has to be a little careful, since [36] uses a definition of symplectic homology (and Viterbo functoriality) which is based on Symplectic Field Theory, and therefore technically somewhat different from that in the original literature (this also explains the restriction on the coefficient field  $\mathbb{K}$ ). However, it is generally expected that the two are equivalent, by an extension of the arguments in [37]; we have tacitly assumed that when stating the result.

We have so far adopted a “black box” approach to symplectic cohomology, but it makes sense to give at least a partial idea of what geometric information it encodes. Let  $M$  be a Liouville domain. The form  $\omega_M|_{\partial M}$  has one-dimensional null space. Let’s assume that all closed characteristics, which means loops tangent to that nullspace, are nondegenerate. Equivalently, these characteristic are periodic orbits of the Reeb field, which is the unique vector field  $R_{\partial M} \in C^\infty(T(\partial M))$  in the null space such that  $\theta_M(R_{\partial M}) = 1$ . Fix a Morse function  $h \in C^\infty(M, \mathbb{R})$  such that  $h^{-1}(1) = 1$  and  $Z_M \cdot h = h$  near the boundary. Symplectic cohomology can be written as the cohomology of a  $\mathbb{Z}/2$ -graded chain complex which has generators of two kinds. First, each critical point of  $h$  gives rise to a generator, whose (mod 2) degree agrees with the Morse index. In fact, this gives rise to a subcomplex isomorphic to the usual Morse complex of  $h$ , hence to a homomorphism (of unital rings)

$$(6.10) \quad H^*(M; \mathbb{K}) \longrightarrow SH^*(M).$$

Secondly, each periodic orbit of  $R_{\partial M}$  (this includes iterates of a prime periodic orbit) gives rise to a pair of generators (of different mod 2 degrees). Of course, this is far from a complete description of symplectic cohomology, since we have not said anything about the differential, which involves pseudo-holomorphic curves. However, it is still fair to say that closed Reeb orbits are the primary ingredient (in fact, symplectic cohomology was originally introduced as a tool to study Reeb dynamics).

### Open symplectic mapping tori

We have discussed the symplectic mapping torus construction for closed symplectic manifolds in Lecture 5. There is a variant which is suitable for the Liouville case. Namely, given a Liouville manifold  $M$  and a Liouville automorphism  $\psi$ , with its compactly supported function  $k$  satisfying  $\psi^*\theta_M = \theta_M + dk$ , define

$$(6.11) \quad \begin{aligned} E &= \mathbb{R}^2 \times M / (p, q, x) \sim (p, q - 1, \psi(x)), \\ \theta_E &= p \, dq + \theta_M + d(qk(x)), \\ \omega_E &= d\theta_E = dp \wedge dq + \omega_M, \end{aligned}$$

which is again a Liouville manifold. It is a general fact that symplectic cohomology splits into a direct sum of pieces indexed by connected components of the free loop space. In



particular, one can write

$$(6.12) \quad SH^*(E) = \bigoplus_{k \in \mathbb{Z}} SH^*(E)^{(k)},$$

where  $k$  measures the degree of loops after projection  $E \rightarrow \mathbb{R} \times S^1$ . Then, there is a long exact sequence

$$(6.13) \quad \dots \rightarrow SH^*(E)^{(0)} \rightarrow SH^*(M) \xrightarrow{id - \psi^*} SH^*(M) \rightarrow \dots$$

This closely related to the results of [140], and can be thought of as the Floer-theoretic analogue of the classical (topological) long exact sequence

$$(6.14) \quad \dots \rightarrow H^*(E) \rightarrow H^*(M) \xrightarrow{id - \psi^*} H^*(M) \rightarrow \dots$$

LEMMA 6.14.  *$SH^*(E)$  vanishes if and only if  $SH^*(M)$  does.*

PROOF. If  $SH^*(M)$  vanishes, so does  $SH^*(E)^{(0)}$ . But for general reasons, the unit element of  $SH^*(E)$  must lie in that summand. In converse direction, if  $SH^*(M)$  is nonzero, then its unit element lies in the kernel of  $id - \psi^*$ , and hence  $SH^*(E)^{(0)} \neq 0$ .  $\square$

One does not expect  $\psi^* : SH^*(M) \rightarrow SH^*(M)$  to carry a lot of information about  $\psi$ . As a concrete example:

CONJECTURE 6.15. *Suppose that  $\dim(M)/2$  is even, and let  $\tau_L$  be the Dehn twist along a Lagrangian sphere. Then,  $\tau_L^2$  acts as the identity on  $SH^*(M)$ .*

Hence,  $SH^*(E)^{(0)}$  should mostly reflect the symplectic topology of  $M$ , rather than that of  $\psi$ . The other summands  $SH^*(E)^{(k)}$ ,  $k \neq 0$ , are much more interesting, and provide a rich tool for studying open symplectic mapping tori. This can be motivated by looking at the underlying Reeb dynamics, which turns out to be related to the dynamics (fixed points, periodic points) of  $\psi$ .

### Nonstandard Liouville structures

We continue our discussion of open mapping tori. For simplicity, let's suppose from now on that  $M$  is simply-connected. In that case, projection yields an isomorphism

$$(6.15) \quad \pi_1(E) \cong \pi_1(\mathbb{R} \times S^1) = \mathbb{Z}.$$

Think of  $E$  and  $M$  as completions of Liouville domains  $E^{trunc}$  and  $M^{trunc}$ . Let's suppose that  $\psi$  is compactly supported, and in fact comes from a symplectic automorphism of  $M^{trunc}$  which is the identity near the boundary. One can construct  $E^{trunc}$  so that it contains a piece  $[-1, 1] \times S^1 \times \partial M^{trunc}$ , with the product contact structure. Then, to a point  $x \in \partial M^{trunc}$  one can associate an isotropic loop  $\{0\} \times S^1 \times \{x\}$  in  $\partial E^{trunc}$ , which generates (6.15). Let's assume that  $\partial M^{trunc}$  is connected. Then, the isotopy class of that loop is independent of all choices. We attach a subcritical Weinstein handle to it, and then complete the result to a simply-connected Liouville domain  $N$ .

LEMMA 6.16. *If  $SH^*(M) \neq 0$ , then  $SH^*(N) \neq 0$ .*

PROOF. In dimensions  $> 4$ , Theorem 6.12 ensures that  $SH^*(N) \cong SH^*(E)$ . This fails in dimension 4, but Viterbo functoriality at least ensures that if  $SH^*(E) \neq 0$ , then also  $SH^*(N) \neq 0$ . Applying Lemma 6.14 completes the argument.  $\square$

EXAMPLE 6.17 (Adapted from [134]). *Suppose that  $n > 2$  is even. Consider the affine hypersurface*

$$(6.16) \quad M = \{x_1^2 + \cdots + x_{n-1}^2 + x_n^3 = 1\} \subset \mathbb{C}^n.$$

*This is the Milnor fibre of the  $(A_2)$  singularity. Its only nontrivial homology group is*

$$(6.17) \quad H_{n-1}(M) \cong \mathbb{Z}^2, \quad \text{with intersection pairing } \begin{pmatrix} 0 & -1 \\ 1 & 0. \end{pmatrix}$$

*In fact, there are two embedded Lagrangian spheres  $L_1, L_2 \subset M$  (vanishing cycles) which represent generators of (6.17), and whose intersection number is  $L_1 \cdot L_2 = 1$ . The existence of these has two consequences. First of all, each sphere gives rise to Viterbo restriction map  $SH^*(M) \rightarrow SH^*(T^*S^{n-1})$ , which shows that  $SH^*(M) \neq 0$ . Secondly, by using the Dehn twists along both spheres, one can every matrix in  $SL_2(\mathbb{Z})$ , which means every automorphism of (6.17), is represented by an automorphism  $\psi$  of  $M$ .*

$$(6.18) \quad \psi_* = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

*then  $id - \psi_*$  is invertible, which by (6.14) implies that  $E$  has the integer homology of  $S^1$ . After attaching a two-handle as explained above, we get a contractible Liouville manifold  $M$  of dimension  $2n$ . That manifold also carries an exhausting Morse function whose critical points all have index  $\leq n$ . It is therefore simply-connected at infinity, and the  $h$ -cobordism theorem applies, showing that it is diffeomorphic to  $\mathbb{R}^{2n}$ . On the other hand, the symplectic cohomology of the Milnor fibre  $M$  is nonzero (this follows from Viterbo functoriality, because  $M$  contains an exact Lagrangian sphere). Hence,  $SH^*(N) \neq 0$  by Lemma 6.16, which implies that  $N$  is not symplectically isomorphic to standard  $\mathbb{R}^{2n}$ .*

In [134], a version of this construction was used to show that there are infinitely many non-isomorphic Liouville manifolds which are diffeomorphic to  $\mathbb{R}^{2n}$ , for any  $n > 2$ . This still uses symplectic cohomology, but involves a closer analysis of the spaces  $SH^*(E)^{(k)}$ ,  $k \neq 0$ , which we will not describe here.

### Symplectic fibrations over surfaces

Suppose that we have a Liouville manifold  $M$ , together with compactly supported Liouville automorphisms  $\psi_1, \dots, \psi_g$ . Take a once-punctured genus  $g$  surface  $S$ , and recall that

$$(6.19) \quad \pi_1(S) = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \rangle$$

is a free group. Let  $E$  be the locally trivial symplectic fibration of  $S$  with fibre  $M$ , whose monodromy around the  $\alpha_i$  is  $\psi_i$ , and with trivial monodromy around the  $\beta_i$ . In particular,

the monodromy around the puncture is again trivial, since a small loop around it is conjugate to  $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$ . One can make  $S$ , and then as before also  $E$ , into a Liouville manifold. Note that the structure of  $E$  at infinity depends only on  $M$  and  $g$ , and not on the choice of the  $\psi_i$  (if one cuts  $E$  down to a Liouville domain  $E^{trunc}$ , then the same would hold for the contact structure on  $\partial E^{trunc}$ ). Finally,  $\pi_1(E)$  is large since (assuming  $M$  connected) it surjects onto  $\pi_1(S)$ , but one could kill the fundamental group by attaching two-dimensional Weinstein handles. As before, let's denote the result of this process by  $N$ .

One can write

$$(6.20) \quad SH^*(E) = SH^*(E)^{(0)} \oplus SH^*(E)^{(1)} \oplus \dots$$

where the (0) summand correspond to loops which project to contractible loops in  $S$ , and the ( $k$ ) summand for  $k > 0$  corresponds to loops which project to a loop going  $k$  times clockwise around the puncture (the other components of the free loop space contribute zero to symplectic cohomology). Then,

$$(6.21) \quad SH^*(E)^{(k)} \cong H^*(S^1; \mathbb{K}) \otimes SH^*(M) \quad \text{for } k > 0.$$

For the remaining summand in (6.20), there is an analogue of (6.13), namely a long exact sequence

$$(6.22) \quad \dots \rightarrow SH^*(E)^{(0)} \rightarrow SH^*(M) \xrightarrow{(id-\psi_1^*, \dots, id-\psi_g^*, 0, \dots, 0)} SH^*(M)^{\oplus 2g} \rightarrow \dots$$

Bearing in mind Conjecture 6.15 and related ideas, one finds that  $SH^*(E)$  only contains a limited amount of information about  $\psi_1, \dots, \psi_g$ . Of course, there are more sophisticated symplectic invariants (notably, ones coming from Symplectic Field Theory), but still, the study of such Liouville manifolds  $E$  (or  $N$ ) remains a challenging issue. In the case  $g = 1$ , one can compactify  $S$  to a torus, and correspondingly partially compactify  $E$ ; that brings one into a situation similar to that from Lecture 5, potentially allowing ideas of flux and non-Hamiltonian isotopies to come into the picture. However, this approach can't be applied to  $N$ .



Part 2

Background



LECTURE 7

## Homological algebra

Out of the many available frameworks for “chain level categories”, we choose to work with the version based on Stasheff’s  $A_\infty$ -algebras [189], mainly because of its natural occurrence in symplectic topology. The motivation for, and history of,  $A_\infty$ -categories is interesting in itself, but outside our scope. We will not even fully reproduce the definitions (there are many available references, among them [106, 120, 176, 108, 116, 28]), but only say enough to fix the notation and conventions. Among other things, we will mention the notion of smoothness for  $A_\infty$ -categories, since that has turned out to be important in the development of noncommutative algebraic geometry. As a reprieve from the steady stream of basic definitions, we briefly consider growth issues for iterated convolution functors.

### First notions

Fix a coefficient field  $\mathbb{K}$ . An  $A_\infty$ -category  $\mathcal{A}$  consists of a set (all our categories are small) of objects  $Ob(\mathcal{A})$ ; morphism spaces  $hom_{\mathcal{A}}(X, Y)$  which are  $\mathbb{Z}$ -graded vector spaces; and composition maps

$$(7.1) \quad \mu_{\mathcal{A}}^d : hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1) \longrightarrow hom_{\mathcal{A}}(X_0, X_d)[2-d] \quad \text{for } d \geq 1,$$

satisfying the  $A_\infty$ -associativity equations. Our sign conventions follow [176], so that the third of these equations (the one that most deservedly bears the name) can be written as

$$(7.2) \quad \begin{aligned} \mu_{\mathcal{A}}^2(a_3, \mu_{\mathcal{A}}^2(a_2, a_1)) - (-1)^{|a_1|} \mu_{\mathcal{A}}^2(\mu_{\mathcal{A}}^2(a_3, a_2), a_1) &= -\mu_{\mathcal{A}}^1(\mu_{\mathcal{A}}^3(a_3, a_2, a_1)) \\ - (-1)^{|a_1|+|a_2|} \mu_{\mathcal{A}}^3(\mu_{\mathcal{A}}^1(a_3), a_2, a_1) + (-1)^{|a_1|} \mu_{\mathcal{A}}^3(a_3, \mu_{\mathcal{A}}^1(a_2), a_1) &- \mu_{\mathcal{A}}^3(a_3, a_2, \mu_{\mathcal{A}}^1(a_1)). \end{aligned}$$

The associated cohomological category  $H(\mathcal{A})$  has the same objects; it has morphisms

$$(7.3) \quad Hom_{H(\mathcal{A})}(X, Y) = H^*(hom_{\mathcal{A}}(X, Y), \mu_{\mathcal{A}}^1);$$

and compositions induced by  $\mu_{\mathcal{A}}^2$ , more precisely

$$(7.4) \quad [a_2] \cdot [a_1] = (-1)^{|a_1|} [\mu_{\mathcal{A}}^2(a_2, a_1)].$$

We always require that  $H(\mathcal{A})$  should have identity morphisms (equivalently, one says that  $\mathcal{A}$  is cohomologically unital), hence is a linear graded category in the classical sense of the word. One can also use the subcategory  $H^0(\mathcal{A})$  which only contains degree 0 morphisms. We say that two objects  $X, Y$  of  $\mathcal{A}$  are quasi-isomorphic if they become isomorphic in  $H^0(\mathcal{A})$ . A little more generally:

DEFINITION 7.1. We say that  $X$  is a homotopy retract of  $Y$  if it becomes a retract in  $H^0(\mathcal{A})$ . This means that there are closed degree 0 morphisms  $\rho \in \text{hom}_{\mathcal{A}}(Y, X)$  and  $\iota \in \text{hom}_{\mathcal{A}}(X, Y)$ , such that  $\mu_{\mathcal{A}}^2(\rho, \iota)$  is homologous to the identity of  $X$  (in more algebraic language, one could say “homotopy direct summand” instead of “homotopy retract”).

DEFINITION 7.2. We say that  $X$  is dependent on  $Y$  if the composition

$$(7.5) \quad H^0(\text{hom}_{\mathcal{A}}(Y, X)) \otimes H^0(\text{hom}_{\mathcal{A}}(X, Y)) \longrightarrow H^0(\text{hom}_{\mathcal{A}}(X, X))$$

is onto, or equivalently contains the identity in its image. If  $\mathcal{A}$  admits finite direct sums, this means that  $X$  is a homotopy retract of a finite direct sum of copies of  $Y$ .

The notion of  $A_{\infty}$ -category is complemented by that of  $A_{\infty}$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ . Such a functor consists of the action on objects, together with maps

$$(7.6) \quad \mathcal{F}^d : \text{hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \text{hom}_{\mathcal{B}}(\mathcal{F}(X_0), \mathcal{F}(X_d))[1-d] \quad \text{for } d \geq 1,$$

satisfying an appropriate homomorphism equation (a polynomial condition on the  $\mathcal{F}^d$ ). The leading order term  $\mathcal{F}^1$  induces a graded linear functor  $H(\mathcal{F}) : H(\mathcal{A}) \rightarrow H(\mathcal{B})$ . One uses that to define quasi-isomorphism and quasi-equivalence for  $A_{\infty}$ -categories, in the obvious way.  $A_{\infty}$ -functors from  $\mathcal{A}$  to  $\mathcal{B}$  themselves form an  $A_{\infty}$ -category  $\text{fun}(\mathcal{A}, \mathcal{B})$ . A significant fact is the following:

LEMMA 7.3. Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  be a quasi-equivalence. Then there is a quasi-equivalence  $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$  such that  $\mathcal{G} \circ \mathcal{F}$  is quasi-isomorphic to the identity in  $\text{fun}(\mathcal{A}, \mathcal{A})$ , and  $\mathcal{F} \circ \mathcal{G}$  is quasi-isomorphic to the identity in  $\text{fun}(\mathcal{B}, \mathcal{B})$ .  $\square$

Some comments and variations on the basic definitions:

- An  $A_{\infty}$ -algebra is the same as an  $A_{\infty}$ -category with one object (and  $A_{\infty}$ -functors then specialize to  $A_{\infty}$ -homomorphisms).

Here is a slight extension, which can sometimes be useful. For any  $m \geq 1$ , consider the semisimple ring  $R = \mathbb{K}^m = \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_m$  (with  $e_i^2 = e_i$ ,  $e_i e_j = 0$  for  $i \neq j$ ). Then, an  $A_{\infty}$ -algebra over  $R$  is the same as an  $A_{\infty}$ -category with  $m$  ordered objects  $(X_1, \dots, X_m)$ . Explicitly, if  $\mathcal{A}$  is an  $A_{\infty}$ -algebra over  $R$ , one turns it into an  $A_{\infty}$ -category by taking  $e_j \mathcal{A} e_i$  to be the space of morphisms  $X_i \rightarrow X_j$ .

- $A_{\infty}$ -categories make sense with  $\mathbb{Z}/N$ -gradings for any even  $N$  (and even for odd  $N$  if  $\text{char}(\mathbb{K}) = 2$ ). Nevertheless, the  $\mathbb{Z}$ -graded version remains our default choice, since it relates more readily to established intuition from homological algebra.

- Any dg (differential graded) category  $\mathcal{A}$  becomes an  $A_{\infty}$ -category by setting

$$(7.7) \quad \mu_{\mathcal{A}}^1(a) = (-1)^{|a|} da, \quad \mu_{\mathcal{A}}^2(a_2, a_1) = (-1)^{|a_1|} a_2 a_1, \quad \mu_{\mathcal{A}}^d = 0 \text{ for } d \geq 3.$$

We will carry out this conversion tacitly, and just consider dg categories as a special case of  $A_{\infty}$ -categories. A frequently example is the dg category  $Ch$  of chain complexes of  $\mathbb{K}$ -vector spaces.



- In converse direction, any  $A_\infty$ -category is quasi-isomorphic to a dg category. One place where this is convenient is when defining the tensor product  $\mathcal{A} \otimes \mathcal{B}$  of  $A_\infty$ -categories. It is possible to approach this directly, but that requires some choice of diagonals, leading to explicit but lengthy combinatorial formulae [165]. However, the situation is straightforward if one of the two factors is a dg category, and one can always reduce to that up to quasi-isomorphism.
- For functors, any dg functor between dg categories gives rise to an  $A_\infty$ -functor. In converse direction, given any  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  between dg categories, one can construct a diagram

$$(7.8) \quad \begin{array}{ccc} & \tilde{\mathcal{B}} & \\ \tilde{\mathcal{F}} \nearrow & & \searrow \mathcal{G} \\ \mathcal{A} & \xrightarrow{\mathcal{F}} & \mathcal{B} \end{array}$$

$\simeq$

which commutes up to quasi-isomorphism in  $\text{fun}(\mathcal{A}, \tilde{\mathcal{B}})$ ; and where  $\tilde{\mathcal{B}}$  is a dg category,  $\tilde{\mathcal{F}}, \mathcal{G}$  are dg functors, and  $\mathcal{G}$  is a quasi-equivalence. Hence formally  $\mathcal{F}$  is the “fraction”  $\mathcal{G}^{-1}\tilde{\mathcal{F}}$ . There is a similar statement with a “right fraction” instead of a left one.

- One says that  $\mathcal{A}$  is *strictly unital* if every object  $X$  has an endomorphism  $e_X \in \text{hom}_{\mathcal{A}}^0(X, X)$  such that

$$(7.9) \quad \begin{aligned} \mu_{\mathcal{A}}^1(e_X) &= 0, \quad \mu_{\mathcal{A}}^2(a, e_X) = a, \quad \mu_{\mathcal{A}}^2(e_X, a) = (-1)^{|a|}a, \\ \mu_{\mathcal{A}}^d(a_d, \dots, a_1) &= 0 \text{ for any } d > 2, \text{ if } a_i = e_X \text{ for some } i. \end{aligned}$$

Any  $A_\infty$ -category is quasi-isomorphic to a strictly unital one, and sometimes strict unitality can simplify the formulation of certain constructions. If  $\mathcal{A}, \mathcal{B}$  are strictly unital, one can consider strictly unital  $A_\infty$ -functors between them. Such functors, and the corresponding notion of strictly unital  $A_\infty$ -transformations, form an  $A_\infty$ -subcategory which is quasi-equivalent to the whole of  $\text{fun}(\mathcal{A}, \mathcal{B})$ .

- $\mathcal{A}$  is called *minimal* if  $\mu_{\mathcal{A}}^1 = 0$ . Any  $A_\infty$ -category is quasi-isomorphic to a minimal one, by the Perturbation Lemma.
- We say that  $\mathcal{A}$  is *proper* if each space (7.3) is of finite total dimension.
- Assume that  $\text{char}(\mathbb{K}) = 0$ . A *strictly cyclic*  $A_\infty$ -category of dimension  $n \in \mathbb{Z}$  is a strictly proper  $A_\infty$ -category together with nondegenerate pairings

$$(7.10) \quad \begin{aligned} \langle \cdot, \cdot \rangle &: \text{hom}_{\mathcal{A}}^*(Y, X) \otimes \text{hom}_{\mathcal{A}}^{n-*}(X, Y) \longrightarrow \mathbb{K}, \\ \langle a_1, a_2 \rangle &= (-1)^{(|a_2|-1)(|a_1|-1)+1} \langle a_2, a_1 \rangle, \end{aligned}$$

such that the expressions  $\xi^{d+1}(a_{d+1}, \dots, a_1) = \langle a_{d+1}, \mu_{\mathcal{A}}^d(a_d, \dots, a_1) \rangle$  are cyclically symmetric with appropriate signs:

$$(7.11) \quad \xi^{d+1}(a_d, \dots, a_1, a_{d+1}) = (-1)^{(|a_{d+1}|-1)(|a_1|+\dots+|a_d|-d)} \xi^{d+1}(a_{d+1}, \dots, a_1).$$

A better notion of cyclicity, which requires only properness, was introduced in [116, Section 10]. Every  $A_\infty$ -category which is cyclic in the sense defined there is quasi-equivalent to a strictly cyclic one.

### Twisted complexes

Any  $A_\infty$ -category  $\mathcal{A}$  has a canonical enlargement  $\mathcal{A}^{tw}$ , the category of twisted complexes, which allows shifts and mapping cones. It is best to introduce this in two steps. First, consider the additive enlargement  $\mathcal{A}^\oplus$ , whose objects are formal sums

$$(7.12) \quad C = \bigoplus_{i \in I} W_i \otimes X_i$$

over some finite set  $I$ , with finite-dimensional graded vector spaces  $W_i$ , and objects  $X_i$  of  $\mathcal{A}$ . The morphisms are matrices whose entries combine linear maps of vector spaces and morphisms of the constituent objects in  $\mathcal{A}$ . The  $A_\infty$ -structure is similarly inherited from  $\mathcal{A}$ .

Now, a twisted complex is a pair  $(C, \delta_C)$ , where  $C$  is as in (7.12) and the differential is an element  $\delta_C \in \text{hom}_{\mathcal{A}^\oplus}^1(C, C)$ , which explicitly means

$$(7.13) \quad \delta_C = (\delta_{C,ij}), \quad \delta_{C,ij} \in \left( \text{Hom}(W_j, W_i) \otimes \text{hom}_{\mathcal{A}}(X_j, X_i) \right)^1.$$

There are two additional conditions. First,  $C$  must admit a finite filtration by subobjects in  $\mathcal{A}^\oplus$ , with respect to which  $\delta_C$  is strictly lower-triangular. Secondly,  $\delta_C$  must satisfy a Maurer-Cartan type equation. Morphism spaces in  $\mathcal{A}^{tw}$  are the same as in  $\mathcal{A}^\oplus$ , but the  $A_\infty$ -structure comes with additional insertions of the differential.

EXAMPLE 7.4. *Given  $a \in \text{hom}_{\mathcal{A}}^0(X_0, X_1)$  which is closed,  $\mu_{\mathcal{A}}^1(a) = 0$ , one can define its mapping cone*

$$(7.14) \quad C = X_0[1] \oplus X_1, \quad \delta_C = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},$$

*which is a twisted complex. Here, the shift operation  $X_0[1]$  means  $\mathbb{K}[1] \otimes X_0$  (where  $\mathbb{K}[1]$  is the one-dimensional vector space placed in degree  $-1$ ).*

Using a more general construction of mapping cones, one equips the category  $H^0(\mathcal{A}^{tw})$  with a triangulated structure in the classical sense. We denote this category by  $D^{tw}(\mathcal{A})$ . It is one of the versions of the “derived category of an  $A_\infty$ -category” (the word derived is probably inappropriate, since no localisation takes place, but by now well-established in the literature).

### Modules

A (right)  $A_\infty$ -module  $M$  is a collection of graded vector spaces  $M(X)$ ,  $X \in \text{Ob}(\mathcal{A})$ , together with operations

$$(7.15) \quad \mu_M^{1;d} : M(X_d) \otimes \text{hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow M(X_0)[1-d], \quad d \geq 0,$$

satisfying a suitable associativity equation. We always assume that  $M$  is cohomologically unital, hence induces a module  $H(M)$  over  $H(\mathcal{A})$ . Such  $A_\infty$ -modules form a dg category  $\mathcal{A}^{mod}$ . In comparison with classical module categories, the bar resolution is built into the definition of morphisms in  $\mathcal{A}^{mod}$ , which in particular leads to the following statement (showing that the two possible definition of “quasi-isomorphism of  $A_\infty$ -modules” are equivalent):

LEMMA 7.5. *If a closed morphism  $M \rightarrow N$  induces an isomorphism  $H(M) \rightarrow H(N)$ , it is an isomorphism in  $H^0(\mathcal{A}^{mod})$ .*  $\square$

Due to the presence of mapping cones, the category  $H^0(\mathcal{A}^{mod})$  is again triangulated; we denote it by  $D^{mod}(\mathcal{A})$ . Additionally, this category is idempotent closed (also called Karoubi complete). This means that, given an  $A_\infty$ -module  $M$  and an idempotent endomorphism  $[\pi] \in H^0(\text{hom}_{\mathcal{A}^{mod}}(M, M))$ , there is another  $A_\infty$ -module which is the corresponding homotopy retract. There is a cohomologically full and faithful  $A_\infty$ -functor, the Yoneda embedding

$$(7.16) \quad \mathcal{A} \longrightarrow \mathcal{A}^{mod}.$$

Concretely, this maps an object  $Y$  to the module  $M = Y^{yon}$  with  $M(X) = \text{hom}_{\mathcal{A}}(X, Y)$ , and with the  $A_\infty$ -module structure induced from the  $A_\infty$ -structure of  $\mathcal{A}$ . One can extend (7.16) to twisted complexes, either by writing down the resulting modules explicitly, or more conceptually through the following diagram:

$$(7.17) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{inclusion}} & \mathcal{A}^{tw} \\ \text{Yoneda} \downarrow & \swarrow \text{---} & \downarrow \text{Yoneda} \\ \mathcal{A}^{mod} & \xleftarrow{\text{restriction}} & (\mathcal{A}^{tw})^{mod} \end{array}$$

Some additional remarks:

- There is a notion of opposite  $A_\infty$ -category  $\mathcal{A}^{opp}$ , and in these terms  $A_\infty$ -modules can be defined as functors into chain complexes:

$$(7.18) \quad \mathcal{A}^{mod} \cong \text{fun}(\mathcal{A}^{opp}, Ch) \cong \text{fun}(\mathcal{A}, Ch^{opp}).$$

Similarly, left  $A_\infty$ -modules (which we had not mentioned so far, but which of course are perfectly sensible) would be functors from  $\mathcal{A}$  itself to  $Ch$ .

- If  $\mathcal{A}$  is a dg category, every dg module is an  $A_\infty$ -module. In general, the spaces of dg module morphisms differ from those of  $A_\infty$ -module homomorphisms, since the latter come “already derived”. More concretely, if  $M$  is  $K$ -projective in the sense of [27] then the dg and  $A_\infty$ -versions of  $\text{hom}(M, -)$  are quasi-isomorphic. In converse direction, every  $A_\infty$ -module is quasi-isomorphic to a dg module.
- If  $\mathcal{A}$  is strictly unital, one can define the notion of strictly unital  $A_\infty$ -module, and strictly unital homomorphisms between such modules. This yields a dg subcategory which is quasi-equivalent to the whole of  $\mathcal{A}^{mod}$ .

- If  $\mathcal{A}$  is minimal, one can consider  $A_\infty$ -modules  $M$  which are themselves minimal, meaning that  $\mu_M^1 = 0$ . Every  $A_\infty$ -module is quasi-isomorphic to a minimal one.
- An  $A_\infty$ -module  $M$  is called proper if the cohomology groups  $H(M(X))$  are of finite total dimension for all  $X$ . Such modules form a full subcategory  $\mathcal{A}^{prop} \subset \mathcal{A}^{mod}$ .
- An  $A_\infty$ -module is called *perfect* if it is a homotopy retract of the Yoneda image of a twisted complex. Perfect modules form a full subcategory  $\mathcal{A}^{perf}$ . By definition, every perfect complex can be constructed by starting from objects of  $\mathcal{A}$  (or rather, their Yoneda images) and taking shifts, mapping cones, and homotopy retracts. In particular, if  $\mathcal{A}$  is proper then so is  $\mathcal{A}^{perf}$ . We write  $D^{perf}(\mathcal{A}) = H^0(\mathcal{A}^{perf})$ .

LEMMA 7.6. *If  $M$  is perfect and  $N$  is proper, then  $H^*(\text{hom}_{\mathcal{A}}(M, N))$  is of finite (total) dimension.*

PROOF. Suppose first that  $M = X^{yon}$  is the Yoneda image of some  $X \in \text{Ob}(\mathcal{A})$ . Then there is a canonical quasi-isomorphism

$$(7.19) \quad N(X) \xrightarrow{\cong} \text{hom}_{\mathcal{A}^{mod}}(M, N).$$

The general case is derived from that by going through the construction steps mentioned above (shifts, mapping cones, and homotopy retracts).  $\square$

EXAMPLE 7.7. *Take a trivial example, the  $A_\infty$ -category  $\mathbb{K}$  (with a single object, and only multiples of the identity as endomorphisms). Then, the cohomologically full and faithful embeddings*

$$(7.20) \quad \mathbb{K}^{tw} \longrightarrow \mathbb{K}^{perf} \longrightarrow \mathbb{K}^{prop}$$

*are all equivalences. Indeed, any finite-dimensional chain complex of vector spaces can be made into an object of  $\mathbb{K}^{tw}$ ; on the other hand, a strictly unital proper  $A_\infty$ -module over  $\mathbb{K}$  is the same as a chain complex of vector space with finite-dimensional (total) cohomology; and the two resulting dg categories of chain complexes are themselves quasi-equivalent. Similarly, the strictly unital version of  $\mathbb{K}^{mod}$  is the dg category of (arbitrary) chain complexes of vector spaces.*

## Bimodules

Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $A_\infty$ -categories. An  $(\mathcal{A}, \mathcal{B})$ -bimodule  $Q$  consists of graded vector spaces  $Q(X, Y)$  for all  $(X, Y) \in \text{Ob}(\mathcal{B}) \times \text{Ob}(\mathcal{A})$ , together with operations

$$(7.21) \quad \begin{aligned} \mu_Q^{s;1;r} : \text{hom}_{\mathcal{A}}(Y_{s-1}, Y_s) \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(Y_0, Y_1) \otimes Q(X_r, Y_0) \\ \otimes \text{hom}_{\mathcal{B}}(X_{r-1}, X_r) \otimes \cdots \otimes \text{hom}_{\mathcal{B}}(X_0, X_1) \longrightarrow Q(X_0, Y_s)[1-r-s], \quad r, s \geq 0 \end{aligned}$$

satisfying an appropriate bimodule equation, and as before, a unitality condition on the cohomology level.  $A_\infty$ -bimodules form a dg category  $(\mathcal{A}, \mathcal{B})^{mod}$ .

Arguably the most important use of bimodules is through their action on modules. Any bimodule  $Q$  gives rise to an  $A_\infty$ -functor (sometimes called “convolution”)

$$(7.22) \quad \Phi_Q : \mathcal{A}^{mod} \longrightarrow \mathcal{B}^{mod}, \quad \Phi_Q(M) = M \otimes_{\mathcal{A}} Q.$$

Of course, bimodules themselves admit a tensor product, which corresponds to the composition of convolution functors. We refer to [200, 174] for more details.

Some examples and comments:

- $\mathcal{A}$  itself is an  $(\mathcal{A}, \mathcal{A})$ -bimodule (called the “diagonal bimodule”). More precisely, the structure maps of the diagonal bimodule are related to the  $A_\infty$ -category structure of  $\mathcal{A}$  by

$$(7.23) \quad \mu_{\mathcal{A}}^{s;1;r}(a''_s, \dots, a''_1; a; a'_r, \dots, a'_1) = (-1)^{|a'_1| + \dots + |a'_r| - r + 1} \mu_{\mathcal{A}}^{s+1+r}(a''_s, \dots, a, a'_r, \dots, a'_1)$$

Tensor product with  $\mathcal{A}$  yields a functor which is quasi-isomorphic to the identity, but not strictly equal to it (one of the less fortunate aspects of the definition of tensor product).

- To any  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  one can associate a  $(\mathcal{A}, \mathcal{B})$ -bimodule  $Q = \text{Graph}(\mathcal{F})$ , called the graph of  $\mathcal{F}$ . One can view this as the result of pulling back the diagonal bimodule of  $\mathcal{B}$  by  $\mathcal{F}$  on the left hand side only. Explicitly,

$$(7.24) \quad \begin{aligned} Q(X, Y) &= \text{hom}_{\mathcal{B}}(X, \mathcal{F}(Y)), \\ \mu_Q^{s;1;r}(a_s, \dots, a_1; q; b_r, \dots, b_1) &= \sum (-1)^{|b_1| + \dots + |b_r| + r - 1} \mu_{\mathcal{B}}^{r+1+k}(\mathcal{F}^{i_k}(a_s, \dots, a_{s-i_k+1}), \\ &\quad \dots, \mathcal{F}^{i_1}(a_{i_1}, \dots, a_1), q, b_r, \dots, b_1), \end{aligned}$$

where the sum is over all partitions  $i_1 + \dots + i_k = s$ . Convolution with  $Q$  fits into a diagram (commutative up to quasi-isomorphism of  $A_\infty$ -functors)

$$(7.25) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{F}} & \mathcal{B} \\ \text{Yoneda} \downarrow & & \downarrow \text{Yoneda} \\ \mathcal{A}^{mod} & \xrightarrow{\Phi_Q} & \mathcal{B}^{mod}. \end{array}$$

This shows how bimodules can be thought of as generalization of functors.

- If  $R$  is a  $\mathcal{B}$ -module, and  $L$  a left  $\mathcal{A}$ -module, one can define an  $(\mathcal{A}, \mathcal{B})$ -bimodule  $Q = L \otimes R$  by

$$(7.26) \quad Q(X, Y) = L(Y) \otimes R(X)$$

with operations  $\mu_Q^{s;1;0}$  and  $\mu_Q^{0;1;r}$  inherited from  $L$  and  $R$ , respectively, and all other operations set to zero. The associated convolution functor takes any module  $M$  to  $(M \otimes_{\mathcal{A}} L) \otimes R$ , where  $M \otimes_{\mathcal{A}} L$  is just a chain complex.

- If  $\mathcal{A}$  and  $\mathcal{B}$  are strictly unital, one can define a dg category of strictly unital  $(\mathcal{A}, \mathcal{B})$ -bimodules, which as usual turns out to be quasi-equivalent to  $(\mathcal{A}, \mathcal{B})^{mod}$ .
- Properness for bimodules is defined exactly as for modules.
- If  $Q$  is a bimodule, then for any fixed object  $Y$  it yields a module  $Q(-, Y)$ . Equivalently up to quasi-isomorphism, this module can be thought of as the tensor product

$$(7.27) \quad Q(-, Y) \simeq Y^{yon} \otimes_{\mathcal{A}} Q.$$

We say that  $Q$  is *right perfect* if  $Q(-, Y)$  is a perfect module for each  $Y$ . For such bimodules,  $\Phi_Q$  maps  $\mathcal{A}^{perf}$  to  $\mathcal{B}^{perf}$ , which is easy to see from (7.27). All graph bimodules are right perfect.

- Even though we have not defined them as such,  $(\mathcal{A}, \mathcal{B})$ -bimodules are the same, up to quasi-equivalence of dg categories, as modules over  $\mathcal{A}^{opp} \otimes \mathcal{B}$ . In particular, there is a Yoneda embedding

$$(7.28) \quad \mathcal{A}^{opp} \otimes \mathcal{B} \longrightarrow (\mathcal{A}, \mathcal{B})^{opp}.$$

Up to quasi-isomorphism, this maps a pair  $(X, Y)$  of objects to the tensor product of Yoneda modules  $X^{yon} \otimes Y^{yon}$ , in the sense of (7.26) (here  $X^{yon}$  is the right Yoneda module over  $\mathcal{A}^{opp}$ , which means the left Yoneda module for  $\mathcal{A}$ ).

One can extend (7.28) to  $(\mathcal{A}^{opp} \otimes \mathcal{B})^{tw}$ , and that is used to define the notion of *perfect bimodule*, just as in the module case. If  $Q$  is perfect, then  $\Phi_Q$  maps proper modules to perfect modules.

EXAMPLE 7.8. *The dual diagonal bimodule  $\mathcal{A}^\vee$  is the bimodule with*

$$(7.29) \quad \mathcal{A}^\vee(X, Y) = \text{hom}_{\mathcal{A}}(Y, X)^\vee,$$

*and with bimodule operations obtained (roughly speaking) by partially dualizing  $\mu_{\mathcal{A}}^{r+1+s}$ . Suppose that  $\mathcal{A}$  is proper. Then*

$$(7.30) \quad \Phi_{\mathcal{A}^\vee} : \mathcal{A}^{perf} \longrightarrow \mathcal{A}^{prop}$$

*is a cochain level implementation of the Serre functor for perfect modules. This means that if  $P$  and  $Q$  are perfect modules, then*

$$(7.31) \quad H^*(\text{hom}_{\mathcal{A}^{mod}}(P, \Phi_{\mathcal{A}^\vee}(Q))) \cong H^*(\text{hom}_{\mathcal{A}^{mod}}(Q, P))^\vee$$

*functorially in both  $P$  and  $Q$ . For instance, if  $\mathcal{A}$  is strictly cyclic of dimension  $n$ , in the sense introduced above, then*

$$(7.32) \quad \mathcal{A}^\vee \cong \mathcal{A}[n],$$

*which means that the Serre functor is quasi-isomorphic to the  $n$ -fold downwards shift. In fact, one often uses the existence of a quasi-isomorphism (7.32) as a weak replacement for cyclicity. Let's call  $A_\infty$ -categories for which such a quasi-isomorphism exists weakly cyclic*

of dimension  $n$  (we are trying to avoid the word “Calabi-Yau” since it has acquired several different connotations).

### Smoothness

The last-mentioned class of bimodules leads to a nontrivial property of  $A_\infty$ -categories:

DEFINITION 7.9.  $\mathcal{A}$  is smooth if the diagonal bimodule is perfect.

If that is the case, the functor  $\Phi_{\mathcal{A}}$ , which is quasi-isomorphic to the identity, maps proper modules to perfect ones. Hence, over a smooth  $A_\infty$ -category, every proper module is perfect. As a consequence of that and Lemma 7.6, the space of (cohomology level) morphisms between two proper morphisms is finite-dimensional.

EXAMPLE 7.10. Consider the coordinate ring of an affine algebraic variety over  $\mathbb{C}$ . This is an algebra, hence a fortiori an  $A_\infty$ -algebra. It is smooth if and only if the variety is smooth in the ordinary geometric sense (this follows from the classical homological criterion for regularity of local rings).

EXAMPLE 7.11. Take an  $A_\infty$ -category  $\mathcal{A}$  with a finite ordered set of objects, say  $(X_1, \dots, X_m)$ . We say that  $\mathcal{A}$  is directed if it is strictly unital and

$$(7.33) \quad \text{hom}_{\mathcal{A}}(X_i, X_j) = \begin{cases} 0 & i > j, \\ \mathbb{K} e_{X_i} & i = j, \\ \text{finite-dimensional} & i < j. \end{cases}$$

Directed  $A_\infty$ -categories are proper by definition, and they are also always smooth. In fact, the diagonal bimodule is a twisted complex over  $\mathcal{A}^{opp} \otimes \mathcal{A}$ , in a canonical way (the prototype for this was Beilinson’s resolution of diagonals for coherent sheaves on projective space). As a consequence,  $\mathcal{A}^{tw}$ ,  $\mathcal{A}^{perf}$ ,  $\mathcal{A}^{prop}$  are all quasi-equivalent.

EXAMPLE 7.12. Take an  $A_\infty$ -category in which the only morphisms are multiples of the identity endomorphisms of the objects. If the set of objects is finite, this is a trivial kind of directed  $A_\infty$ -category. However, if that set is infinite, the category is not smooth.

$A_\infty$ -categories which are proper and smooth have received considerable attention, as a possible fruitful framework for noncommutative algebraic geometry. There are sophisticated results about modules over such a category, as well as the “moduli space” of such categories itself [197, 196]; and there are even deeper issues, which aim at Hodge theory and a theory of motives in the noncommutative context [98].

### Iterated convolution

Let  $\mathcal{A}$  be an  $A_\infty$ -algebra, and  $P$  a perfect  $\mathcal{A}$ -module. By definition,  $P$  can be built (up to quasi-isomorphism) by starting with a finite number of shifted copies of the free module  $\mathcal{A}$ ,

then combining and modifying these copies by mapping cones and homotopy retracts. Define the *size*  $\|P\|$  to be the minimal number of initial copies of  $\mathcal{A}$  needed for this construction. For instance,  $\|P\| = 0$  iff  $P$  is quasi-isomorphic to zero;  $\|P\| \leq 1$  if and only if  $P$  is a homotopy retract of  $\mathcal{A}[k]$  for some  $k \in \mathbb{Z}$ ; and  $\|P\| \leq 2$  if and only if  $P$  is a homotopy retract of the mapping cone of a map  $\mathcal{A}[k] \rightarrow \mathcal{A}[l]$ . We then have the following easy quantitative refinement of Lemma 7.6:

LEMMA 7.13. *Let  $P$  be a perfect module, and  $M$  a proper module. Then*

$$(7.34) \quad \dim H^*(\text{hom}_{\mathcal{A}\text{-mod}}(P, M)) \leq \|P\| \cdot \dim H^*(M).$$

□

LEMMA 7.14. *Let  $P$  be a perfect module, and  $Q$  a bimodule which is right perfect. Then*

$$(7.35) \quad \|\Phi_Q(P)\| \leq \|P\| \cdot \|Q\|.$$

PROOF. Since  $P$  can be built starting from  $\|P\|$  copies of  $\mathcal{A}$ ,  $\Phi_Q(P) = P \otimes_{\mathcal{A}} Q$  can be built starting from  $\|P\|$  copies of  $\Phi_Q(\mathcal{A}) \simeq Q$ . But each of these copies of  $Q$  can be built from  $\|Q\|$  copies of  $\mathcal{A}$ . □

One can apply this to the iterated tensor product  $Q^{\otimes_{\mathcal{A}} k} = Q \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} Q$ , and see that  $\|Q^{\otimes_{\mathcal{A}} k}\| \leq \|Q\|^k$  for all  $k \geq 0$ . Combining this with Lemma 7.13 yields

$$(7.36) \quad \dim H^*(\text{hom}_{\mathcal{A}\text{-mod}}(P \otimes_{\mathcal{A}} Q^{\otimes_{\mathcal{A}} k}, M)) \leq \|P\| \cdot \|Q\|^k \cdot \dim H^*(M).$$

Suppose that  $\mathcal{A}$  is proper. Then one clearly has  $\dim H^*(Q) \leq \|Q\| \cdot \dim H^*(\mathcal{A})$ , hence the dimension of  $H^*(Q^{\otimes_{\mathcal{A}} k})$  grows at most exponentially in  $k$ . Moreover, if  $\mathcal{A}$  is proper and smooth, there is a constant  $C = C(\mathcal{A})$  such that

$$(7.37) \quad \dim H^*(\text{hom}_{(\mathcal{A}, \mathcal{A})\text{-mod}}(\mathcal{A}, Q^{\otimes_{\mathcal{A}} k})) \leq C \|Q\|^k.$$

## Quotients

Let  $\mathcal{A}$  be an  $A_\infty$ -category and  $\mathcal{B} \subset \mathcal{A}$  a full subcategory. By a *quotient  $A_\infty$ -category* we mean an  $A_\infty$ -category  $\mathcal{C}$  together with an  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C}$  such that:

- Any object of  $\mathcal{C}$  is quasi-isomorphic to one in the image of  $\mathcal{F}$  (essential surjectivity).
- The image of any object of  $\mathcal{B}$  under  $\mathcal{F}$  is quasi-isomorphic to zero (passage to  $\mathcal{C}$  kills  $\mathcal{B}$ ).
- Given any other  $A_\infty$ -category  $\mathcal{D}$ , composition with  $\mathcal{F}$  yields an  $A_\infty$ -functor

$$(7.38) \quad \cdot \circ \mathcal{F} : \text{fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \text{fun}(\mathcal{A}, \mathcal{D})$$

which is cohomologically full and faithful, and whose image up to quasi-isomorphism is the subcategory of functors  $\mathcal{A} \rightarrow \mathcal{D}$  which kill  $\mathcal{B}$ .



It is easy to see that the pair  $(\mathcal{C}, \mathcal{F})$  is essentially unique, so we will usually just write  $\mathcal{C} = \mathcal{A}/\mathcal{B}$ . The existence of quotients was first established by Keller [105]; alternative constructions were given in [62, 126].

EXAMPLE 7.15 (Taken from [62]). *Suppose that  $\mathcal{A}$  is a dg category. Define  $\mathcal{C}$  as a dg category with  $Ob(\mathcal{C}) = Ob(\mathcal{A})$ , and*

$$(7.39) \quad \begin{aligned} & hom_{\mathcal{C}}(X, Y) = hom_{\mathcal{A}}(X, Y) \\ & \oplus \bigoplus_{Z_0 \in Ob(\mathcal{B})} hom_{\mathcal{A}}(Z_0, Y) \otimes hom_{\mathcal{A}}(X, Z_0)[1] \\ & \oplus \bigoplus_{Z_0, Z_1 \in Ob(\mathcal{B})} hom_{\mathcal{A}}(Z_1, Y) \otimes hom_{\mathcal{B}}(Z_0, Z_1) \otimes hom_{\mathcal{A}}(X, Z_0)[2] \\ & \oplus \cdots \end{aligned}$$

The differential combines the given ones on each  $hom_{\mathcal{A}}$  space with composition (as in the classical bar construction). In particular, if  $X \in Ob(\mathcal{B})$ , then the identity endomorphism of  $X$  becomes nullhomologous in  $\mathcal{C}$ , because it lies in the image of the composition  $hom_{\mathcal{B}}(X, X) \otimes hom_{\mathcal{B}}(X, X) \rightarrow hom_{\mathcal{B}}(X, X)$ .

EXAMPLE 7.16. *For any proper  $A_{\infty}$ -category, one has  $\mathcal{A}^{perf} \subset \mathcal{A}^{prop}$ . Define the singular derived category of  $\mathcal{A}$  to be the cohomological category associated to  $\mathcal{A}^{prop}/\mathcal{A}^{perf}$ . This is a more abstract version of the basic definitions from [42, 147].*

We want to consider the computation of morphism spaces in quotient categories. Start with (7.39), which can be written more succinctly as

$$(7.40) \quad hom_{\mathcal{A}/\mathcal{B}}(X, Y) \simeq Cone(Y^{yon} \otimes_{\mathcal{B}} X^{yon} \longrightarrow hom_{\mathcal{A}}(X, Y)).$$

Here  $Y^{yon}$  is the Yoneda module which we then restrict to  $\mathcal{B}$ ;  $X^{yon}$  is the same thing with left modules; and the tensor product is that of  $A_{\infty}$ -modules over  $\mathcal{B}$  (while the discussion in Example 7.15 was for dg categories, the general case follows by quasi-isomorphism). In parallel with the use of resolutions in classical homological algebra, there are cases when (7.40) simplifies further. To formulate the problem in general, take  $Y \in Ob(\mathcal{A})$ , map it to  $\mathcal{A}/\mathcal{B}$ , take the associated Yoneda module in  $(\mathcal{A}/\mathcal{B})^{mod}$ , and then pull that back to  $\mathcal{A}^{mod}$ , denoting the result by  $\bar{Y}^{yon}$ . By construction,

$$(7.41) \quad \bar{Y}^{yon}(X) = hom_{\mathcal{A}/\mathcal{B}}(X, Y)$$

and the  $A_{\infty}$ -module structure of  $\bar{Y}^{yon}$  expresses the functoriality of (7.41) in  $X \in Ob(\mathcal{A})$ .

LEMMA 7.17. *Suppose that the object  $Y$  has the property that  $H^*(hom_{\mathcal{A}}(X, Y)) = 0$  for all  $X \in Ob(\mathcal{B})$ . Then  $\bar{Y}^{yon}$  is quasi-isomorphic to the ordinary Yoneda module  $Y^{yon}$ , and in particular  $H^*(hom_{\mathcal{A}/\mathcal{B}}(X, Y)) \cong H^*(hom_{\mathcal{A}}(X, Y))$  for any  $X$ .*

This follows from (7.40) since the Yoneda module of  $Y$  restricts to an acyclic module over  $\mathcal{B}$ . Actually, it serves only a warmup for the following more general:

LEMMA 7.18. *Suppose that there is a sequence of objects and morphisms in  $H^0(\mathcal{A})$ ,*

$$(7.42) \quad Y = Y_0 \xrightarrow{[t_1]} Y_1 \xrightarrow{[t_2]} Y_2 \longrightarrow \cdots$$

with the following properties:

- For each  $k$ ,  $\text{Cone}(t_k)$  in  $\text{Ob}(\mathcal{A}^{tw})$  is quasi-isomorphic to an object of  $\mathcal{B}^{tw}$ .
- For any  $X \in \text{Ob}(\mathcal{B})$  we have  $\varinjlim_k H^*(\text{hom}_{\mathcal{A}}(X, Y_k)) = 0$ .

Then  $\bar{Y}^{yon}$  is quasi-isomorphic to the homotopy direct limit of the sequence

$$(7.43) \quad Y_0^{yon} \longrightarrow Y_1^{yon} \longrightarrow Y_2^{yon} \dots$$

In particular,  $H^*(\text{hom}_{\mathcal{A}/\mathcal{B}}(X, Y)) \cong \varinjlim_k H^*(\text{hom}_{\mathcal{A}}(X, Y_k))$ .

Without loss of generality, we may assume that  $\mathcal{A}$  is strictly unital. The homotopy direct limit is defined via the telescope construction, which one can describe as follows. Replace  $Y_k$  by the quasi-isomorphic twisted complex  $C_k$  indicated by the following diagram:

$$(7.44) \quad \begin{array}{ccccccc} & Y_0 & & Y_1 & & \cdots & & Y_{k-1} & & Y_k \\ & \uparrow & & \nearrow & & \uparrow & & \uparrow & & \nearrow \\ -e_{Y_0} & & t_1 & & -e_{Y_1} & & & -e_{Y_{k-1}} & & t_k \\ & Y_0[1] & & Y_1[1] & & \cdots & & Y_{k-1}[1] & & \end{array}$$

Then, the  $[t_k]$  correspond to inclusions  $C_{k-1} \hookrightarrow C_k$ . There are corresponding inclusions of the Yoneda complexes, and the homotopy direct limit is the union

$$(7.45) \quad M = \varinjlim_k C_k^{yon}.$$

Now if one similarly considers the expressions

$$(7.46) \quad \text{Cone}(C_k^{yon} \otimes_{\mathcal{B}} X^{yon} \longrightarrow \text{hom}_{\mathcal{A}^{tw}}(X, C_k))$$

there are inclusions of the  $(k-1)$ -st one into the  $k$ -th one, and the first assumption in Lemma 7.18 says that the inclusions are all quasi-isomorphisms. One can therefore pass to the limit and consider

$$(7.47) \quad \text{Cone}(M \otimes_{\mathcal{B}} X^{yon} \longrightarrow M(X)).$$

The second Assumption in the Lemma says that  $M$  becomes acyclic when restricted to  $\mathcal{B}$ , hence (7.47) is quasi-isomorphic to  $M(X)$ . To summarize, we have shown that  $\text{hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$  is quasi-isomorphic to  $M(X)$ . One can realize this quasi-isomorphism as the leading order term of an  $A_\infty$ -module map  $\bar{Y}^{yon} \simeq M$ , which is the desired statement.

Via mapping cones, quotients can also be used to define the localisation of  $\mathcal{A}$  with respect to a set of morphisms  $S = \{S(X_0, X_1)\}$ ,  $S(X_0, X_1) \subset H^0(\text{hom}_{\mathcal{A}}(X_0, X_1))$ . Namely:

**DEFINITION 7.19.** *Let  $\mathcal{B} \subset \mathcal{A}^{tw}$  be the full  $A_\infty$ -subcategory consisting of mapping cones over elements of  $S$ . Then,  $S^{-1}\mathcal{A}$  is defined to be the image of  $\mathcal{A}$  under  $\mathcal{A} \rightarrow \mathcal{A}^{tw} \rightarrow \mathcal{A}^{tw}/\mathcal{B}$ .*

## LECTURE 8

# Hochschild homology

Following [104, 187], let's consider the idea of a homology theory for  $A_\infty$ -categories. The discussion here is only partly formalized: we start with a reasonable-looking list of properties for such a theory, and extract some useful consequences (for readers interested in a more systematic discussion, one possible approach is [191]). In any case, we ultimately have a single specific such theory in mind, namely Hochschild homology, and the axiomatic approach is just one way to understand its role while keeping computations to a minimum.

### First properties

We again work over a fixed field  $\mathbb{K}$ . Let's say that a homology theory  $\mathbb{H}_*$  should associate to an  $A_\infty$ -category  $\mathcal{A}$  a graded vector space  $\mathbb{H}_*(\mathcal{A})$ , which is functorial under  $A_\infty$ -functors. More precisely, quasi-isomorphic  $A_\infty$ -functors should induce the same map, which in particular implies that  $\mathbb{H}_*(\mathcal{A})$  is invariant under quasi-equivalences. We require the following basic properties:

- (*Morita invariance*) The Yoneda embedding  $\mathcal{A} \rightarrow \mathcal{A}^{perf}$  induces an isomorphism  $\mathbb{H}_*(\mathcal{A}) \cong \mathbb{H}_*(\mathcal{A}^{perf})$ .
- (*Künneth formula*) There is a canonical isomorphism  $\mathbb{H}_*(\mathcal{A} \otimes \mathcal{B}) \cong \mathbb{H}_*(\mathcal{A}) \otimes \mathbb{H}_*(\mathcal{B})$ .
- (*Opposite property*) There is a canonical isomorphism  $\mathbb{H}_*(\mathcal{A}^{opp}) \cong \mathbb{H}_*(\mathcal{A})$ .
- (*Normalisation*) There is a fixed isomorphism

$$(8.1) \quad \mathbb{H}_*(\mathbb{K}) \cong \begin{cases} \mathbb{K} & * = 0, \\ 0 & * \neq 0. \end{cases}$$

Moreover, for any  $P \in Ob(\mathbb{K}^{perf})$  and its associated functor  $\mathbb{K} \rightarrow \mathbb{K}^{perf}$  (see Example 7.7, in which the notation used here is decoded), the induced map  $\mathbb{H}_*(\mathbb{K}) \rightarrow \mathbb{H}_*(\mathbb{K}^{perf}) \cong \mathbb{H}_*(\mathbb{K})$  is multiplication with the Euler characteristic of  $P$ .

This formulation is internally incomplete, since there are various compatibility requirements between these isomorphisms, which have not been stated explicitly; the single exception to that is the normalisation property, in whose statement we have included a compatibility requirement with Morita invariance, since that is particularly important for our purpose.

As pointed out in [187], the properties above already yield the existence of a Chern character for perfect  $A_\infty$ -modules. Namely, a  $P \in \text{Ob}(\mathcal{A}^{perf})$  corresponds to an  $A_\infty$ -functor  $\mathbb{K} \rightarrow \mathcal{A}^{perf}$ , which induces a map  $\mathbb{K} = \mathbb{H}_0(\mathbb{K}) \rightarrow \mathbb{H}_0(\mathcal{A}^{perf}) \simeq \mathbb{H}_0(\mathcal{A})$ . We denote the image of 1 under that map by

$$(8.2) \quad [P]_{\mathbb{H}} \in \mathbb{H}_0(\mathcal{A}).$$

Because of functoriality, this is an invariant of the quasi-isomorphism class of  $P$ . Moreover, passing from  $P$  to  $P[k]$  changes (8.2) by  $(-1)^k$ , because of normalisation. Dually, given a proper module  $M \in \text{Ob}(\mathcal{A}^{prop})$ , which is an  $A_\infty$ -functor  $\mathcal{A}^{opp} \rightarrow \mathbb{K}^{prop} \cong \mathbb{K}^{perf}$ , we get a class

$$(8.3) \quad [M]_{\mathbb{H}}^{\vee} \in \mathbb{H}_0(\mathcal{A})^{\vee} = \text{Hom}(\mathbb{H}_0(\mathcal{A}), \mathbb{K}).$$

If we compose the functors corresponding to  $P$  and  $M$ , the result is a functor  $\mathbb{K} \cong \mathbb{K}^{opp} \rightarrow \mathbb{K}^{prop}$  which maps the unique object of  $\mathbb{K}$  to the chain complex  $\text{hom}_{\mathcal{A}^{mod}}(P, M)$ . The induced map on our homology groups can be described in two ways: as the canonical pairing between (8.2) and (8.3); or directly in terms of the normalization property. Comparing these two yields the following (which on one hand is almost a tautology, and on the other hand important as a first version of a ‘‘Cardy relation’’):

LEMMA 8.1. *For any perfect module  $P$  and proper module  $M$ ,*

$$(8.4) \quad \langle [M]_{\mathbb{H}}^{\vee}, [P]_{\mathbb{H}} \rangle = \chi(H^*(\text{hom}_{\mathcal{A}^{mod}}(P, M))). \quad \square$$

Here is a slightly different perspective on the same construction. There is a natural  $A_\infty$ -functor  $\mathcal{A}^{prop} \otimes (\mathcal{A}^{perf})^{opp} \rightarrow \mathbb{K}^{prop}$ , which takes a pair  $(M, P)$  to the chain complex  $\text{hom}_{\mathcal{A}^{mod}}(P, M)$ . This induces a pairing

$$(8.5) \quad \mathbb{H}_*(\mathcal{A}^{prop}) \otimes \mathbb{H}_*(\mathcal{A}^{perf}) \cong \mathbb{H}_*(\mathcal{A}^{prop} \otimes (\mathcal{A}^{perf})^{opp}) \rightarrow \mathbb{H}_*(\mathbb{K}^{prop}) \cong \mathbb{K}.$$

Any proper module has an associated class  $[M]_{\mathbb{H}} \in \mathbb{H}_0(\mathcal{A}^{prop})$ . Under the pairing (8.5), the image of that class is (8.3), and (8.4) is then obvious.

Consider the convolution functor  $\Phi_Q : \mathcal{A}^{perf} \rightarrow \mathcal{B}^{perf}$  associated to a right perfect  $(\mathcal{A}, \mathcal{B})$ -bimodule. By Morita invariance, this induces a map

$$(8.6) \quad \mathbb{H}_*(\Phi_Q) : \mathbb{H}_*(\mathcal{A}) \rightarrow \mathbb{H}_*(\mathcal{B}),$$

which depends only on the quasi-isomorphism type of  $Q$ , and is functorial under tensor product of bimodules. If one takes  $Q$  to be the graph bimodule of an  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ , then (8.6) agrees with the map induced by  $\mathcal{F}$ . Again, there is a universal viewpoint on this. Write  $(\mathcal{A}, \mathcal{B})^{r-perf} \subset (\mathcal{A}, \mathcal{B})^{mod}$  for the full subcategory of bimodules which are right perfect. Tensor product yields a map

$$(8.7) \quad \mathbb{H}_*(\mathcal{A}^{perf}) \otimes \mathbb{H}_*((\mathcal{A}, \mathcal{B})^{r-perf}) \rightarrow \mathbb{H}_*(\mathcal{B}^{perf}),$$

and the maps (8.6) are obtained by specializing this to a given class

$$(8.8) \quad [Q]_{\mathbb{H}} \in \mathbb{H}_0((\mathcal{A}, \mathcal{B})^{r-perf}).$$

### The smooth proper case

Let  $Q$  be an  $(\mathcal{A}, \mathcal{B})$ -bimodule which is perfect. Then, one can improve on (8.8) and associate to it a class

$$(8.9) \quad [Q]_{\mathbb{H}} \in \mathbb{H}_*(\mathcal{A}^{opp} \otimes \mathcal{B}) \cong \mathbb{H}_*(\mathcal{A}) \otimes \mathbb{H}_*(\mathcal{B}).$$

The action of  $\Phi_Q : \mathcal{A}^{prop} \rightarrow \mathcal{B}^{perf}$  on  $\mathbb{H}_*$  is then given by contraction with (8.9), using (8.5), hence has finite rank. We can apply this to the diagonal bimodule, and obtain the following:

LEMMA 8.2 ([188]). *Suppose that  $\mathcal{A}$  is smooth. Then the map  $\mathbb{H}_*(\mathcal{A}^{prop}) \rightarrow \mathbb{H}_*(\mathcal{A})$  induced by inclusion  $\mathcal{A}^{prop} \rightarrow \mathcal{A}^{perf}$  has finite rank.*  $\square$

Let  $\mathcal{A}$  be a proper  $A_\infty$ -category. Since any perfect module over  $\mathcal{A}$  is proper, one can restrict (8.5) to get a pairing

$$(8.10) \quad (\cdot, \cdot)_{\mathbb{H}} : \mathbb{H}_*(\mathcal{A}) \otimes \mathbb{H}_*(\mathcal{A}) \longrightarrow \mathbb{K}.$$

THEOREM 8.3 ([187], where the result is attributed to Kontsevich-Soibelman). *Suppose that  $\mathcal{A}$  is smooth and proper. Then  $\mathbb{H}_*(\mathcal{A})$  is of finite total dimension, and (8.5) is nondegenerate.*

PROOF. Finite-dimensionality follows from the previous Lemma. For the second part, we inspect more closely the argument given there, which says that the identity map on  $\mathbb{H}_*(\mathcal{A})$  factors as

$$(8.11) \quad \mathbb{H}_*(\mathcal{A}) \xrightarrow{[\mathcal{A}]_{\mathbb{H}} \otimes id} \mathbb{H}_*(\mathcal{A}) \otimes \mathbb{H}_*(\mathcal{A}) \otimes \mathbb{H}_*(\mathcal{A}) \xrightarrow{id \otimes \text{pairing}} \mathbb{H}_*(\mathcal{A}).$$

This immediately shows that the pairing is nondegenerate (on the left, and therefore also on the right).  $\square$

### Exactness and homotopy invariance

Some additional properties of a homology theory can be formulated by prescribing the value it attains for specific examples of categories, generalizing the normalisation condition. Take the following:

- (*Weak exactness*) Let  $\mathcal{A}$  be a directed  $A_\infty$ -category with objects  $(X_1, \dots, X_m)$ . Then  $\mathbb{H}_*(\mathcal{A}) \cong \mathbb{K}^m$ , concentrated in degree zero.

The Yoneda modules associated to the objects  $X_i$  have classes (8.2). They are also proper, hence have dual classes (8.3). By Lemma 8.1, the composition of the two resulting maps

$$(8.12) \quad \mathbb{K}^m \longrightarrow \mathbb{H}_0(\mathcal{A}) \longrightarrow \mathbb{K}^m$$

is given by the matrix with entries  $\chi(\text{hom}_{\mathcal{A}}(X_i, X_j))$ , which is invertible by directedness. Hence, both maps in (8.12) are isomorphisms.

At this point, we need the definition of the Grothendieck group of an  $A_\infty$ -category:  $K_0(\mathcal{A})$  is the abelian group generated by classes  $[C]_K$  for any  $C \in \text{Ob}(\mathcal{A}^{tw})$ , with the relations

$[Cone(C_1 \rightarrow C_2)]_K = [C_2]_K - [C_1]_K$  (this implies that the classes of objects of  $\mathcal{A}$  themselves are already generators). It may be tempting to think of  $K_0(\mathcal{A})$  itself as a homology theory, but it does not satisfy Morita invariance: passage to idempotent completions (Karoubi completions) generally makes the Grothendieck group much larger, a phenomenon that has been studied in depth [195].

LEMMA 8.4. *If  $\mathbb{H}_*(\cdot)$  has the weak exactness property, the assignment (8.2) defines a group homomorphism  $K_0(\mathcal{A}^{perf}) \rightarrow \mathbb{H}_0(\mathcal{A})$ . Similarly, (8.3) yields a map  $K_0(\mathcal{A}^{prop}) \rightarrow \mathbb{H}_0(\mathcal{A})^\vee$ . Finally, the maps (8.6) only depend on  $[Q]_K \in K_0((\mathcal{A}, \mathcal{B})^{r-perf})$ .  $\square$*

PROOF. Consider first the simplest non-semisimple directed category. This category, which we call  $\mathcal{A}_2^\rightarrow$ , has two objects  $X_1, X_2$  and a one-dimensional morphism space

$$(8.13) \quad \text{hom}_{\mathcal{A}_2^\rightarrow}(X_1, X_2) = \mathbb{K}, \quad \text{concentrated in degree zero}$$

(these properties, together with directedness, determine  $\mathcal{A}_2^\rightarrow$ ). It is the *universal model for a morphism*, in the following rather obvious way. Given any  $A_\infty$ -category  $\mathcal{A}$  and a morphism in  $H^0(\mathcal{A})$ , there is an  $A_\infty$ -functor (unique up to quasi-isomorphism)  $\mathcal{A}_2 \rightarrow \mathcal{A}$  which maps the generator (8.13) to that morphism.

A simple Euler characteristic computation shows that the classes of the objects  $X_1, X_2$  and  $Cone(X_1 \rightarrow X_2)$  in  $(\mathcal{A}_2^\rightarrow)^{perf}$  go to  $(1, 0)$ ,  $(1, 1)$  and  $(0, 1)$  under the second map in (8.12). Since that map is an isomorphism, we have

$$(8.14) \quad [Cone(X_1 \rightarrow X_2)]_{\mathbb{H}} = [X_2]_{\mathbb{H}} - [X_1]_{\mathbb{H}}.$$

By the universal property, the same will hold for cones of morphisms in any  $A_\infty$ -category. Applying that to  $\mathcal{A}^{perf}$  proves the first statement; applying it  $\mathcal{A}^{prop}$  and then using (8.5) proves the second statement; and similarly with  $(\mathcal{A}, \mathcal{B})^{r-perf}$  and (8.7) for the last one.  $\square$

There is a somewhat stronger property one could require:

- (*Exactness*) Let  $\mathcal{B} \hookrightarrow \mathcal{A}$  be a full  $A_\infty$ -subcategory, and  $\mathcal{A}/\mathcal{B}$  the quotient. Then there is a long exact sequence

$$(8.15) \quad \cdots \rightarrow \mathbb{H}_*(\mathcal{B}) \longrightarrow \mathbb{H}_*(\mathcal{A}) \longrightarrow \mathbb{H}_*(\mathcal{A}/\mathcal{B}) \longrightarrow \mathbb{H}_{*+1}(\mathcal{B}) \rightarrow \cdots$$

This implies weak exactness, by an inductive argument: if  $\mathcal{A}$  is directed and  $\mathcal{B} \cong \mathbb{K}$  is the full subcategory corresponding to the first object, then  $\mathcal{A}/\mathcal{B}$  is directed with one less object than  $\mathcal{A}$ . More importantly, (8.15) provides a link between the different degrees of  $\mathbb{H}_*$ , which had been conspicuously missing from our discussion so far. Note that even though we use a subscript following standard notational conventions, our grading of  $\mathbb{H}_*$  is in fact cohomological, as witnessed by the fact that the differential in (8.15) raises the degree.

So far, our discussion centered on what, in an analogy to the Eilenberg-Steenrod axioms, would be a combination of excision and the long exact sequence. What about homotopy invariance? The fact that quasi-isomorphic functors induce the same map on  $\mathbb{H}_*(\cdot)$  can be taken as answering that, but there is also a totally different possible viewpoint:

- (*Homotopy invariance*) For the polynomial algebra  $\mathbb{K}[t]$  considered as an  $A_\infty$ -algebra in the obvious way, the inclusion of constants  $\mathbb{K} \rightarrow \mathbb{K}[t]$  induces an isomorphism on  $\mathbb{H}_*(\cdot)$ .

However, it is doubtful whether there are homology theories which satisfy this as well as the previous conditions (periodic cyclic homology, which has homotopy invariance, does not have the normalisation property in the form we have stated it).

### The definition

After this long preliminary discussion, we finally introduce the theory of interest, together with a twisted version. Given any  $\mathcal{A}$ -bimodule  $Q$ , define the Hochschild homology of  $\mathcal{A}$  with coefficients in  $Q$  to be

$$(8.16) \quad HH_*(\mathcal{A}, Q) = H^*(Q \otimes_{\mathcal{A}^{opp} \otimes \mathcal{A}} \mathcal{A}),$$

where we think of  $Q$  as a right module over  $\mathcal{A}^{opp} \otimes \mathcal{A}$ ; of  $\mathcal{A}$  as a left module over the same; and tensor them together to get a chain complex. The standard Hochschild chain complex, somewhat simpler than the one suggested by (8.16), has this form:

$$(8.17) \quad CC_*(\mathcal{A}, Q) = \bigoplus Q(X_d, X_0) \otimes hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1)[d],$$

where the direct sum is over all  $d \geq 0$  and objects  $(X_0, \dots, X_d)$ . The differential is

$$(8.18) \quad \begin{aligned} \partial(q) &= \mu_Q^{0;1;0}(q), \\ \partial(q \otimes a) &= q \otimes \mu_{\mathcal{A}}^1(a) + (-1)^{|a|-1} \mu_Q^{0;1;0}(q) \otimes a + \mu_Q^{0;1;1}(q; a) + (-1)^{(|a|-1)|q|} \mu_Q^{0;1;1}(a; q), \\ &\dots \end{aligned}$$

Clearly, (8.16) is covariantly functorial in  $Q$ . Moreover, for any  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  and any  $\mathcal{B}$ -bimodule  $Q$  we get a map

$$(8.19) \quad HH_*(\mathcal{A}, \mathcal{F}^*Q) \longrightarrow HH_*(\mathcal{B}, Q)$$

where  $\mathcal{F}^*$  is pullback on both sides.

**REMARK 8.5.** *If  $\mathcal{A}$  and  $Q$  are strictly unital, there is a quasi-isomorphic reduced version of (8.17), where each  $hom_{\mathcal{A}}(X, X)$  factor that occurs gets quotiented out by  $\mathbb{K}e_X$ . This slightly smaller complex, denoted by  $CC_*^{red}(\mathcal{A}, Q)$ , can sometimes be useful in computations.*

Ordinary Hochschild homology  $HH_*(\mathcal{A}, \mathcal{A})$  is obtained by specializing to the diagonal bimodule  $Q = \mathcal{A}$ . Any  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  comes with a canonical bimodule homomorphism  $\mathcal{A} \rightarrow \mathcal{F}^*\mathcal{B}$ . In view of the previous discussion, it follows that  $HH_*(\mathcal{A}, \mathcal{A})$  is covariantly functorial; alternatively, the maps induced by functors can be defined directly in terms of (8.17). We will now consider its other properties as a ‘‘homology theory’’ in the previously explained sense:

- (*Morita invariance*) This is straightforward from (8.16): the inclusion  $\mathcal{A} \rightarrow \mathcal{A}^{perf}$  induces a restriction map from  $\mathcal{A}^{perf}$ -bimodules to  $\mathcal{A}$ -bimodules, which is a quasi-equivalence and sends the diagonal to the diagonal.
- (*Künneth formula*) This is a consequence of (8.16), obviously best proved by passing to quasi-isomorphic dg categories. For explicit formulae see [124, Section 4.2].
- (*Opposite property*), (*Normalisation*), (*Weak exactness*) These are straightforward in terms of (8.17) (using the reduced complex in the last two cases).
- (*Exactness*) This is a nontrivial result, proved in [104].
- (*Homotopy invariance*) Hochschild homology does not have this property. Instead, as a special case of the Hochschild-Kostant-Rosenberg theorem [90], one has:

$$(8.20) \quad HH_*(\mathbb{K}[s], \mathbb{K}[s]) \cong \begin{cases} \mathbb{K}[s] & * = 0, -1, \\ 0 & \text{otherwise.} \end{cases}$$

The failure of homotopy invariance is actually useful, since it allows for the existence of “Lefschetz trace” type formulae. As a toy model, suppose that we have a chain complex  $C$  of vector spaces with finite-dimensional cohomology, together with an endomorphism  $c$  of that complex. These data can be described by an  $A_\infty$ -functor  $\mathbb{K}[s] \rightarrow \mathbb{K}^{perf}$ , mapping  $s$  to  $c$ . By explicit computation, one shows that the induced map on Hochschild homology is

$$(8.21) \quad \begin{aligned} \mathbb{K}[s] &\cong HH_0(\mathbb{K}[s], \mathbb{K}[s]) \longrightarrow HH_0(\mathbb{K}^{perf}) \cong \mathbb{K}, \\ s^k &\longmapsto \text{Str}(c^k). \end{aligned}$$

If one thinks of this map as an element of  $HH_0(\mathbb{K})[[u]] \cong \mathbb{K}[[u]]$ , where  $u^k$  is dual to  $s^k$ , then the expression can be written as  $\sum_k u^k \text{Str}(c^k)$ . Now suppose that over some  $A_\infty$ -category  $\mathcal{A}$ , we have a perfect module  $P$  together with a (closed degree zero) endomorphism  $p \in \text{hom}_{\mathcal{A}^{mod}}(P, P)$ . In the same way as before, this gives rise to an element

$$(8.22) \quad [p]_{HH} \in HH_0(\mathcal{A}, \mathcal{A})[[u]],$$

and the relevant “Lefschetz trace” formula says that for any proper  $\mathcal{A}$ -module  $M$ ,

$$(8.23) \quad \langle [M]_{HH}^\vee, [p]_{HH} \rangle = \sum_k u^k \text{Str}(\cdot [p]^k : H^*(\text{hom}_{\mathcal{A}^{mod}}(P, M)) \rightarrow H^*(\text{hom}_{\mathcal{A}^{mod}}(P, M))).$$



## LECTURE 9

# Hochschild cohomology

Hochschild cohomology is to Hochschild homology as *Ext* is to *Tor*. However, its internal structure is far richer.  $HH^*(\mathcal{A}, \mathcal{A})[1]$  carries the structure of a graded Lie algebra, reflecting its role in the deformation theory of the  $A_\infty$ -structure on  $\mathcal{A}$  (Hochschild cohomology itself parametrizes first order deformations, and the bracket yields the second order obstruction). Furthermore,  $HH^*(\mathcal{A}, \mathcal{A})$  carries the structure of a graded commutative algebra, which is the endomorphism algebra of the diagonal bimodule  $\mathcal{A}$ . Both operations combine to form a Gerstenhaber algebra structure [73].

On the chain level, the complex  $CC^*(\mathcal{A}, \mathcal{A})[1]$  carries the structure of a dg Lie algebra, which transfers (non-canonically) to an  $L_\infty$ -structure on  $HH^*(\mathcal{A}, \mathcal{A})[1]$ , encoding the deformation theory of the  $A_\infty$ -algebra  $\mathcal{A}$  to arbitrarily high order. Similarly, the dg algebra structure on  $CC^*(\mathcal{A}, \mathcal{A})$  induces an  $A_\infty$ -structure on  $HH^*(\mathcal{A}, \mathcal{A})$ , but that loses the information about its homotopy commutativity. To recover a fuller picture, one needs to use a version of Deligne's conjecture (for which now many proofs are available), which says that  $CC^*(\mathcal{A}, \mathcal{A})$  carries the structure of an algebra over the (chain level) little disc operad. Since that operad is formal [192], it follows that  $HH^*(\mathcal{A}, \mathcal{A})$  carries the structure of a homotopy Gerstenhaber algebra in an appropriate sense.

Our aim in this lecture is much more modest, and is limited to considering the two classical structures (Lie bracket and product) on Hochschild cohomology essentially separately. We will be particularly interested in the following: the role of Hochschild cohomology in classifying  $A_\infty$ -structures with fixed cohomology [97]; and the deformation theory of the diagonal bimodule, which is related to the commutativity of the product.

### The Lie bracket

Let  $\mathcal{A}$  be a graded vector space over a field  $\mathbb{K}$ , assumed to be of characteristic 0 (see Remark 9.10 for a discussion of this assumption). Write

$$(9.1) \quad T(\mathcal{A}[1]) = \mathbb{K} \oplus \mathcal{A}[1] \oplus \mathcal{A}^{\otimes 2}[2] \oplus \dots$$

We turn (9.1) into a coalgebra, with coproduct

$$(9.2) \quad a_d \otimes \dots \otimes a_1 \longmapsto \sum_{i=0}^d (a_d \otimes \dots \otimes a_{i+1}) \otimes (a_i \otimes \dots \otimes a_1)$$

and counit which is the projection to  $\mathbb{K}$ . Write

$$(9.3) \quad CC^*(\mathcal{A}, \mathcal{A}) = \text{Hom}(T(\mathcal{A}[1]), \mathcal{A}) = \prod_{d \geq 0} \text{Hom}(\mathcal{A}^{\otimes d}, \mathcal{A})[-d].$$

There is an isomorphism between this and the graded space of coderivations of our coalgebra (see e.g. [60, Proposition 4.19]). More precisely,

$$(9.4) \quad \text{Coder}(T(\mathcal{A}[1])) \cong CC^*(\mathcal{A}, \mathcal{A})[1].$$

In one direction, one takes a coderivation and composes it with the projection  $T(\mathcal{A}[1]) \rightarrow \mathcal{A}[1]$ . In the opposite direction, one takes a homogeneous element  $\gamma = (\gamma^d)_{d \geq 0}$  of  $CC^*(\mathcal{A}, \mathcal{A})$  and associates to it the coderivation

$$(9.5) \quad \Gamma(a_d \otimes \cdots \otimes a_1) = \sum_{i,j} (-1)^{(|\gamma|-1)(|a_1|+\cdots+|a_i|-i)} a_d \otimes \cdots \gamma^j(a_{i+j}, \dots, a_{i+1}) \cdots \otimes a_1.$$

Here, the degrees  $|a|$  and  $|\gamma|$  are taken in  $\mathcal{A}$  and  $CC^*(\mathcal{A}, \mathcal{A})$ , which means without taking shifts into account. While we are on the subject of signs, note that we think of coderivations as acting from the right, which affects (9.5) through the Koszul sign conventions. For instance, if  $\gamma^0 = 0$  then

$$(9.6) \quad \Gamma(a_2 \otimes a_1) = (-1)^{(|\gamma|-1)(|a_1|-1)} \Gamma(a_2) \otimes a_1 + a_2 \otimes \Gamma(a_1),$$

where  $|\gamma| - 1$  is the degree of the coderivation  $\Gamma$ , and  $|a_1| - 1$  the degree of  $a_1$  in  $T(\mathcal{A}[1])$ .

$\text{Coder}(T(\mathcal{A}[1]))$  is naturally a graded Lie algebra. The corresponding structure (the Gerstenhaber bracket) on  $CC^*(\mathcal{A}, \mathcal{A})[1]$  is

$$(9.7) \quad [\beta_2, \beta_1] = \beta_2 \circ \beta_1 - (-1)^{(|\beta_2|-1)(|\beta_1|-1)} \beta_1 \circ \beta_2,$$

where

$$(9.8) \quad (\beta_2 \circ \beta_1)^d(a_d, \dots, a_1) = \sum_{i,j} (-1)^{(|\beta_1|-1)(|a_1|+\cdots+|a_i|-i)} \beta_2^{d-j+1}(a_d, \dots, \beta_1^j(a_{i+j}, \dots, a_{i+1}), \dots, a_1).$$

(9.8) by itself has no meaning in terms of coderivations, but will become important later on.

In parallel with what we've said for coderivations, a coalgebra endomorphism of  $T(\mathcal{A}[1])$  is determined by its composition with the projection to  $\mathcal{A}[1]$ , which means that it can be described by a sequence of maps  $\mathcal{G}^d : \mathcal{A}^{\otimes d} \rightarrow \mathcal{A}[1 - d]$ . The actual analogue of (9.4) is:

LEMMA 9.1. *A sequence  $\mathcal{G} = (\mathcal{G}^d)$  defines an endomorphism of  $T(\mathcal{A}[1])$  if and only if  $\mathcal{G}^0 = 0$ . If moreover  $\mathcal{G}^1$  is invertible as a linear map, then the endomorphism is an automorphism.*

PROOF. Suppose that  $\epsilon \in T(\mathcal{A}[1])$  is a coaugmentation. This means that its composition with the counit is 1, and that the following diagram commutes:

$$(9.9) \quad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\epsilon} & T(\mathcal{A}[1]) \\ id \downarrow & & \downarrow \text{comultiplication} \\ \mathbb{K} \otimes \mathbb{K} & \xrightarrow{\epsilon \otimes \epsilon} & T(\mathcal{A}[1]) \otimes T(\mathcal{A}[1]). \end{array}$$

If we expand it as

$$(9.10) \quad \epsilon = 1 + \epsilon^1 + \epsilon^2 + \cdots \in \mathbb{K} \oplus \mathcal{A}^1 \oplus (\mathcal{A} \otimes \mathcal{A})^2 \oplus \cdots$$

then (9.9) says that  $\epsilon^2 = \epsilon^1 \otimes \epsilon^1$ ,  $\epsilon^3 = \epsilon^2 \otimes \epsilon^1 = \epsilon^1 \otimes \epsilon^2 = \cdots$ , and hence that  $\epsilon^d = \epsilon^1 \otimes \cdots \otimes \epsilon^1$  for any  $d$ . Because (9.10) has to terminate, it follows that  $\epsilon^1 = 0$ , hence there is no coaugmentation other than the trivial one. Every endomorphism has to preserve that coaugmentation, which shows that necessarily  $\mathcal{G}^0 = 0$ . On the other hand, given any such  $\mathcal{G}$ , the associated endomorphism is

$$(9.11) \quad a_d \otimes \cdots \otimes a_1 \mapsto \sum \mathcal{G}^{r_k}(a_d, \dots, a_{d-r_k+1}) \otimes \cdots \otimes \mathcal{G}^{r_1}(a_{r_1}, \dots, a_1),$$

where the sum is over all partitions  $d = r_1 + \cdots + r_k$ .  $\square$

The issue with (9.10) in the case  $\epsilon^1 \neq 0$  is one of “convergence” (for the same reason,  $T(\mathcal{A}[1])$  is not a cofree coalgebra in the strict sense; see the discussion after [60, Definition 4.17], or [127, Section II.3.7]). Let’s temporarily ignore such problems, and make a formal attempt to integrate a derivation  $\gamma \in CC^1(\mathcal{A}, \mathcal{A})$  to an automorphism  $\mathcal{G} = \exp(\gamma)$  of  $T(\mathcal{A}[1])$ . This yields the formulae

$$(9.12) \quad \begin{aligned} \mathcal{G}^0 &= \gamma^0 + \frac{1}{2}\gamma^1(\gamma^0) + \frac{1}{3}\gamma^2(\gamma^0, \gamma^0) + \frac{1}{6}\gamma^1(\gamma^1(\gamma^0)) + \cdots, \\ \mathcal{G}^1 &= id_{\mathcal{A}} + \gamma^1 + \frac{1}{2}\gamma^1(\gamma^1) + \frac{1}{2}\gamma^2(\cdot, \gamma^0) + \frac{1}{2}\gamma^2(\gamma^0, \cdot) + \cdots, \\ &\dots \end{aligned}$$

The summands are indexed by planar trees with  $d$  inputs and one output. The constant in front of a tree with  $k$  vertices is  $1/k!$  times the number of ways in which the vertices can be ordered compatibly with the input-to-output orientation (the same concept of ordered planar trees describes the face structure of permutohedra). Of course, since  $\mathcal{G}^0$  is nonzero, and all the formulae contain infinite sums, this is not meaningful in general.

To get around this problem, one can proceed in two different ways.  $CC^*(\mathcal{A}, \mathcal{A})$  has a complete decreasing filtration, the *length filtration*, by the subspaces  $F^p CC^*(\mathcal{A}, \mathcal{A})$  of maps which vanish on  $\mathcal{A}^{\otimes k}$ ,  $k < p$ . This is compatible with the Lie bracket up to a shift by 1:

$$(9.13) \quad [F^p CC^*(\mathcal{A}, \mathcal{A}), F^q CC^*(\mathcal{A}, \mathcal{A})] \subset F^{p+q-1} CC^*(\mathcal{A}, \mathcal{A}).$$

In particular,  $F^2 CC^1(\mathcal{A}, \mathcal{A})$  is a pro-nilpotent Lie algebra. If one allows only  $\gamma$  lying in that subspace, then all sums in (9.12) are finite, leading to a well-defined automorphism  $\mathcal{G}$  with

$$(9.14) \quad \begin{aligned} \mathcal{G}^0 &= 0, \\ \mathcal{G}^1 &= id, \\ \mathcal{G}^2 &= \gamma^2, \\ \mathcal{G}^3 &= \gamma^3 + \frac{1}{2}\gamma^2(\cdot, \gamma^2) + \frac{1}{2}\gamma^2(\gamma^2, \cdot), \\ &\dots \end{aligned}$$

Slightly more generally, one could allow  $\gamma^1$  to be nontrivial as long as it is nilpotent. There is another case which can be treated in the same way. Namely, suppose that  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and that  $\mathcal{A}$  is finite-dimensional in each degree. Take  $\gamma \in F^1 CC^1(\mathcal{A}, \mathcal{A})$ . Then (9.12) converges

componentwise in the standard topology, yielding

$$\begin{aligned}
(9.15) \quad & \mathcal{G}^0 = 0, \\
& \mathcal{G}^1 = \exp(\gamma^1), \\
& \mathcal{G}^2 = \sum_{i,j,k} \frac{1}{(i+j+k+1)!} \binom{j+k}{j} (\gamma^1)^i \gamma^2((\gamma^1)^j, (\gamma^1)^k), \\
& \dots
\end{aligned}$$

The other (more obvious) approach is to introduce a formal parameter  $u$ . One can then consider formal one-parameter families of automorphisms of  $T(\mathcal{A}[1])$  which specialize to the identity at  $u = 0$ . Such an automorphism is given by a sequence of maps  $\mathcal{G}_u^d : \mathcal{A}^{\otimes d} \rightarrow (\mathcal{A}[[u]])[1-d]$  such that  $\mathcal{G}_u^1 = id + o(1)$ , and  $\mathcal{G}_u^d = o(1)$  for  $d \neq 1$ , where  $o(1)$  means of order  $u$  or higher. In particular, one may now have a nonzero term  $\mathcal{G}_u^0$ , as long as it is  $o(1)$  (with respect to the situation in Lemma 9.1, the difference is that the  $u$ -adically completed tensor algebra  $\hat{T}(\mathcal{A}[[u]])[1]$  admits many augmentations). Given any  $\gamma_u \in uCC^1(\mathcal{A}, \mathcal{A})[[u]]$  (note this is  $o(1)$  by assumption), one can define such a family by taking

$$(9.16) \quad \mathcal{G}_u = \exp(\gamma_u).$$

The formulae in (9.12) are now all  $u$ -adically convergent, since there are only finitely many terms which are nonzero at a fixed power of  $u$ .

### $A_\infty$ -structures

We will consider  $A_\infty$ -structures on  $\mathcal{A}$ , but for now without any unitality requirement, which means  $\mu_{\mathcal{A}} = \{\mu_{\mathcal{A}}^1, \mu_{\mathcal{A}}^2, \dots\}$  satisfying the  $A_\infty$ -associativity equations. One can think of  $\mu_{\mathcal{A}}$  as an element of  $F^1CC^1(\mathcal{A}, \mathcal{A})$ , and then the equations are

$$(9.17) \quad \frac{1}{2}[\mu_{\mathcal{A}}, \mu_{\mathcal{A}}] = 0.$$

Equivalently,  $\mu_{\mathcal{A}}$  is a differential on the coalgebra  $T(\mathcal{A}[1])$  whose composition with the coaugmentation  $\mathbb{K} \rightarrow T(\mathcal{A}[1])$  vanishes. Given any such  $\mu_{\mathcal{A}}$ , one can introduce a differential  $\partial = [\mu_{\mathcal{A}}, \cdot]$  on  $CC^*(\mathcal{A}, \mathcal{A})$ . The resulting complex is called the Hochschild cochain complex of  $\mathcal{A}$ , and its cohomology is the Hochschild cohomology  $HH^*(\mathcal{A}, \mathcal{A})$ . For general reasons,  $HH^*(\mathcal{A}, \mathcal{A})[1]$  inherits a Lie bracket from (9.7).

REMARK 9.2. *It is instructive to compare this with (8.17), which for the present case of an  $A_\infty$ -algebra would be written as  $CC_*(\mathcal{A}, \mathcal{A}) = \mathcal{A} \otimes T(\mathcal{A}[1])$ . In parallel with that situation, there is a generalization to Hochschild cohomology with coefficients in an  $A_\infty$ -bimodule  $Q$ , which is the cohomology of the complex  $CC^*(\mathcal{A}, Q) = \text{Hom}(T(\mathcal{A}[1]), Q)$  with differential*

$$\begin{aligned}
(9.18) \quad & (\partial\beta)^d(a_d, \dots, a_1) = - \sum_{s,r} (-1)^{|\beta|(|a_1|+\dots+|a_r|-r)} \mu_Q^{s;1;r}(a_d, \dots, a_{d-s+1}; \\
& \beta^{d-s-r}(a_{d-s}, \dots, a_{r+1}); a_r, \dots, a_1) \\
& + \sum_{i,j} (-1)^{|\beta|+|a_1|+\dots+|a_i|-i} \beta^{d-i-j+1}(a_d, \dots, \mu_{\mathcal{A}}^j(a_{i+j}, \dots, a_{i+1}), \dots, a_1).
\end{aligned}$$

In view of (7.23), this indeed reduces to the previous definition if  $Q = \mathcal{A}$  is the diagonal bimodule.

REMARK 9.3. Another recurring feature (compare Remark 8.5) is that for strictly unital  $\mathcal{A}$ , there is a reduced version  $CC_{red}^*(\mathcal{A}, \mathcal{A})$  of the Hochschild complex, consisting of multilinear maps which vanish when any one of the inputs is the identity. This encodes deformations of  $\mathcal{A}$  inside the class of  $A_\infty$ -algebras with the same strict unit. Actually, since the inclusion  $CC_{red}^*(\mathcal{A}, \mathcal{A}) \hookrightarrow CC^*(\mathcal{A}, \mathcal{A})$  is a quasi-isomorphism, any deformation of  $\mathcal{A}$  is equivalent to such a strictly unital one.

We now consider the role of Hochschild cohomology in the deformation theory of  $A_\infty$ -structures. Let  $\mathbb{K}_\epsilon = \mathbb{K}[\epsilon]/\epsilon^2$ .

DEFINITION 9.4. A first order deformation of the  $A_\infty$ -algebra  $\mathcal{A}$  is a  $\mathbb{K}_\epsilon$ -linear  $A_\infty$ -structure  $\mu_{\mathcal{A}_\epsilon}$  on  $\mathcal{A}_\epsilon = \mathcal{A} \otimes \mathbb{K}_\epsilon$ , which reduces to  $\mu_{\mathcal{A}}$  if one sets  $\epsilon = 0$ .

Explicitly, such a deformation can be written as

$$(9.19) \quad \mu_{\mathcal{A}_\epsilon} = \mu_{\mathcal{A}} + \epsilon \beta \quad \text{for } \beta \in F^1 CC^2(\mathcal{A}, \mathcal{A}).$$

From (9.17) one sees that  $\beta$  is a Hochschild cocycle. There is an obvious notion of isomorphism of deformations, and this corresponds to being cohomologous in  $F^1 CC^*(\mathcal{A}, \mathcal{A})$ . Hence,  $H^2(F^1 CC^*(\mathcal{A}, \mathcal{A}))$  classifies first order infinitesimal deformations of  $\mathcal{A}$  up to isomorphism. Now suppose that we have a deformation which extends to second order, meaning over  $\mathbb{C}[\epsilon]/\epsilon^3$ . Writing that as

$$(9.20) \quad \mu_{\mathcal{A}} + \epsilon \beta + \epsilon^2 \gamma$$

and taking the Taylor expansion of (9.17), one finds that

$$(9.21) \quad \partial \gamma + \frac{1}{2}[\beta, \beta] = 0.$$

This gives a (necessary and sufficient) cohomological obstruction for extending a given first order deformation to second order. All of this is an instance of the standard Maurer-Cartan formalism of deformation theory [79], which means that it is a consequence of writing the  $A_\infty$ -equations in the form (9.17).

REMARK 9.5. One can also allow curved deformations, where  $\mu_{\mathcal{A}_\epsilon}^0$  can be nonzero as long as it is  $o(1)$ , and the notion of equivalence of deformations is similarly generalized, allowing all of  $HH^*(\mathcal{A}, \mathcal{A})$  to appear in the deformation theory. This is particularly important in a categorical context, as we will now explain (a similar discussion would apply if one remained within the world of  $A_\infty$ -algebras, but considered them up to Morita invariance).

Hochschild cohomology for  $A_\infty$ -categories can be built in the same way as Hochschild homology (8.17), using composable chains of morphism spaces (in fact, one could formulate the entirety of this lecture on that level of generality, even though we have not done so for the

sake of brevity). Concretely, this means that the underlying chain complex is

$$(9.22) \quad CC^*(\mathcal{A}, \mathcal{A}) = \prod_d \prod_{X_0, \dots, X_d} \text{Hom}(\text{hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \\ \cdots \otimes \text{hom}_{\mathcal{A}}(X_0, X_1), \text{hom}_{\mathcal{A}}(X_0, X_d))[-d].$$

The resulting cohomology theory  $HH^*(\mathcal{A}, \mathcal{A})$  is invariant under quasi-equivalences. Moreover, the inclusion  $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{perf}}$  yields a quasi-isomorphism  $HH^*(\mathcal{A}^{\text{perf}}, \mathcal{A}^{\text{perf}}) \rightarrow HH^*(\mathcal{A}, \mathcal{A})$ . Hence, if  $\mathcal{A}^{\text{perf}}$  is quasi-equivalent to  $\mathcal{B}^{\text{perf}}$ , then curved deformations of  $\mathcal{A}$ , up to isomorphism, correspond bijectively to such deformations of  $\mathcal{B}$ . This would fail if one removed the word ‘‘curved’’. Indeed, an ordinary deformation of  $\mathcal{A}$  generally gives rise to a curved deformation of  $\mathcal{A}^{\text{tw}}$ , because not every twisted complex over  $\mathcal{A}$  can be deformed to a twisted complex over  $\mathcal{A}_\epsilon$ .

One can apply this deformation theory to classify  $A_\infty$ -structures with a fixed cohomology [97]. For that, take a graded algebra  $A$ , and view it as an  $A_\infty$ -algebra with only one nontrivial operation. Consider  $A_\infty$ -deformations of that algebra are defined formally to all orders, meaning over  $\mathbb{K}[[\epsilon]]$ . There is an additional grading in this situation, which we want to exploit. Namely, give elements of  $A$  their natural degrees; give  $\epsilon$  degree one; and require that  $\mathcal{A}_\epsilon$  be homogeneous in this sense. Then, the only nontrivial components of the deformation are

$$(9.23) \quad A^{\otimes d} \longrightarrow \epsilon^{d-2} A[2-d], \quad d \geq 3.$$

If we set  $\epsilon = 1$ , then these components define an  $A_\infty$ -structure, which extends the algebra structure of  $A$  by higher order operations. Write  $CC^s(A, A[t])$  for the space of maps  $A^{\otimes s} \rightarrow A$  of degree  $t$ . The Hochschild differential arising from the algebra structure on  $A$  has bidegree  $(1, 0)$  in this respect, leading to bigraded cohomology groups  $HH^s(A, A[t])$ . To first order, the homogeneous deformation theory introduced above is governed by  $HH^d(A, A[2-d])$  with  $d \geq 3$ . By combining this idea and the Perturbation Lemma (see Lecture 7), one obtains for instance the following:

**PROPOSITION 9.6** (Intrinsic formality, an analogue of [88]). *Suppose that  $HH^d(A, A[2-d]) = 0$  for all  $d \geq 3$ . Then, any  $A_\infty$ -algebra  $\mathcal{A}$  whose cohomology is isomorphic to  $A$  (as a graded algebra), is actually quasi-isomorphic to  $A$  (as an  $A_\infty$ -algebra).  $\square$*

**PROPOSITION 9.7** (See for instance [172, Section 3]). *Suppose that there is exactly one value of  $d > 2$  such that  $HH^d(A, A[2-d])$  is nonzero. Then, any  $A_\infty$ -algebras  $\mathcal{A}$  together with a cohomology level isomorphism  $H(\mathcal{A}) \cong A$  determines an element of  $HH^d(A, A[2-d])$ . Moreover, from those elements one can recover the  $A_\infty$ -structure up to quasi-isomorphism.  $\square$*

### The Yoneda product

We return to (9.8), which defines a product on  $CC^*(\mathcal{A}, \mathcal{A})[1]$  that is neither commutative nor associative. Its failure to be commutative is measured by (9.7). Its failure to be associative

takes on the following form:

$$(9.24) \quad (\gamma \circ \alpha) \circ \beta - \gamma \circ (\beta \circ \alpha) = (-1)^{(|\alpha|-1)(|\beta|-1)} \langle \gamma, \alpha, \beta \rangle + \langle \gamma, \beta, \alpha \rangle,$$

where we introduce the notation (only of temporary importance)

$$(9.25) \quad \langle \gamma, \alpha, \beta \rangle^d(a_d, \dots, a_1) = \sum_{i,j,k,l} (-1)^* \gamma^{d-l-j+2}(a_d, \dots, \alpha^l(a_{k+l}, \dots, a_{k+1}), \dots, \beta^j(a_{i+j}, \dots, a_{i+1}), \dots, a_1),$$

with the sign given by  $*$  =  $(|\beta| - 1)(|a_1| + \dots + |a_i| - i) + (|\alpha| - 1)(|a_1| + \dots + |a_k|)$ . There is one particular case where the terms (9.25) cancel, namely

$$(9.26) \quad \gamma \circ [\alpha, \beta] = (\gamma \circ \alpha) \circ \beta - (-1)^{(|\alpha|-1)(|\beta|-1)} (\gamma \circ \beta) \circ \alpha.$$

Now suppose that  $\mathcal{A}$  carries an  $A_\infty$ -algebra structure. Define a product on  $CC^*(\mathcal{A}, \mathcal{A})$  by

$$(9.27) \quad \beta * \gamma = \langle \mu_{\mathcal{A}}, \beta, \gamma \rangle.$$

This is compatible with the differential, in the same sense as in the  $A_\infty$ -associativity equations (we omit that computation). Moreover,

$$(9.28) \quad \begin{aligned} & (-1)^{|\gamma|} \beta * \gamma - (-1)^{(|\gamma|-1)|\beta|} \gamma * \beta = (-1)^{|\gamma|} ((\mu_{\mathcal{A}} \circ \gamma) \circ \beta - \mu_{\mathcal{A}} \circ (\gamma \circ \beta)) \\ & = (-1)^{|\gamma|} \partial \gamma \circ \beta - (\gamma \circ \mu_{\mathcal{A}}) \circ \beta - (-1)^{|\gamma|} \partial(\gamma \circ \beta) - (-1)^{|\beta|} (\gamma \circ \beta) \circ \mu_{\mathcal{A}} \\ & = (-1)^{|\gamma|} (\partial \gamma \circ \beta - (-1)^{|\gamma|} \gamma \circ \partial \beta - \partial(\gamma \circ \beta)), \end{aligned}$$

where the last step uses (9.26). Hence, the product on  $HH^*(\mathcal{A}, \mathcal{A})$  induced by  $(-1)^{|\gamma|} \beta * \gamma$  is graded commutative. Note that by definition, that product makes the canonical map  $HH^*(\mathcal{A}, \mathcal{A}) \rightarrow H^*(\mathcal{A})$  (induced by the projection onto the first term of the Hochschild complex) into a homomorphism of graded algebras.

We want to take a more roundabout way which gives an alternative, and more conceptual, explanation for homotopy commutativity. The starting point for that is the interpretation of Hochschild cohomology as bimodule homomorphisms; from this point onwards, we again assume that  $\mathcal{A}$  is cohomologically unital. For any bimodule  $Q$  there is a quasi-isomorphism

$$(9.29) \quad CC^*(\mathcal{A}, Q) \longrightarrow \text{hom}_{(\mathcal{A}, \mathcal{A})^{mod}}(\mathcal{A}, Q),$$

From now on, we will only use the case of the diagonal bimodule  $Q = \mathcal{A}$ . In that case, the cohomology level isomorphism induced by (9.29) identifies the product (9.27) with the Yoneda product (composition of bimodule maps). The commutativity of the Yoneda product in this particular situation is a consequence of a general categorical Eckmann-Hilton argument [63], which we will now explain. Take two endomorphisms  $[\phi]$ ,  $[\psi]$  of the diagonal bimodule in the category  $H^*((\mathcal{A}, \mathcal{A})^{mod})$ . Using the quasi-isomorphism

$$(9.30) \quad \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A},$$

we construct the diagram

$$(9.31) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{[\psi]} & \mathcal{A} & \xrightarrow{[\phi]} & \mathcal{A} \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \mathcal{A} & & \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} & & \mathcal{A} \\ \uparrow [e_{\mathcal{A}} \otimes \psi] & & \uparrow [\phi \otimes e_{\mathcal{A}}] & & \uparrow \\ \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} & \xrightarrow{[\phi \otimes \psi]} & \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} & \xrightarrow{[\phi \otimes \psi]} & \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \mathcal{A} & \xrightarrow{[\phi]} & \mathcal{A} & \xrightarrow{[\psi]} & \mathcal{A} \end{array}$$

This is commutative (up to, in the case of the bottom triangle, the Koszul sign  $(-1)^{|\phi||\psi|}$ ), so by going around its outer edges one obtains the desired result.

There is also a deformation-theoretic version of the argument, which is particularly simple in degree 1. To emphasize that this is different from our previous discussion of the deformation theory of  $A_\infty$ -algebras, and also for compatibility with similar notation in the rest of the book, we denote the formal variable by  $u$ , so that the analogue of Definition 9.4 is this:

**DEFINITION 9.8.** *Take an  $\mathcal{A}$ -bimodule  $Q$ . A first order deformation of  $Q$  is an  $\mathcal{A}$ -bimodule structure on  $Q_u = Q \otimes \mathbb{K}_u$  which is  $\mathbb{K}_u$ -linear and reduces to  $\mu_Q$  if one sets  $u = 0$ .*

If one writes  $\mu_{Q_u} = \mu_Q + u\phi$ , then it follows that  $\phi$  defines a class

$$(9.32) \quad [\phi] \in H^1(\text{hom}_{(\mathcal{A}, \mathcal{A})\text{mod}}(Q, Q)).$$

Conversely, this class determines the deformation up to isomorphism. The second order obstruction is given by the Yoneda square  $[\phi]^2 \in H^2(\text{hom}_{(\mathcal{A}, \mathcal{A})\text{mod}}(Q, Q))$ . Slightly more generally, consider the graded Lie bracket

$$(9.33) \quad \begin{aligned} H^*(\text{hom}_{\mathcal{A}\text{mod}}(Q, Q))^{\otimes 2} &\longrightarrow H^*(\text{hom}_{\mathcal{A}\text{mod}}(Q, Q)), \\ [\phi] \otimes [\psi] &\longmapsto [\phi][\psi] - (-1)^{|\phi||\psi|}[\psi][\phi]. \end{aligned}$$

Its deformation-theoretic meaning, in degree 1, is as follows. Suppose that  $[\phi]$  and  $[\psi]$  classify deformations  $Q_u$  and  $Q_v$  over  $\mathbb{K}_u$  and  $\mathbb{K}_v$ , respectively. Then (9.33) is the obstruction to combining them into a deformation over  $\mathbb{K}[u, v]/(u^2, v^2)$ .

Note that there is a related problem which is always unobstructed. Namely, if  $Q_u$  is a first order deformation of  $Q$ , and  $R_v$  a first order deformation of  $R$ , then  $Q_u \otimes R_v$  is a deformation of  $Q \otimes R$  over  $\mathbb{K}[u, v]/(u^2, v^2)$ . For  $Q = R = \mathcal{A}$ , the tensor product is a deformation of  $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}$ . However, one can transfer that deformation back to  $\mathcal{A}$  via (9.30). Hence (9.33) must vanish in that case. By using deformations whose formal variables have nonzero degrees, one can



extend this argument to give another proof of graded commutativity of the product on  $HH^*(\mathcal{A}, \mathcal{A})$ .

The preceding discussion looks somewhat like the beginning of a  $T^1$ -lifting argument [102]. The general aim of such arguments is to show unobstructedness of formal deformations to all orders. There is indeed such a result in our case, but it is best approached in a different way:

LEMMA 9.9. *For any element  $[\beta] \in HH^1(\mathcal{A}, \mathcal{A})$ , there is a deformation of the diagonal bimodule over  $\mathbb{K}[[u]]$ , whose reduction to  $\mathbb{K}[u]/u^2$  is classified by  $[\beta]$ .*

Fix a cochain representative  $\beta$ , and form the formal one-parameter family of  $A_\infty$ -homomorphisms  $\mathcal{G}_u = \exp(u\beta)$  in the sense of (9.16) (these may have a curvature term  $\mathcal{G}_u^0$ , which is of order  $o(1)$ , meaning infinitesimally small). One can associate to  $\mathcal{G}_u$  an  $\mathcal{A}$ -bimodule

$$(9.34) \quad \mathcal{O}^{inf} = \text{Graph}(\mathcal{G}_u),$$

which we call the *infinitesimal orbit bimodule*. Its underlying graded vector space  $\mathcal{A}[[u]]$ , and the  $A_\infty$ -bimodule structure defined as in (7.24). Expansion in  $u$  yields

$$(9.35) \quad \begin{aligned} \mu_{\mathcal{O}^{inf}}^{s;1;r}(a''_s, \dots, a''_1; a; a'_r, \dots, a'_1) &= (-1)^{|a'_r| + \dots + |a'_1| + r - 1} \left( \mu_{\mathcal{A}}^{s+1+r}(a''_s, \dots, a, \dots, a'_1) \right. \\ &\quad \left. + u \sum_{i,j} \mu_{\mathcal{A}}^{r+s+2-j}(a''_s, \dots, \gamma^j(a''_{i+j}, \dots, a''_{i+1}), \dots, a, \dots, a'_1) + \dots \right) \end{aligned}$$

Since the order  $u$  term is the image of  $\gamma$  under (9.29), this explicit construction proves Lemma 9.9.

REMARK 9.10. *The definition of Hochschild cohomology works over a field  $\mathbb{K}$  of arbitrary characteristic. The same is true of its role in the first deformation theory of  $A_\infty$ -structures, even though that theory can then no longer interpreted as part of the general Maurer-Cartan formalism. More concretely, one rewrites (9.17) without denominators as  $\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}} = 0$ . For a deformation  $\mathcal{A}_\epsilon$ , one then Taylor-expands*

$$(9.36) \quad \begin{aligned} \mu_{\mathcal{A}_\epsilon} \circ \mu_{\mathcal{A}_\epsilon} &= (\mu_{\mathcal{A}} + \epsilon\beta + \epsilon^2\gamma + \dots) \circ (\mu_{\mathcal{A}} + \epsilon\beta + \epsilon^2\gamma + \dots) \\ &= \epsilon \cdot \partial\beta + \epsilon^2(\partial\gamma + \beta \circ \beta) + \dots \end{aligned}$$

*In particular, Propositions 9.6 and 9.7 continue to hold in arbitrary characteristic. On the other hand, the discussion of formal exponentiation of derivations, (9.16), and therefore also Lemma 9.9, require the assumption that  $\text{char}(\mathbb{K}) = 0$ .*



## The Fukaya category of a surface

Setting up Fukaya categories for general symplectic manifolds is quite an involved process (see [71]; we got a taste of that in Lecture 5, when discussing Lagrangian Floer cohomology). There are several ways of simplifying the problem, by drastically shrinking the level of generality. Among them are:

- Considering only Weinstein manifolds, where the symplectic form is exact, and imposing similar conditions on the Lagrangian submanifolds. This allows one to avoid Novikov fields as well as “curved”  $A_\infty$ -structures, hence to stay on a level of difficulty comparable to the early Floer theory literature (there are other possibilities which lead to similar simplifications, such as suitable “monotonicity” conditions, e.g. [141, 204], but we will not consider them here).
- Restricting to the lowest-dimensional case (of a surface). In that dimension, Lagrangian submanifolds are just (arbitrary) embedded curves, and there are no genuinely symplectic phenomena. Correspondingly, one can avoid pseudo-holomorphic curves, and argue in a purely combinatorial or topological framework. This still comes at a price: pseudo-holomorphic curve theory allows a large class of perturbations of the relevant equations; the combinatorial theory lacks that flexibility, which makes some formal workarounds necessary.

To make things as simple as possible, we adopt both these restrictions at the same time, which means that our symplectic manifolds are open (punctured) surfaces. These are also “symplectically Calabi-Yau”, meaning that their first Chern class vanishes, allowing us to make Fukaya categories  $\mathbb{Z}$ -graded.

We proceed in several iterations, gradually increasing the level of sophistication. Fix the surface  $M$  (with some auxiliary structures) and a coefficient field  $\mathbb{K}$ .

- The *Donaldson-Fukaya category*  $DF(M)$  is a  $\mathbb{Z}$ -graded category linear over  $\mathbb{K}$ , encoding “cohomology level” data, which means the Floer cohomology groups of simple closed loops (with some auxiliary structures), and the triangle product on those groups. The combinatorial approach, of which we give a version, is explained in detail in [56] (which of course has a lot of precursors, e.g. computations in [113]).
- *Directed Fukaya categories*  $Fuk^{\rightarrow}(L_1, \dots, L_m)$  are associated to finite ordered collections of simple closed curves  $(L_1, \dots, L_m)$ . They partially refine the cohomology

level structure to an  $A_\infty$ -structure, using directedness to avoid technical complications with self-intersections. They depend on the choice of objects and their ordering, hence are not invariants of  $M$ . The combinatorial aspects of this are described in [176, Section 13] or [23].

- The *Fukaya category* is an  $A_\infty$ -category  $Fuk(M)$  with  $DF(M) \cong H(Fuk(M))$ . We construct it using a formal quotient trick (there is at least one alternative approach to this technical issue, namely: to introduce the Fukaya category as an  $A_\infty$ -pre-category in the sense of [115]; and then either work with it directly as in [3], or else apply a rectification procedure as in [206]).

There are several variations and generalizations of these ideas. Maybe the conceptually most important one is this:

- The *wrapped Fukaya category*  $\mathcal{W}(M)$  contains  $Fuk(M)$  as a full  $A_\infty$ -subcategory, but enlarges it by allowing non-compact curves, which go towards the punctures.

For general Weinstein manifolds, the definition of  $\mathcal{W}(M)$  involves Hamiltonian dynamics “at infinity” [8], but in the two-dimensional case there is a quotient trick similar to our definition of  $\mathcal{F}(M)$ , which we will outline briefly. We end by discussing  $\mathcal{F}(M)$  and  $\mathcal{W}(M)$  in the simplest nontrivial example, which is when  $M$  is a cylinder.

*Acknowledgments.* The quotient construction of Fukaya categories is borrowed from (currently unpublished) joint work of Mohammed Abouzaid and the author.

### Geometric setup

Let  $M$  be a punctured surface (obtained by removing a nonempty finite set of points from a closed oriented surface). We equip it with a symplectic form, which here means an everywhere positive two-form  $\omega_M$ , as well as a primitive  $\theta_M$ . Additionally, we choose a nowhere vanishing vector field  $Y_M$ .

REMARK 10.1.  $\theta_M$  matters only up to adding exact one-forms, so the overall amount of choice is an affine space over  $H^1(M; \mathbb{R})$ . Similarly,  $Y_M$  only matters up to homotopy, so the effective choices are an affine space over  $H^1(M; \mathbb{Z})$ . Another way to think of  $Y_M$  is to choose a positively oriented complex structure  $J_M$ . Then, there is a unique nowhere vanishing  $C^\infty$  complex one-form  $\eta_M$  such that  $\eta_M(Y_M) = 1$ . In higher dimensions, the corresponding choice is that of a  $C^\infty$  complex volume form, for some compatible almost complex structure  $J_M$ .

We will consider oriented simple closed loops  $L \subset M$  with the following conditions and decorations:

- (*Exactness*)  $[\theta_M|_L] \in H^1(L; \mathbb{R})$  must vanish. More concretely,  $\int_L \theta_M = 0$ .

- (*Grading*) We want to choose a one-parameter family of nowhere zero sections  $(Y_{L,r})_{0 \leq r \leq 1}$  of  $TM|L$  such that  $Y_{L,0}$  is positively tangent to  $L$ , and  $Y_{L,1} = Y_M|L$ . In fact, this family only matters up to deformations rel endpoints. The obstruction to its existence is the rotation number of  $L$  with respect to  $Y_M$ ; and the possible choices form an affine space over  $\mathbb{Z}$ . Following [169], one calls this a *grading* of  $L$ .
- (*Local coefficients*)  $L$  should also come with a flat  $\mathbb{K}$ -vector bundle  $\xi_L \rightarrow L$ . Up to isomorphism, this is classified by a conjugacy class in  $GL_r(\mathbb{K})$ , where  $r$  is the rank.

Borrowing terminology from physics, let's call such objects *closed exact Lagrangian branes*.

REMARK 10.2. *Since we ask for all our categories to be small, the flat vector bundles should be taken from some fixed "repertoire", in order to avoid set-theoretic issues. We leave it to the reader to flesh out this issue.*

A collection of closed exact Lagrangian branes is said to be in *general position* if: any two of them intersect transversally; and there are no triple intersection points. Of course, this can always be achieved by a small perturbation (within the same isotopy class of branes). It is maybe worth while to spell out what we mean by an isotopy between two closed exact Lagrangian branes  $L$  and  $L'$ . Take a closed connected surface  $I \subset M \times [0, 1]$  which intersects each  $M \times \{r\}$  transversally, and with  $\partial I = I \cap (M \times \{0, 1\}) = (L \times \{0\}) \cup (L' \times \{1\})$ ; this implies that  $I$  is an annulus. We want the isotopy to be Hamiltonian, meaning that the pullback of  $\omega_M$  to  $I$  is exact; we need it to carry an orientation and grading compatible with those of  $L$  and  $L'$ ; and we also want to have a flat bundle  $\xi_I \rightarrow I$  together with *specified* isomorphisms

$$(10.1) \quad \begin{aligned} \xi_I|L \times \{0\} &\cong \xi_L, \\ \xi_I|L \times \{1\} &\cong \xi_{L'}. \end{aligned}$$

REMARK 10.3. *In the general construction of Fukaya categories (at least if the coefficient field is of characteristic  $\text{char}(\mathbb{K}) \neq 2$ , making signs meaningful), an additional Spin condition on the Lagrangian submanifolds is necessary. In the surface case, we implicitly assume that each curve comes with the trivial Spin structure (the one coming from the unique trivialization of  $TS^1$ ). Instead of changing the Spin structure to the opposite one, one then changes the sign of the holonomy of  $\xi_L$ .*

Also interesting is the corresponding class of symplectic automorphisms  $\phi : M \rightarrow M$ :

- (*Exactness*)  $\phi$  should be exact with respect to  $\theta_M$ , meaning that  $\phi^*\theta_M - \theta_M$  is an exact one-form.
- (*Grading*)  $\eta_M$  and  $\phi^*\eta_M$  should lie in the same connected component of the space of nowhere vanishing one-forms, and in fact fix a path connecting them (as before, only the homotopy class of the path matters).
- (*Local coefficients*) We additionally fix a flat  $\mathbb{K}$ -line bundle  $\xi_\phi \rightarrow M$ .

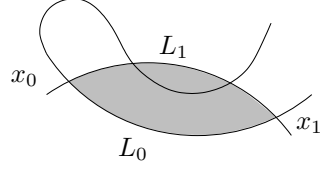


FIGURE 1.

Such *extended automorphisms* act on closed exact Lagrangian branes. For instance, if we take  $\phi = id_M$ , with the trivial grading and a general flat line bundle  $\xi_\phi$ , the action on objects twists the flat vector bundles

$$(10.2) \quad \xi_L \longrightarrow \xi_L \otimes \xi_\phi|_L.$$

### Floer cohomology

Suppose that  $(L_0, L_1)$  are closed exact Lagrangian branes which are in general position. We can assign to them a  $\mathbb{Z}$ -graded chain complex  $CF^*(L_0, L_1)$ ; Floer cohomology  $HF^*(L_0, L_1)$  is defined as the cohomology of that complex. To any intersection point  $x \in L_0 \cap L_1$  one associates a degree (or *absolute Maslov index*)  $i(x) \in \mathbb{Z}$ , using the gradings of both  $L_0$  and  $L_1$  (we refer to [176, Section 13] for the detailed conventions). Then

$$(10.3) \quad CF^k(L_0, L_1) \stackrel{\text{def}}{=} \bigoplus_{i(x)=k} Hom(\xi_{L_0,x}, \xi_{L_1,x}).$$

The differential on (10.3) is obtained as a sum of contributions associated to orientation-preservingly immersed *bigons* or *lunes* with boundary sides on  $L_0, L_1$ , and (convex) corners at  $x_0, x_1$  (Figure 1). To any such bigon  $u$  one associates a sign  $\sigma(u) \in \pm 1$ , which depends on the orientations of  $(L_0, L_1)$  (see again [176, Section 13]) as well as an isomorphism

$$(10.4) \quad \tau(u) : Hom(\xi_{L_0,x_1}, \xi_{L_1,x_1}) \longrightarrow Hom(\xi_{L_0,x_0}, \xi_{L_1,x_0}),$$

obtained by using parallel transport along the boundary sides. The contribution of  $u$  to the differential is  $\sigma(u)\tau(u)$ . Elementary combinatorial arguments show that the resulting operation has degree 1 and squares to zero.

Similarly, if  $(L_0, L_1, L_2)$  are in general position, there is a canonical map

$$(10.5) \quad CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_2),$$

obtained by counting immersed triangles (as indicated in Figure 2) with signs and weights similar to (10.4). One can show by elementary means that (10.5) is a chain map, and induces a product on Floer cohomology which is associative for *quadruples*  $(L_0, L_1, L_2, L_3)$  which are in general position. This means that in that case, the two ways of bracketing yield the same map

$$(10.6) \quad HF^*(L_2, L_3) \otimes HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \longrightarrow HF^*(L_0, L_3)$$

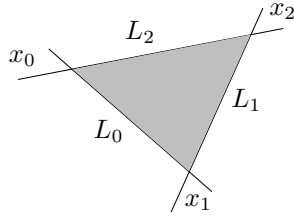


FIGURE 2.

(note that there are less generic situations, such as when  $L_0 = L_2$ , where one can make sense of (10.6) by bracketing in a particular way).

Because of the general position requirement, the structures defined so far not quite constitute a category. To make up for that deficiency, we need some additional Floer-theoretic data. Suppose that  $(L, L')$  are in general position and related by an isotopy  $I$ . That isotopy gives rise to a distinguished *continuation element*

$$(10.7) \quad c_I \in HF^0(L, L').$$

REMARK 10.4.  $c_I$  depends on the choice of (10.1). For instance, multiplying those maps by constants  $b_k \in \mathbb{K}^\times$  changes  $c_I$  by  $b_1/b_0$ . There is one special case where the isomorphisms are unique, namely when  $\mathbb{K} = \mathbb{Z}/2$  and the  $\xi_{L_k}$  have rank 1; restricting to that case would simplify the construction of  $DF(M)$  a little.

Here are the fundamental properties of continuation elements:

- (*Deformation invariance*)  $c_I$  is invariant under deformation of  $I$  rel endpoints;
- (*Composition*) If we have  $I$  as well as another isotopy  $I'$  from  $L'$  to  $L''$ , and  $(L, L', L'')$  are in general position, then the image of  $c_{I'} \otimes c_I$  under the cohomology level product  $HF^*(L', L'') \otimes HF^*(L, L') \rightarrow HF^*(L, L'')$  agrees with the element given by the composite isotopy.
- (*Pseudo-invertibility*) Suppose that  $L_1$  is isotopic to  $L'_1$  by an isotopy  $I_1$ , and that  $(L_0, L_1, L'_1)$  is in general position. Then left multiplication with  $c_{I_1}$  yields an isomorphism  $HF^*(L_0, L_1) \cong HF^*(L_0, L'_1)$ . Similarly, if  $L_0$  is isotopic to  $L'_0$  by an isotopy  $I_0$ , and  $(L_0, L'_0, L_1)$  is in general position, right multiplication with  $c_{I_0}$  yields an isomorphism  $HF^*(L'_0, L_1) \cong HF^*(L_0, L_1)$ .

We will not describe the combinatorial definition of the  $c_I$ . The prototypical case is when  $M$  is an annulus and  $L$  its meridian; the general situation can be reduced to that one by passing to a suitable covering. The only part which involves more complicated topology is the proof of pseudo-invertibility, which can be done (assuming that the other properties have been established first) by breaking up the isotopy into small pieces.

LEMMA 10.5. *Suppose that  $L_1$  is isotopic to  $L'_1$  by an isotopy  $I_1$ , and that  $(L_0, L_1, L'_1)$  is in general position. Let  $\bar{I}_1$  be the result of reversing  $I_1$ . Then the maps*

$$(10.8) \quad \begin{array}{ccc} & \xrightarrow{c_{I_1}} & \\ HF^*(L_0, L_1) & & HF^*(L_0, L'_1) \\ & \xleftarrow{c_{\bar{I}_1}} & \end{array}$$

are inverses of each other.

This is not immediately obvious: the composition of  $I_1$  and its inverse is an isotopy between  $L_1$  and itself, which has no associated element (10.7) because the endpoints coincide, hence are not in general position. The following argument is a typical workaround for such situations (for instance, the same argument yields the corresponding result for left multiplication).

PROOF. Choose an auxiliary small perturbation  $L_1^*$  of  $L_1$ , such that  $(L_0, L_1, L'_1, L_1^*)$  are in general position. Take the isotopy  $I^*$  from  $L_1$  to  $L_1^*$ . By the composition property and deformation invariance,

$$(10.9) \quad (c_{I^*} c_{\bar{I}_1}) c_{I_1} = c_{I^*} \in HF^0(L_1, L_1^*).$$

The bracketing on the left hand side of (10.9) ensures that all the products are defined. As a consequence, for all  $x \in HF^*(L_0, L_1)$  we have:

$$(10.10) \quad \begin{aligned} c_{I^*} x &= ((c_{I^*} c_{\bar{I}_1}) c_{I_1}) x \\ &= (c_{I^*} c_{\bar{I}_1})(c_{I_1} x) \\ &= c_{I^*}(c_{\bar{I}_1}(c_{I_1} x)). \end{aligned}$$

Here, associativity is applied twice, for the quadruples of objects are  $(L_0, L_1, L'_1, L_1^*)$  and  $(L_0, L'_1, L_1, L_1^*)$ , which are in general position. Finally, left multiplication with  $c_{I^*}$  is an isomorphism  $HF^*(L_0, L_1) \rightarrow HF^*(L_0, L_1^*)$ , hence  $x = c_{\bar{I}_1}(c_{I_1} x)$ , as desired.  $\square$

We are now ready to define the Donaldson-Fukaya category  $DF(M)$ . Objects are exact closed Lagrangian branes. The morphisms are

$$(10.11) \quad Hom_{DF(M)}^*(L_0, L_1) \stackrel{\text{def}}{=} HF^*(L_0, L'_1),$$

where  $L'_1$  is isotopic to  $L_1$  by a fixed isotopy, and transverse to both  $L_0$  and  $L_1$ . It is important to remember that  $L'_1$  depends on  $L_0$  as well as  $L_1$  (even though the notation does not reflect that). Composition of morphisms is given by

$$(10.12) \quad \begin{aligned} Hom_{DF(M)}^*(L_1, L_2) \otimes Hom_{DF(M)}^*(L_0, L_1) &= HF^*(L_1, L'_2) \otimes HF^*(L_0, L'_1) \\ &\cong HF^*(L'_1, L'_2) \otimes HF^*(L_0, L'_1) \xrightarrow{\text{product}} HF^*(L_0, L'_2) \\ &\cong HF^*(L_0, L''_2) = Hom_{DF(M)}^*(L_0, L_2). \end{aligned}$$

Here,  $(L_0, L'_1)$ ,  $(L_1, L'_2)$  and  $(L_0, L''_2)$  are the pairs whose Floer cohomology defines the morphisms in the Donaldson-Fukaya category. One chooses an auxiliary  $L_2^*$  isotopic to  $L_2$ , so that each of the triples

$$(10.13) \quad (L_0, L'_1, L_2^*), (L_1, L'_2, L_2^*), (L_1, L'_1, L_2^*), (L_0, L''_2, L_2^*)$$



is in general position. The isomorphisms in (10.12) are all given by multiplication with continuation elements. Of course, one needs to prove that this is independent of the choice of perturbations (which involves Lemma 10.5) and associative. Having defined  $DF(M)$  in this way, a further use of the continuation elements shows that isotopic closed exact Lagrangian branes give rise to isomorphic objects. We will not pursue the details of this further. Eventually, the perturbations become part of the basic machinery underlying Floer theory: one writes  $HF^*(L_0, L_1)$  for any pair  $(L_0, L_1)$ , where (10.11) is understood, and similarly for the product.

EXAMPLE 10.6. *The endomorphism ring of any object  $L$  is*

$$(10.14) \quad HF^*(L, L) \cong H^*(L; \text{Hom}(\xi_L, \xi_L)),$$

where the right hand is ordinary cohomology with twisted coefficients in the endomorphism bundle. This isomorphism is canonical, and compatible with the ring structure. More generally, if we have two objects  $L_0, L_1$  which share the same underlying curve  $L$  (including the orientation and grading), but carry different flat bundles, then

$$(10.15) \quad HF^*(L_0, L_1) \cong H^*(L; \text{Hom}(\xi_{L_0}, \xi_{L_1})).$$

All these results carry over to higher-dimensional closed exact Lagrangian submanifolds (with different proofs, of course).

### Directed Fukaya categories

Let  $(L_1, \dots, L_m)$  be a finite ordered collection of closed exact Lagrangian branes, which are in general position. One can associate to such a collection an  $A_\infty$ -category  $\mathcal{A} = \text{Fuk}^\rightarrow(L_1, \dots, L_m)$  which has the  $L_i$  as objects, and is directed (Example 7.11). The non-trivial morphism spaces are  $\text{hom}_{\mathcal{A}}(L_i, L_j) = CF^*(L_i, L_j)$  for  $i < j$ . Correspondingly, the nontrivial  $A_\infty$ -compositions (i.e. those that are not uniquely determined by the strict unitality assumption) are

$$(10.16) \quad \mu_{\mathcal{A}}^d : CF^*(L_{i_{d-1}}, L_{i_d}) \otimes \cdots \otimes CF^*(L_{i_0}, L_{i_1}) \longrightarrow CF^*(L_{i_0}, L_{i_d})[2-d]$$

for  $d \geq 1$  and  $1 \leq i_0 < \cdots < i_d \leq m$ . One defines them by a count of immersed  $(d+1)$ -gons. For  $d = 1, 2$  this reduces to the previously defined differential and product (up to the usual slight differences in sign conventions (7.7)).

At this point, it is worth while to make a strategic observation. As we have just seen, directedness simplifies the technical issues involved in defining Fukaya categories considerably, since it removes the need to define morphisms from an object to itself geometrically. The disadvantage is that in a directed  $A_\infty$ -category, no two distinct objects can be quasi-isomorphic. Our plan for defining  $\text{Fuk}(M)$  is to start with an algebraic structure which has an ordering property similar to directedness, and then add the “missing” quasi-isomorphisms in a purely algebraic way, by localisation.

### Defining the Fukaya category

For any exact closed Lagrangian brane  $L$ , fix a family of perturbations  $L^{(1)}, L^{(2)}, \dots$  (each  $L^{(k)}$  being isotopic to  $L$ ) with the following property: *any finite subset  $(L_0^{(k_0)}, \dots, L_m^{(k_m)})$ , with  $k_0, \dots, k_m$  pairwise distinct, is in general position*. This may seem tricky at first glance, but it can be achieved, for instance by choosing all such perturbations from a sufficiently large countable set of simple closed curves. One chooses the first perturbations  $L^{(1)}$  from that set, for all  $L$  (with no particular restriction); then the second perturbations, making sure that each of them is in general position with respect to all the  $L^{(1)}$  (which yields only countably many conditions, hence can be satisfied); and so on.

We will first construct an  $A_\infty$ -category  $\mathcal{A}$  whose objects are the  $L^{(k)}$  (but really considered as pairs consisting of  $L$  and  $k$ ). Morphisms are

$$(10.17) \quad \text{hom}_{\mathcal{A}}(L_0^{(k_0)}, L_1^{(k_1)}) = \begin{cases} CF^*(L_0^{(k_0)}, L_1^{(k_1)}) & k_0 < k_1, \\ \mathbb{K} \cdot e_{L^{(k)}} & L_0^{(k_0)} = L_1^{(k_1)} \\ 0 & \text{otherwise.} \end{cases}$$

We should emphasise that the middle case of (10.17) really applies *only to the endomorphisms of a given object* (it can happen that  $L_0^{(k_0)}$  has the same underlying curve as  $L_1^{(k_1)}$ , with the same grading and isomorphic flat bundles; but that does not matter unless the objects are actually the same element of the set  $Ob(\mathcal{A})$ ). We define a strictly unital  $A_\infty$ -structure on  $\mathcal{A}$  exactly as in the directed case.

Let's fix, for each  $L$  and  $k$ , an isotopy from  $L^{(k)}$  to  $L^{(k+1)}$ . Denote the associated continuation elements by

$$(10.18) \quad c_{L,k} \in HF^0(L^{(k)}, L^{(k+1)}) = H^0(\text{hom}_{\mathcal{A}}(L^{(k)}, L^{(k+1)})).$$

Take the set  $S$  of all such morphisms, and form the localisation  $S^{-1}\mathcal{A}$  in the sense of Lecture 7. In the localised category,  $L^{(k)}$  and  $L^{(l)}$  becomes quasi-isomorphic for any  $(k, l)$ . Moreover:

LEMMA 10.7. *For any two objects,*

$$(10.19) \quad H^*(\text{hom}_{S^{-1}\mathcal{A}}(L_0^{(k_0)}, L_1^{(k_1)})) \cong HF^*(L_0, L_1).$$

PROOF. For an object  $L_1^{(k_1)}$ , consider the sequence

$$(10.20) \quad L_1^{(k_1)} \xrightarrow{c_{L_1, k_1}} L_1^{(k_1+1)} \xrightarrow{c_{L_1, k_1+1}} L_1^{(k_1+2)} \dots$$

Given any other  $L_0^{(k_0)}$ , we have

$$(10.21) \quad \begin{aligned} \varinjlim_{k_1} H^*(\text{hom}_{\mathcal{A}}(L_0^{(k_0+1)}, L_1^{(k_1)})) &\cong \varinjlim_{k_1} HF^*(L_0^{(k_0+1)}, L_1^{(k_1)}) \\ &\cong \varinjlim_{k_1} HF^*(L_0^{(k_0)}, L_1^{(k_1)}) \cong \varinjlim_{k_1} H^*(\text{hom}_{\mathcal{A}}(L_0^{(k_0)}, L_1^{(k_1)})), \end{aligned}$$

where the direct limits are formed with respect to (10.20), and the nontrivial map between them is given by multiplication with  $c_{L_0, k_0}$  on the right. These are precisely the conditions

needed to apply Lemma 7.18, which shows that

$$(10.22) \quad H^*(\text{hom}_{S^{-1}\mathcal{A}}(L_0^{(k_0)}, L_1^{(k_1)})) \cong \varinjlim_{k_1} H^*(\text{hom}_{\mathcal{A}}(L_0^{(k_0)}, L_1^{(k_1)})) \cong \varinjlim_{k_1} HF^*(L_0^{(k_0)}, L_1^{(k_1)}).$$

Because of the isotopy invariance of Floer cohomology, the direct limit stabilizes as soon as  $k_1 > k_0$ . For the same reason, the outcome is isomorphic to  $HF^*(L_0, L_1)$ .  $\square$

With this in mind, we *define*

$$(10.23) \quad \text{Fuk}(M) \stackrel{\text{def}}{=} S^{-1}\mathcal{A}.$$

One can show that this is independent of the choice of perturbations  $L^{(k)}$ , and of the isotopies between  $L^{(k)}$  and  $L^{(k+1)}$ , up to quasi-equivalence; and that (10.19) is compatible with products, which means that  $H^*(\text{Fuk}(M))$  is canonically equivalent to  $DF(M)$ , as required.

REMARK 10.8. *The standard definition of Fukaya category is based on pseudo-holomorphic curves, even in the case of surfaces, see e.g. [176]. If one denotes that construction by  $\mathcal{C}$ , then it is not hard to show that there is an  $A_\infty$ -functor  $\mathcal{A} \rightarrow \mathcal{C}$  which maps the elements of  $S$  to quasi-isomorphisms, and which therefore (by the universal property of categorical quotients) induces a quotient  $A_\infty$ -functor  $S^{-1}\mathcal{A} \rightarrow \mathcal{C}$ . Using Lemma 10.7, one sees easily that the quotient functor is a quasi-equivalence, which proves the compatibility of the two approaches.*

### The wrapped version

Suppose now that  $M$  is actually a (two-dimensional) Weinstein manifold. This means that its structure at infinity is that of a finite disjoint union of cylindrical ends

$$(10.24) \quad [1, \infty) \times S^1 \hookrightarrow M, \quad \theta_M = p \, dq, \quad \omega_M = dp \wedge dq.$$

We want to fix such coordinates on the ends. Along with the previous objects, we now allow certain non-compact curves, namely properly embedded (connected oriented)  $L \subset \mathbb{R}$  which are exact rel infinity, as defined in Lecture 6. Concretely, this means that  $L$  has two ends, which look like half-infinite lines  $\{q = \text{const}\}$  in the coordinates (10.24), and that the (compactly supported) one-form  $\theta_M|_L$  has zero integral. We choose a grading and flat bundle  $\xi_L$  as before. Let's call the resulting larger class of objects *exact Lagrangian branes*.

If two such branes  $(L_0, L_1)$  are in general position, the intersection  $L_0 \cap L_1$  must be finite, and it is straightforward to define  $HF^*(L_0, L_1)$ . However, a simple perturbation such as (10.11) no longer yields a unique result, because isotopies may create or cancel intersection points at infinity. The solution to that is to include an “infinite reservoir” of intersection points at infinity into the definition. Concretely, one defines the *wrapped Floer cohomology*

$$(10.25) \quad HW^*(L_0, L_1) = \varinjlim_t HF^*(\phi_H^t(L_0), L_1)$$

where  $H$  is a Hamiltonian function on  $M$  which satisfies  $H(p, q) = p$  for  $p \gg 0$  on the ends (10.24), and whose Hamiltonian vector field is therefore  $X_H = \partial_q$  at infinity. If one considers the flow  $\phi_H^t$  for a generic choice of  $t$ , then  $\phi_H^t(L_0) \cap L_1$  will be compact, and

therefore  $HF^*(\phi_H^t(L_0), L_1)$  can be defined by a further compactly supported perturbation (which we have suppressed from the notation). The direct limit in (10.25) is taken over larger and larger such  $t$ . The maps between the different groups are defined using canonical continuation elements

$$(10.26) \quad c_{L, t_1, t_0} \in HF^*(\phi_H^{t_1}(L), \phi_H^{t_0}(L)), \quad t_1 > t_0.$$

These are partially similar to (10.7), but taking the product with them does not usually induce an isomorphism, which means that the direct limit (10.25) fails to stabilize (indeed, while each of the Floer cohomology groups on the right hand side is finite-dimensional, wrapped Floer cohomology is infinite-dimensional in many cases).

One can define a wrapped version of the Donaldson-Fukaya category, with exact Lagrangian branes as objects and (10.25) as morphisms. On the chain level, there is a wrapped version  $\mathcal{W}(M)$  of the Fukaya category; in analogy with our construction of  $\mathcal{F}(M)$ , this can be defined by starting with an “ordered” version and inverting (10.26).

### The cylinder

Take  $M = \mathbb{R} \times S^1$ , with  $\theta_M$  and  $\omega_M$  as in (10.24), and with a constant (in  $(p, q)$ -coordinates) vector field  $Y_M$ . For simplicity, we assume that the coefficient field  $\mathbb{K}$  is algebraically closed.

Any exact simple closed curve on  $M$  is necessarily isotopic (through such curves) to the meridian  $\{0\} \times S^1$ . The meridian has a standard orientation and grading, which we adopt. The remaining choice is that of local coefficient system (or flat bundle), which is determined up to isomorphism by its monodromy  $A \in GL_r(\mathbb{K})$ . Denote the resulting exact closed Lagrangian brane by  $L_A$  (or by  $L_a$  for  $r = 1$ , in which case  $A$  consists of a single element  $a \in \mathbb{K}^\times$ ). After a suitable perturbation, the chain complex  $CF^*(L_{A_0}, L_{A_1})$  can be identified with this:

$$(10.27) \quad \begin{aligned} \text{Hom}(\mathbb{K}^{r_0}, \mathbb{K}^{r_1}) &\longrightarrow \text{Hom}(\mathbb{K}^{r_0}, \mathbb{K}^{r_1}), \\ X &\mapsto A_1 X - X A_0. \end{aligned}$$

PROPOSITION 10.9.  *$D^{tw}(\text{Fuk}(M))$  is equivalent to  $D^b \text{Coh}_{cpt}(M^\vee)$ , the bounded derived category of compactly supported coherent sheaves on the affine algebraic curve  $M^\vee = \mathbb{G}_m$  (the punctured affine line).*

If one is willing to ignore the triangulated structure, this could be proved by establishing a one-to-one match between isomorphism classes of objects, as in [154]. The object  $L_A$  corresponds to the sheaf  $X_A$  defined by

$$(10.28) \quad 0 \longrightarrow \mathcal{O}_{M^\vee}^{\oplus r} \xrightarrow{A - wI} \mathcal{O}_{M^\vee}^{\oplus r} \longrightarrow X_A \longrightarrow 0,$$

where  $w$  is the coordinate on  $M^\vee$ , and  $I$  the identity matrix. Note that, because  $M^\vee$  is one-dimensional, any object of the derived category is isomorphic to the direct sum of its cohomology sheaves. Moreover, each compactly supported coherent sheaf is a finite direct sum of (10.28).

On the other hand, there is an underlying cochain level statement, which implies the cohomology level statement in a form that includes the triangulated structure, and whose proof is simpler and more conceptual. It is based on the following chain of observations:

- Each  $L_A$  is quasi-isomorphic to a twisted complex constructed out of  $L_a$ . Hence, it is sufficient to consider the full  $A_\infty$ -subcategory with objects  $\{L_a\}$ .
- There are no morphisms from  $L_a$  to  $L_b$  for  $a \neq b$ . Hence, it is sufficient to consider each object  $L_a$  separately.
- $HF^*(L_a, L_a) \cong H^*(L_a; \mathbb{K})$  is an exterior algebra in one variable. One can show (either as a consequence of the general classification theory from Lecture 9, or by a more direct argument involving passing to a strictly unital model) that the underlying  $A_\infty$ -structure must be quasi-isomorphic to the trivial (formal) one, which we denote by  $\Lambda$ .

As a consequence,

$$(10.29) \quad Fuk(M)^{tw} \cong \bigoplus_{a \in \mathbb{C}^*} \Lambda^{tw}.$$

The same argument applies on the coherent sheaf side.

Proposition 10.9 is a simple example of Kontsevich's Homological Mirror Symmetry (HMS) [113], with  $M^\vee$  the mirror of  $M$ . However, in view of (10.29) the content of that statement is maybe disappointing: the categories involved do not seem to reflect the geometry of the underlying manifolds, and they are also not particularly well-behaved (not smooth, for instance; compare Example 7.12). Therefore, the following wrapped version is maybe more meaningful:

PROPOSITION 10.10.  $D^{tw}(\mathcal{W}(M))$  is equivalent to  $D^b Coh(M^\vee)$ .

This reduces to computing the endomorphisms of the object  $L = \mathbb{R} \times \{0\}$  (). One shows that

$$(10.30) \quad HW^*(L, L) \cong \mathbb{K}[w, w^{-1}],$$

concentrated in degree 0. The underlying  $A_\infty$ -structure is then again necessarily formal (for degree reasons). Under the mirror equivalence, this object corresponds to the structure sheaf  $\mathcal{O}_{M^\vee}$ .



LECTURE 11

## A four-dimensional example

We return to local mirror symmetry, in an example which is close to the one that already appeared in Lecture 3. However, this time we want to go into more depth concerning the symplectic topology, and specifically the Fukaya category, or our manifold. This gives us the chance to demonstrate several of the approaches to such categories that have been explored in recent years (the discussion stops short of an actual proof of Kontsevich’s Homology Symmetry Conjecture, even though such a proof is in principle within reach of the existing technology).

*Acknowledgments.* Mohammed Abouzaid and Nicholas Sheridan graciously allowed me to include their proofs of Lemma 11.6, and so did Ailsa Keating for Remark 11.11. Obviously, any errors in the presentation are my own.

### Affine conic fibrations and their mirrors

To place the following example in context, we recall one of the early approaches to constructing local mirrors [92, 84]. Take an integer polytope  $P \subset \mathbb{R}^{n-1}$ . This singles out a class of Laurent polynomials

$$(11.1) \quad \begin{aligned} W : (\mathbb{C}^*)^{n-1} &\longrightarrow \mathbb{C}, \\ W(y_1, \dots, y_{n-1}) &= \sum_{\nu \in P \cap \mathbb{Z}^{n-1}} z_\nu y^\nu. \end{aligned}$$

Within the linear space of such  $W$  (or equivalently the space of possible coefficients  $\{z_\nu\}$ ), there is a Zariski open subset of “nondegenerate” ones (these have the property that  $W^{-1}(0)$  is smooth, and satisfy additional nondegeneracy conditions at infinity, in the manner of [41]). We choose a Laurent polynomial in that subset, and form the conic fibration

$$(11.2) \quad M = \{(x, y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^{n-1} : x_1 x_2 + W(y_1, \dots, y_{n-1}) = 0\}.$$

As the name suggests, projection  $p : M \rightarrow (\mathbb{C}^*)^{n-1}$  has conic fibres, which degenerate exactly along  $W^{-1}(0)$ . If one equips  $M$  with an appropriate Kähler form, it becomes a Liouville manifold. We also equip it with the trivialization of the canonical bundle given by

$$(11.3) \quad \eta_M = \operatorname{res}_M \frac{dx_1 \wedge dx_2 \wedge dy_1/y_1 \wedge \cdots \wedge dy_{n-1}/y_{n-1}}{x_1 x_2 + W(y_1, \dots, y_{n-1})}.$$

Suppose that we have a maximal triangulation of  $P$ , which means a decomposition into integer simplices of minimal volume. There is an additional condition (the triangulation

must be induced from a piecewise affine convex function  $P \rightarrow \mathbb{R}$ ). Form the cone over  $P$ , which is the set of points in  $\mathbb{R}^{n-1} \times \mathbb{R}$  of the form  $(tp, t)$  for  $p \in P$  and  $t \geq 0$ . The triangulation gives rise to a fan structure on the cone, which determines a smooth toric Calabi-Yau variety  $N$ . The projection  $\mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$  corresponds to a toric function  $F : N \rightarrow \mathbb{C}$ . If we remove  $F^{-1}(0)$ , then this function is just the projection  $(\mathbb{C}^*)^n \rightarrow \mathbb{C}^*$ . However, instead we want to remove the fibre over a nonzero point, say  $F^{-1}(1) \cong (\mathbb{C}^*)^{n-1}$ :

$$(11.4) \quad M^\vee = N \setminus F^{-1}(1).$$

This is a non-compact smooth Calabi-Yau variety, which we call the *naive mirror* of  $M$ . The word “naive” is appropriate, because the construction is heuristic. A better motivated version can be carried out in the general context of Strominger-Yau-Zaslow mirror symmetry with instanton corrections, either in the algebro-geometric framework of the Gross-Siebert program [86, 85] or by counting holomorphic discs [20, 48]. In particular, this provides a precise mirror map, which should replace the use of nondegenerate Laurent polynomials in the definition of  $M$ . However, we will stick to the “naive” version for the sake of simplicity.

There are at least two possible versions of HMS in this context:

$$(11.5) \quad D^{\text{perf}} \text{Fuk}(M) \stackrel{?}{\cong} D^b \text{Coh}_{\text{cpt}}(M^\vee),$$

$$(11.6) \quad D^{\text{perf}} \mathcal{W}(M) \stackrel{?}{\cong} D^b \text{Coh}(M^\vee).$$

These are obviously analogues of Propositions 10.9 and 10.10 (which can actually be thought of as a special case, even if a somewhat degenerate one, with  $n = 0$ ). The reservations expressed there, namely that the categories involved do not reflect the geometry appropriately, also apply to (11.5), making (11.6) the more formally satisfying statement.

REMARK 11.1. *One difference between the formulations of HMS here and in Lecture 10 is the appearance of  $D^{\text{perf}}$  instead of  $D^{\text{tw}}$ . We recall that, for any  $A_\infty$ -category  $\mathcal{A}$ ,  $D^{\text{perf}}(\mathcal{A})$  is the Karoubi completion (obtained by formally introducing retracts of idempotent endomorphisms) of  $D^{\text{tw}}(\mathcal{A})$ . This completion is unwelcome from a symplectic geometry viewpoint, since it leads us further away from the original objects (Lagrangian submanifolds). But it seems indispensable, both because the categories on the right hand side of (11.5), (11.6) are already Karoubi complete, and because of the role played by split-generation results [176, 4] in understanding the Fukaya category.*

*It is maybe worth while mentioning that, due to a general algebraic result of [195], the difference between  $D^{\text{tw}}(\mathcal{A})$  and  $D^{\text{perf}}(\mathcal{A})$  is essentially one of Grothendieck groups. Namely, if  $\mathcal{T}$  is any triangulated category which is Karoubi complete, and  $\mathcal{S} \subset \mathcal{T}$  a full triangulated subcategory whose Karoubi completion is  $\mathcal{T}$ , then  $\mathcal{S}$  can be completely reconstructed from the subgroup  $K_0(\mathcal{S}) \subset K_0(\mathcal{T})$ . In particularly simple examples, this may allow one to prove that  $D^{\text{tw}}$  and  $D^{\text{perf}}$  of the Fukaya category coincide (this was the case for the annulus).*

EXAMPLE 11.2. *Take  $P = [0, 1] \subset \mathbb{R}$ . For  $W(y) = y - 1$ , one gets*

$$(11.7) \quad M = \{(x_1, x_2) \in \mathbb{C}^2 : x_1 x_2 \neq 1\},$$



with the holomorphic volume form

$$(11.8) \quad \eta_M = \frac{dx_1 \wedge dx_2}{1 - x_1 x_2}.$$

It is important to write down  $\eta_M$  because  $H^1(M) \cong \mathbb{Z}$ , which means that there are different homotopy classes of  $C^\infty$  trivializations of  $K_M$  (leading to different  $\mathbb{Z}$ -graded Fukaya categories). A suitable choice of Kähler form is  $\omega_M = d\theta_M = d(-d^c h)$ , where the Kähler potential is

$$(11.9) \quad h = \frac{1}{4}|x_1|^2 + \frac{1}{4}|x_2|^2 + \frac{1}{2}(\log|1 - x_1 x_2|)^2.$$

On the other side of the mirror, the associated toric variety is  $N = \mathbb{C}^2$ , and  $F(w_1, w_2) = w_1 w_2$ . Hence,

$$(11.10) \quad M^\vee = \{(w_1, w_2) \in \mathbb{C}^2 : w_1 w_2 \neq 1\}$$

is isomorphic to  $M$  (a low-dimensional coincidence).

EXAMPLE 11.3. Take  $P$  to be a minimal integer simplex. This yields

$$(11.11) \quad M = \{(x_1, x_2, y_1, \dots, y_n) \in \mathbb{C}^2 \times (\mathbb{C}^*)^n : x_1 x_2 = y_1 + y_2 + \dots + y_n - 1\}.$$

This time, the mirror has quite different topology:

$$(11.12) \quad M^\vee = \{w_1 \cdots w_n \neq 1\} = \mathbb{C}^n \setminus (\mathbb{C}^*)^{n-1}.$$

EXAMPLE 11.4. Let's again start from Example 11.2, but generalize that in a different direction. Namely, take  $P^{(m)} = [0, m] \subset \mathbb{R}$ . For a suitable choice of Laurent polynomial, one gets an  $M^{(m)}$  which is a  $\mathbb{Z}/m$ -covering of the space  $M$  from (11.7). On the other side, for the unique decomposition of  $P^{(m)}$ , the associated toric variety  $N^{(m)}$  is the minimal resolution of the  $(A_{m-1})$  quotient singularity  $\mathbb{C}^2/G$ , where  $G \cong \mathbb{Z}/m$  is generated by  $\text{diag}(\zeta, \zeta^{-1})$ ,  $\zeta = e^{2\pi i/m}$ . To form  $M^{(m),\vee}$  one removes the preimage of the conic  $\{w_1 w_2 = 1\} \subset \mathbb{C}^2/G$ . This time,  $M^{(m),\vee}$  is still diffeomorphic to  $M^{(m)}$ , but carries a different complex structure (the  $m = 2$  case is the example of local mirror symmetry which appeared in Lecture 3, with the notations for the two manifolds switched around).

On the categorical level, the relation to Example 11.2 is expressed as follows. Take  $M^\vee$  as in that Example. Thanks to the McKay correspondence [100], one has

$$(11.13) \quad D^b \text{Coh}(M^{(m),\vee}) \cong D^b \text{Coh}_G(M^\vee),$$

and similarly for the compactly supported versions. On the other hand,  $G$  acts on the Fukaya category of  $M$  (by tensoring objects with flat line bundles on  $M$  whose holonomy is a root of unity), and this leads to a relationship with the Fukaya category of  $M^{(m)}$  analogous to (11.13). Making this precise requires one to admit certain immersed Lagrangian submanifolds as objects of the Fukaya category of  $M$ . We will return to this issue later on, when carrying out concrete computations.

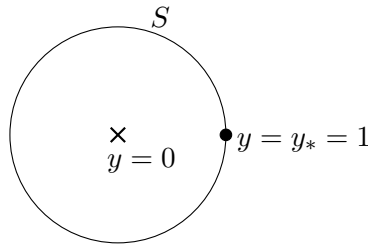


FIGURE 1.

### Lagrangian spheres

From now on, we concentrate on the simplest of the examples introduced above, namely the manifold  $M$  from Example 11.2. In this case, the conic fibration

$$(11.14) \quad y = 1 - x_1x_2 : M \longrightarrow \mathbb{C}^*$$

has a single singular point, lying over  $y_* = 1 \in \mathbb{C}^*$ . We had mentioned in Lecture 3 that there is a way to go from paths in the base to ( $S^1$ -invariant) Lagrangian surfaces in the total space. Explicitly, let's start with an immersed path  $\gamma : I \rightarrow \mathbb{C}^*$ , whose domain is either  $I = S^1$  or  $I = [0, 1]$ . In the first case,  $\gamma$  should avoid  $y_*$ ; in the second case we want  $\gamma^{-1}(y_*) = \{0, 1\}$ , and  $\gamma'(0)$  should not be a positive multiple of  $\gamma'(1)$  (but can be a negative multiple). One associates to  $\gamma$  a Lagrangian immersion, whose image is

$$(11.15) \quad \{x_1x_2 = 1 - \gamma(t) \text{ for some } t \in I, \text{ and } |x_1| = |x_2|\} \subset M.$$

The domain of this immersion is a torus if  $I = S^1$ , and a sphere if  $I = [0, 1]$ . In the first case, the pullback of  $\theta_M$  to the torus is exact if and only if

$$(11.16) \quad \int_I \gamma^*(d^c(\log |y|)^2) = 0.$$

Let's start by taking  $I = [0, 1]$ , and with  $\gamma$  that has no selfintersections other than at the endpoints, and goes once around  $y = 0$  (Figure 1). This gives rise to a Lagrangian sphere with a single self-intersection point, which we denote by  $S$ . This sphere is automatically exact, and one can equip it with some grading as well as the unique *Spin* structure. Generally speaking, immersed Lagrangian submanifolds can be accommodated into the Fukaya category only after a substantial effort [13]. However, in this particular case the sphere lifts to an embedded one in the universal cover of  $M$ , which makes the definition of Floer cohomology unproblematic [186]. Throughout this lecture, we use Floer cohomology with complex coefficients.

LEMMA 11.5.  $HF^*(S, S) \cong \Lambda^*(\mathbb{C}^2)$  is an exterior algebra with two generators.

SKETCH OF PROOF. In such a situation (an exact Lagrangian submanifold with a single self-intersection point, which disappears when lifting to the universal cover) one has [12]

$$(11.17) \quad HF^*(S, S) \cong H^*(S; \mathbb{C}) \oplus \mathbb{C}a \oplus \mathbb{C}a^*,$$

where the extra generators  $a, a^*$  have degrees that add up to  $n = \dim(S)$ , and can be chosen so that the ring structure on  $HF^*(S, S)$  satisfies

$$(11.18) \quad \begin{aligned} a a^* &= [point] \in H^n(S; \mathbb{C}), \\ a^* a &= (-1)^{|a| \cdot |a^*|} [point]. \end{aligned}$$

In our case, we only need to check that  $a, a^*$  have degree 1, and then the ring structure is completely determined by (11.18) and degree arguments. The relevant degree is the absolute Maslov index between the two branches of  $S$  at the selfintersection point. In our case, if we take  $\gamma$  to be exactly the unit circle, then  $S$  becomes special Lagrangian, which makes the computation particularly simple, compare [194].  $\square$

LEMMA 11.6 (Abouzaid). *The  $A_\infty$ -structure underlying  $HF^*(S, S)$  is formal (quasi-isomorphic to that with vanishing higher order products).*

PROOF. This is an application of the general computational method developed in [6] (some of the underlying ideas go back to [160]). The *Abouzaid model* for the cochain level structure is given by the dga

$$(11.19) \quad \mathcal{A} = \Omega^*(S) \oplus \Omega^*(U)[-1] \oplus \Omega_c^*(U)[1].$$

The first summand in (11.19) consists of (complex-valued) differential forms on  $S \cong S^2$ . For the rest, we take an open disc  $U$  and embeddings  $\iota_+, \iota_- : U \rightarrow S$  whose images are neighbourhoods of the two preimages of the selfintersection point, and which have opposite orientations. The ring structure is given by

$$(11.20) \quad \begin{aligned} (\alpha_2, \beta_2, \gamma_2)(\alpha_1, \beta_1, \gamma_1) &= (\alpha_2 \wedge \alpha_1) + (-1)^{|\gamma_1|} \iota_{-,*}(\beta_2 \wedge \gamma_1) + (-1)^{|\beta_1|} \iota_{+,*}(\gamma_2 \wedge \beta_1), \\ \iota_-^*(\alpha_2) \wedge \beta_1 + (-1)^{|\alpha_1|} \beta_2 \wedge \iota_+^*(\alpha_1), &\iota_+^*(\alpha_2) \wedge \gamma_2 + (-1)^{|\alpha_1|} \gamma_2 \wedge \iota_-^*(\alpha_1). \end{aligned}$$

Here,  $\iota_\pm^*$  is pullback, and  $\iota_{\pm,*}$  is extension of compactly supported differential forms (by zero). (11.20) satisfies the usual sign conventions for dga's (determining the signs is a bit tricky, because the last two summands in (11.19) are shifted from their natural gradings).

Take  $1 \in \Omega^0(S)$ , an  $\eta \in \Omega_c^2(U)$  which is homologically nontrivial, and a  $\xi \in \Omega^1(S)$  such that

$$(11.21) \quad \begin{aligned} \iota_+^* \xi &= \iota_-^* \xi, \\ d\xi &= \iota_{+,*} \eta + \iota_{-,*} \eta. \end{aligned}$$

Then, the elements

$$(11.22) \quad \begin{aligned} (1, 0, 0), (\iota_{+,*} \eta, 0, 0), (\iota_{-,*} \eta, 0, 0), (\xi, 0, 0), \\ (0, 1, 0), (0, \iota_\pm^* \xi, 0), (0, \eta, 0), (0, 0, \eta) \end{aligned}$$

span a quasi-isomorphic dg subalgebra of  $\mathcal{A}$ . There is a two-sided differential ideal in that subalgebra, spanned by

$$(11.23) \quad \begin{aligned} (\xi, 0, 0), (\iota_{+,*} \eta + \iota_{-,*} \eta, 0, 0), \\ (0, \iota_\pm^* \xi, 0), (0, \eta, 0). \end{aligned}$$

That ideal is acyclic, and quotienting out by it yields a quasi-isomorphic to  $H^*(\mathcal{A})$ .  $\square$

ALTERNATIVE PROOF (Sheridan). This is based on a modified version of the classification theory for  $A_\infty$ -structures from Lecture 9. Let's first see why such a modification is necessary. We generalize slightly, and write  $A = \Lambda(V)$  for the exterior algebra on an  $n$ -dimensional vector space  $V$ . Its Hochschild cohomology is computed by a version of Hochschild-Konstant-Rosenberg theorem:

$$(11.24) \quad HH^s(A, A[t]) \cong \text{Sym}^s(V^\vee) \otimes \Lambda^{s+t}(V).$$

This is bad news, since the spaces classifying first order deformations of the  $A_\infty$ -structure ( $s > 2$ ,  $s + t = 2$ ) are large. Moreover, these deformations are not obstructed to higher order. In fact, they can be explicitly realized in terms of non-commutative deformations of the Koszul dual algebra (which is the ring of functions on  $\hat{V}$ , the formal neighbourhood of the origin in  $V$ ), using Kontsevich's Formality Theorem [114].

Switching to a general discussion, consider  $A_\infty$ -algebras  $\mathcal{A}$  which come together with a bimodule homomorphism  $\phi : \mathcal{A} \rightarrow \mathcal{A}^\vee[n]$  from the diagonal bimodule to its dual (of some fixed degree  $n$ ). There is a cohomology theory which captures first order deformations of the pair  $(\mathcal{A}, \phi)$ , in a way analogous to Hochschild cohomology. The cohomology groups of that theory, here denoted just by  $K^*$  since there is no established terminology for them, sit in a long exact sequence

$$(11.25) \quad \dots \rightarrow HH^{*+n-2}(\mathcal{A}, \mathcal{A}^\vee) \rightarrow K^* \rightarrow HH^*(\mathcal{A}, \mathcal{A}) \rightarrow HH^{*+n-1}(\mathcal{A}, \mathcal{A}^\vee) \rightarrow \dots$$

The map  $K^2 \rightarrow HH^2(\mathcal{A}, \mathcal{A})$  corresponds to forgetting  $\phi$ , and just considering deformations of  $\mathcal{A}$ . The other map  $HH^n(\mathcal{A}, \mathcal{A}^\vee) \rightarrow K^2$  corresponds to deformations of  $\phi$  alone, keeping  $\mathcal{A}$  constant. The remaining piece of information is the connecting map, which we *conjecture* to be the map induced by  $\phi$ ,  $HH^*(\mathcal{A}, \mathcal{A}) \rightarrow HH^{*+n}(\mathcal{A}, \mathcal{A}^\vee)$ , composed with Connes'  $B$  operator  $HH^*(\mathcal{A}, \mathcal{A}^\vee) \rightarrow HH^{*-1}(\mathcal{A}, \mathcal{A}^\vee)$ .

We now specialize to the case when  $A$  is a graded algebra which is Frobenius of degree  $n$ , hence comes with a bimodule isomorphism  $A \rightarrow A^\vee[n]$ . The conjecture made above can easily be verified in this case. Moreover, the Hochschild cohomology groups with coefficients in  $A$  and  $A^\vee$  are obviously isomorphic. Finally, there is a bigraded refinement  $K^{*,*}$  of the groups above. One then gets a version of (11.25) of the form

$$(11.26) \quad \dots \rightarrow HH^{s-2}(A, A[t]) \rightarrow K^{s,t} \rightarrow HH^s(A, A[t]) \rightarrow HH^{s-1}(A, A[t]) \rightarrow \dots$$

where the connecting map is just the Connes operator. The classification of  $A_\infty$ -structures  $\mathcal{A}$  with  $H(\mathcal{A}) \cong A$ , together with  $A_\infty$ -bimodule maps  $\phi$  which on cohomology induce the given isomorphism, is governed by the groups

$$(11.27) \quad K^{s,t}, \quad s + t = 2, \quad s > 2,$$

in the same sense as in Propositions 9.6, 9.7.

With this general theory at hand, we return to the case of the exterior algebra  $A = \Lambda(V)$ , where (11.26) can be written as

$$(11.28) \quad \begin{aligned} \cdots \rightarrow \text{Sym}^{s-1}(V^\vee) \otimes \Lambda^{n-s-t+1}(V^\vee) \xrightarrow{d} \text{Sym}^{s-2}(V^\vee) \otimes \Lambda^{n-s-t+2}(V^\vee) \rightarrow K^{s,t} \\ \rightarrow \text{Sym}^s(V^\vee) \otimes \Lambda^{n-s-t}(V^\vee) \xrightarrow{d} \text{Sym}^{s-1}(V^\vee) \otimes \Lambda^{n-s-t+1}(V^\vee) \rightarrow \cdots \end{aligned}$$

Here, we have used a choice of nonzero element of  $\Lambda^n(V)$  to identify  $\Lambda^t(V) \cong \Lambda^{n-t}(V^\vee)$ , since that allows us to write the Connes boundary operator as the de Rham differential  $d$  (this is a well-known addition to the classical Hochschild-Kostant-Rosenberg theorem). Now let's specialize back to  $n = \dim(V) = 2$ , and to the case  $s + t = 2$  relevant to (11.27). Then (11.28) turns into

$$(11.29) \quad \begin{aligned} \cdots \rightarrow \text{Sym}^{s-1}(V^\vee) \otimes V^\vee \xrightarrow{d} \text{Sym}^{s-2}(V^\vee) \otimes \Lambda^2(V^\vee) \rightarrow K^{s,t} \\ \rightarrow \text{Sym}^s(V^\vee) \xrightarrow{d} \text{Sym}^{s-1}(V^\vee) \otimes V^\vee \rightarrow \cdots \end{aligned}$$

The first de Rham differential in (11.29) is onto; the second one is injective except in the case  $s = 0$ , which is irrelevant for us. The outcome is this: *if  $\mathcal{A}$  is any  $A_\infty$ -algebra with  $H(\mathcal{A}) \cong \Lambda(\mathbb{C}^2)$  and which comes with a quasi-isomorphism of bimodules  $\mathcal{A} \cong \mathcal{A}^\vee[-2]$ , inducing the standard duality  $\Lambda^i(\mathbb{C}^2) \cong \Lambda^{2-i}(\mathbb{C}^2)^\vee$  on cohomology, then  $\mathcal{A}$  is formal.*

The rest of the proof is again completely general: as a weak form of cyclicity for Fukaya categories, the  $A_\infty$ -structure underlying  $HF^*(S, S)$  is weakly cyclic in the sense of Example 7.8, which means that it comes with the required bimodule quasi-isomorphism (see for instance the proof of [179, Proposition 5.1]).  $\square$

REMARK 11.7. *There is a variant of this argument using the deformation theory for cyclic  $A_\infty$ -structures [150], which is based on cyclic cohomology. After collapsing the bigrading for the sake of simplicity, the cyclic cohomology of an  $n$ -dimensional exterior algebra is [124, Chapter 3]*

$$(11.30) \quad HC^*(A) \cong \ker(d : \hat{\Omega}^{n-*} \rightarrow \hat{\Omega}^{n+1-*}) \oplus \hat{H}^{n-*+2} \oplus \hat{H}^{n-*+4} \oplus \cdots,$$

where  $\hat{\Omega}^*$  is the space of differential forms in a formal neighbourhood of 0, and  $\hat{H}^*$  the cohomology of the de Rham complex of such forms. In particular,

$$(11.31) \quad HC^2(\Lambda(\mathbb{C}^2)) \cong \mathbb{C},$$

and once one re-introduces the bigrading, this one-dimensional space does not belong to the part relevant for classifying cyclic  $A_\infty$ -structures. On the other hand, the geometric application of this theory is harder, since it requires one to show that Fukaya  $A_\infty$ -structures can be made cyclic (the general issue is discussed in [70], but the case under consideration here would be less complicated).

From the point of view of (11.5), it is plausible to expect that  $S$  should correspond to the skyscraper sheaf at the origin  $(w_1, w_2) = (0, 0)$  in the mirror (11.10) (there is obviously an ambiguity in setting up the mirror map, so this is heuristic reasoning about what should happen for a particular choice of map). This structure sheaf  $E$  satisfies  $\text{Ext}_{M^\vee}^*(E, E) \cong$

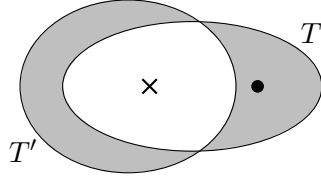


FIGURE 2.

$\Lambda^*(\mathbb{C}^2)$ , and one can use explicit Koszul resolutions to show that the underlying cochain level structure is formal (alternatively, the second proof given here also goes through in the algebro-geometric context).

### Lagrangian tori

We want to consider two exact Lagrangian tori  $T, T' \subset M$ , which are related to the Clifford torus and Chekanov torus in affine space. This follows closely Auroux' paper [20], even though we avoid the use of wall-crossing formulae (making those rigorous requires more advanced pseudo-holomorphic curve techniques); see also [47].

The first torus  $T \subset M$  is obtained from an embedded loop in the base  $\mathbb{C}^*$  which winds once around the origin and around  $y_* = 1$ . The second torus  $T'$  is defined in the same way, but using a loop which only winds around the origin (see Figure 2). Because the loops have to satisfy the exactness condition (11.16), they must intersect. Both  $T$  and  $T'$  can be equipped with gradings, and with their trivial (compatible with the standard trivialization of the tangent bundle of the torus) *Spin* structures. Finally, we choose flat complex line bundles  $\xi, \xi'$  on them, which are classified by their holonomies  $(s_0, s_1), (s'_0, s'_1) \in (\mathbb{C}^*)^2$  (this involves conventions that will be explained below).

PROPOSITION 11.8. *HF<sup>\*</sup>(T, T') is nonzero if and only if*

$$(11.32) \quad \begin{aligned} s'_0 &= s_0, \\ s'_1 &= s_1(1 + s_0). \end{aligned}$$

*In fact, if this equation holds, then (up to a shift) T and T' are quasi-isomorphic objects of Fuk(M).*

PROOF. One can arrange (see again Figure 2) that  $T$  and  $T'$  intersect cleanly in two circles,  $C_+$  and  $C_-$ . Then (assuming that the gradings have been chosen appropriately) an application of basic Morse-Bott methods yields a long exact sequence

$$(11.33) \quad \cdots \rightarrow HF^*(T, T') \rightarrow H^*(C_+; \text{Hom}(\xi|_{C_+}, \xi'|_{C_+})) \rightarrow H^*(C_-; \text{Hom}(\xi|_{C_-}, \xi'|_{C_-})) \rightarrow \cdots$$

When introducing the holonomies, we have implicitly assumed that  $s_0$  is the holonomy around a circle of  $T$  lying in the fibre of  $M \rightarrow \mathbb{C}^*$ , and similarly for  $s'_0$ . Hence, the holonomy of  $\text{Hom}(\xi|_{C_\pm}, \xi'|_{C_\pm})$  is  $s'_0/s_0$ . If that is nontrivial, the cohomology of  $C_\pm$  with these twisted

coefficients vanishes, which shows that the condition  $s'_0 = s_0$  is necessary. We will assume from now that this is satisfied, so that the two last terms in (11.33) can be identified with the ordinary cohomology groups  $H^*(C_{\pm}; \mathbb{C}) \cong H^*(S^1; \mathbb{C})$ .

The map between these groups (11.33) is given by counting Floer trajectories that project to one of the two shaded regions in Figure 2. One region contains no critical point, and (with suitable conventions) it contributes

$$(11.34) \quad -id : H^*(C_+; \mathbb{C}) \rightarrow H^*(C_-; \mathbb{C}).$$

The contribution from the other region is a map that occurs in the long exact sequence [173]. The detailed analysis in [176, Section 17] shows that there are two homotopy classes of Floer trajectories, whose boundary loops differ from each other by going once around the fibre circle. Assuming as before that  $s_0 = s'_0$ , the resulting contribution is

$$(11.35) \quad (s'_1/s_1 + s_0 s'_1/s_1) id : H^*(C_+; \mathbb{C}) \rightarrow H^*(C_-; \mathbb{C}).$$

(This vanishes if and only if  $s_0 = -1$ ; this choice is equivalent to equipping our tori with trivial flat bundles but changing their *Spin* structures, which is what one does in the context of the long exact sequence). Returning to our application, the Floer cohomology in (11.33) is nonzero if and only if (11.34) and (11.35) add up to zero, which is exactly the second condition in (11.32).

What we have shown so far is that if (11.32) is not satisfied,  $HF^*(T, T')$  vanishes; and if it satisfies,  $HF^*(T, T')$  is isomorphic (as a graded vector space) to the ordinary cohomology of a two-torus. In the second case, the same analysis of moduli spaces shows that the “cap product” maps

$$(11.36) \quad \begin{aligned} H^2(T'; \mathbb{C}) \otimes HF^0(T, T') &\longrightarrow HF^2(T, T'), \\ HF^0(T, T') \otimes H^2(T; \mathbb{C}) &\longrightarrow HF^2(T, T') \end{aligned}$$

are nonzero. For general reasons of Poincaré duality in Floer cohomology, these maps are dual to the triangle products

$$(11.37) \quad \begin{aligned} HF^0(T', T) \otimes HF^0(T, T') &\longrightarrow HF^0(T, T) \cong H^0(T; \mathbb{C}), \\ HF^0(T, T') \otimes HF^0(T', T) &\longrightarrow HF^0(T', T') \cong H^0(T'; \mathbb{C}). \end{aligned}$$

Since all the Floer cohomology groups appearing in (11.37) are one-dimensional, it follows that  $T$  and  $T'$  are indeed isomorphic in  $H^0(Fuk(M))$ .  $\square$

From the point of mirror symmetry, the putative interpretation is as follows.  $T$  and  $T'$  correspond to the structure sheaves of the points

$$(11.38) \quad \begin{aligned} (w_0, w_1) &= ((s_0 + 1)s_1, s_1^{-1}), \\ (w_0, w_1) &= (s'_1, (s'_0 + 1)/s'_1). \end{aligned}$$

(One can take this as the starting point for a SYZ construction of the mirror, see [47]). The formulae (11.38) define two charts  $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow M^V$ , which are precisely related by the coordinate transformation (11.32). Neither chart contains the origin, and that predicts the following easily checked statement:

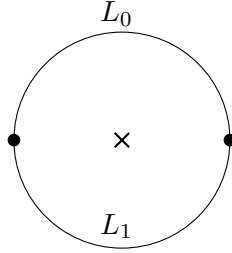


FIGURE 3.

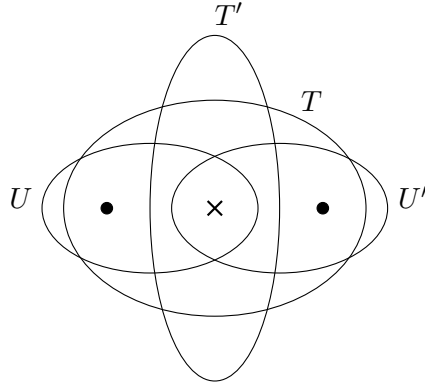


FIGURE 4.

PROPOSITION 11.9.  $HF^*(T, S)$  and  $HF^*(T', S)$  both vanish, for any choice of line bundles on  $T, T'$ .  $\square$

Conversely, one can use mirror symmetry to classify objects of the Fukaya category with a given behaviour. A notable example of this process in the literature is [9]. For our particular example, one has:

PROPOSITION 11.10. *Suppose that the Homological Mirror Symmetry conjecture holds in the form (11.6) for (11.7), and that the equivalence of categories involved does have the properties listed in the previous discussion:*

- *The Lagrangian sphere  $S$  corresponds to the structure sheaf of the origin in  $M^\vee$ ;*
- *The tori  $T$  and  $T'$  correspond to the structure sheaves of points (11.38).*

*Then, if  $L \subset M$  is any exact Lagrangian torus which admits a grading,  $[L] \in H_2(M)$  is nonzero and primitive.*

PROOF. One has a ring isomorphism

$$(11.39) \quad H^*(\text{hom}_{\text{Fuk}(M)}(L, L)) = HF^*(L, L) \cong H^*(L; \mathbb{C}).$$



Let  $X$  be the putative mirror of  $L$  under (11.6), which is an object of  $D^b \text{Coh}_{cpt}(M^\vee)$  whose endomorphism ring must match (11.39). Let  $H^*$  be the cohomology sheaves of  $X$ .

*Claim 2. There is only one nonzero  $H^k$ .*

There is a spectral sequence

$$(11.40) \quad E_2^{pq} = \bigoplus_r \text{Ext}_{M^\vee}^p(H^r, H^{r+q}) \implies \text{Hom}_{D^b \text{Coh}(M^\vee)}^*(X, X).$$

For dimension reasons, each group  $\text{Ext}_{M^\vee}^1(H^k, H^k)$  survives to the  $E_\infty$  page. On the other hand, if  $H^k$  is nonzero, that  $\text{Ext}^1$  group has rank at least 2 (corresponding to the first order deformation which move the point(s) where  $H^k$  is located). Comparison with the degree 1 part of (11.39) then leads to the desired conclusion.

*Claim 3. Up to a shift,  $X$  is isomorphic to the structure sheaf of a point  $w \in M^\vee$ .*

The previous step that, after possibly shifting the grading, we may assume that  $X$  is a single sheaf. It is also indecomposable, hence supported at a single point. The fact that  $\text{Hom}_{M^\vee}(X, X)$  is one-dimensional then leads directly to the desired conclusion.

By the rest of the assumptions in the Proposition,  $X$  is isomorphic to the mirror of  $S$ ,  $T$ , or  $T'$ . Hence  $L$  is isomorphic to  $S$ ,  $T$  or  $T'$  in  $H^0(\text{Fuk}(M))$ . It is a general result that two such isomorphic objects must have the same class in  $H_2(M; \mathbb{C})$ .  $\square$

There are further questions, which don't have obvious answers even if one assumes suitable forms of mirror symmetry (notably, whether there is an exact Lagrangian torus which is quasi-isomorphic to  $S$  in the Fukaya category; and whether there are Lagrangian tori which are exact but do not admit gradings).

REMARK 11.11 (Keating). *It is worth while to consider briefly one possible generalization, namely the  $m = 2$  case of Example 11.4. As already discussed in Lecture 3, that manifold is a conic fibration over  $\mathbb{C}^*$  with two singular fibres. The paths in Figure 3 yield the two Lagrangian spheres  $S_0, S_1$  from (3.47), and the loops from Figure 4 describe four Lagrangian tori  $T, T', U, U'$ , which however are not geometrically independent: up to Hamiltonian isotopy, two of them differ by a Dehn twist*

$$(11.41) \quad U' \simeq \tau_{S_0}(U).$$

*Under mirror symmetry, these tori should correspond to three different rational maps from  $(\mathbb{C}^*)^2$  to the mirror. Here, "different" means that the coordinate transformation between any two of them is not of monomial form. For generic choices of flat line bundles (but not for all choices) one has  $HF^*(S_0, U) = 0$ , in which case  $U$  and  $\tau_{S_0}(U)$  are quasi-isomorphic. Because of this and (11.41), the rational maps corresponding to  $Z$  and  $Z'$  won't be different in the previously explained sense.*

### Wrapped Floer cohomology

In [149], Pascaleff considered a certain non-compact Lagrangian submanifold  $\mathbb{R}^2 \cong O \subset M$ , and computed its wrapped Floer cohomology, which is the ring of cohomology level endomorphisms in the wrapped Fukaya category:

$$(11.42) \quad HW^*(O, O) = H^*(\text{hom}_{\mathcal{W}(M)}(O, O)).$$

The computation uses methods similar to Proposition 11.8 (but is more complicated). The outcome is:

THEOREM 11.12 ([149, Theorem 1.3]). *There is a ring isomorphism*

$$(11.43) \quad HW^*(O, O) \cong \mathbb{C}[M^\vee].$$

The obvious mirror symmetry explanation is that under (11.6),  $O$  should correspond to a locally free sheaf, let's say the structure sheaf of  $M^\vee$ . Note that the underlying  $A_\infty$ -structure  $\mathcal{A} = \text{hom}_{\mathcal{W}(M)}(O, O)$  is necessarily formal, since its cohomology is concentrated in degree zero. One can therefore construct an  $A_\infty$ -functor

$$(11.44) \quad \text{hom}_{\mathcal{W}(M)}(O, -) : \text{Fuk}(M) \longrightarrow \mathcal{A}^{\text{mod}} \cong \mathbb{C}[M^\vee]^{\text{mod}},$$

where the right side is the derived category of quasi-coherent sheaves on  $M^\vee$ . This is of course one way to set up the desired mirror functor, with the main remaining task being to show that it is full and faithful.

One can show that the Floer cohomology of  $O$  with any of the Lagrangian submanifolds  $S, T, T'$  is one-dimensional. Hence, the image of each of these submanifolds under (11.44) would indeed be the structure sheaf of a point of  $M^\vee$ . If one used (11.44) as a mirror functor, one could then rigorously verify the previously stated formulae relating choices of flat bundles with coordinates on the mirror, by computing the multiplicative structures on Floer cohomology.

There is an alternative approach to the computation of wrapped Floer cohomology groups (and the wrapped Fukaya categories) in  $M$ , which has the following two ingredients: first, general conjectures on wrapped Fukaya categories, from [178, 174]; and second, mirror symmetry considerations from [20, Section 5], made concrete in this case in [149] (see also [49] for a discussion of the implications of these ideas, which is similar to the following one). Consider

$$(11.45) \quad \begin{aligned} Q : M &\longrightarrow \mathbb{C}, \\ Q(x_1, x_2) &= x_1 + \frac{x_2^2}{x_1 x_2 - 1}. \end{aligned}$$

This is a Lefschetz fibration (has only nondegenerate critical points, and no singularities at infinity). In fact, if we consider a chart  $(x_1, x_2) = (r_1 + r_2, r_2^{-1})$  for  $(r_1, r_2) \in (\mathbb{C}^*)^2$  (this is as in (11.38) but with the role of  $M$  and its mirror exchanged), then

$$(11.46) \quad Q(r_1, r_2) = r_1 + r_2 + \frac{1}{r_1 r_2}$$

is the well-known toric mirror to  $\mathbb{C}P^2$ . The smooth fibres of (11.46) are three-punctured elliptic curves, and adding  $M \setminus (\mathbb{C}^*)^2 \cong \mathbb{C}$  corresponds to filling in one of the punctures. Homological mirror symmetry for (11.46) was first considered in [170] (see also [23]), where it was shown that the Fukaya category  $Fuk(Q)$  of this Lefschetz fibration is derived equivalent to the derived category of coherent sheaves on  $\mathbb{C}P^2$ . A crucial observation from [22] is that this result remains unchanged under (partial or full) fibrewise compactification.

We have not previously mentioned Fukaya categories of Lefschetz fibrations, and will not engage into a detailed discussion now. For our immediate purpose, it is enough to know that the non-compact Lagrangian submanifold  $O$  mentioned above can be thought of as a Lefschetz thimble for  $Q$ , hence an object of  $Fuk(Q)$ . The general conjecture from [174] says that (11.42) can be computed as follows:

$$(11.47) \quad HW^*(O, O) \cong \varinjlim_p H^*(\mathcal{S}^p(O), O).$$

Here,  $\mathcal{S}$  is the Serre functor. The direct limit in (11.47) uses a natural transformation from the Serre functor to the identity, which is an additional piece of Floer-theoretic information obtained from the geometry of the Lefschetz fibration. For (11.45), this additional piece of information can be computed either using the general theory of [181], or else directly geometrically [49]; and then (11.43) will follow.

This approach relies on (11.47), which we have presented as a conjecture, but in fact considerable progress has been made towards establishing it. On the level of graded vector spaces, one can derive that identity from Symplectic Field Theory surgery formulae [36, Appendix], and this approach can at least in principle be extended to include ring structures [35]. On the other hand, there is work in progress towards a more direct proof, which will yield results on the level of  $A_\infty$ -structures [7].



## Symplectic cohomology

We have already encountered symplectic cohomology  $SH^*(M)$  as an invariant of a Liouville-type symplectic manifold  $M$  (in Lecture 6). At this point, we want to dig a little deeper into its structure and significance, specifically in relation with Fukaya categories. This has been the focus of much recent attention [171, 4, 72, 158], in particular as regards the wrapped version of the Fukaya category.

To see some of the motivation for this, suppose for simplicity that  $M$  is *Spin*, and consider  $M \times \bar{M}$ , where the sign of the symplectic form on the second factor has been reversed. The diagonal  $\Delta_M \subset M \times \bar{M}$  can be made into an object of the wrapped Fukaya category, and one then has a ring isomorphism

$$(12.1) \quad SH^*(M) \cong HW^*(\Delta_M, \Delta_M).$$

Because of the formal analogy between the right hand side and Hochschild cohomology, one is led to the following conjecture (which no longer depends on the *Spin* assumption):

$$(12.2) \quad SH^*(M) \stackrel{?}{\cong} HH^*(\mathcal{W}(M), \mathcal{W}(M)).$$

See [72] for further discussion. Of course, there is a similar conjecture for the conjecture for closed symplectic manifolds, involving quantum cohomology instead of symplectic cohomology [113], but that seems unlikely to hold in general, whereas (12.2) is plausible at least for Stein manifolds (with their natural Liouville structures; there are Liouville manifolds which are not Stein [132], but little is known about their Fukaya categories). Here, we do not try to establish any connection at this level of depth; instead, the discussion is limited to describing the *open-closed string maps* which link symplectic cohomology and Lagrangian Floer cohomology, and some of their properties.

*Acknowledgments.* I would like to thank Sheel Ganatra and Yiannis Vlassopoulos for very useful discussions about higher Hochschild cohomology and open Calabi-Yau structures; their unpublished writings [72, 117] have strongly influenced the exposition here.

### Geometric setup

To harmonize the formalism here with that in Lectures 10–11, we work with Liouville manifolds  $(M^{2n}, \omega_M = d\theta_M)$  which have vanishing first Chern class, and choose a trivialization of their canonical bundle  $K_M = \Lambda_{\mathbb{C}}^n TM$ , which means a  $C^\infty$  complex volume form  $\eta_M$  for

some compatible almost complex structure. This makes  $SH^*(M)$  into a  $\mathbb{Z}$ -graded vector space (over an arbitrary coefficient field  $\mathbb{K}$ ), refining the  $\mathbb{Z}/2$ -grading from Lecture 6.

REMARK 12.1. *As already mentioned in Lecture 6, there is a canonical decomposition*

$$(12.3) \quad SH^*(M) = \bigoplus_c SH^*(M)^{(c)}$$

*indexed by connected components  $c$  of the free loop space  $\mathcal{L}M$ , or equivalently conjugacy classes in  $\pi_1(M)$ . Suppose that we change the trivialization of  $K_M$  by multiplying it with some function  $M \rightarrow \mathbb{C}^*$ , which represents a class  $\gamma \in H^1(M; \mathbb{Z})$ . The effect on symplectic cohomology is to shift the grading of  $SH^*(M)^{(c)}$  by  $2\langle \gamma, c \rangle$ . In particular, if we restrict to the trivial summand (as is often done elsewhere in the literature), the choice of  $\eta_M$  is irrelevant.*

REMARK 12.2. *In analogy with what we've done for fixed point Floer cohomology on closed manifolds (in Lecture 5), one can define symplectic cohomology with coefficients in a flat  $\mathbb{K}$ -vector bundle over the free loop space  $\mathcal{L}M$ . For simplicity, we will not make use of this, but it would be convenient at several points; for instance, if one wished to remove the assumptions of Spinness (or even orientability) from (6.7).*

Finally, we define symplectic homology  $SH_*(M)$  to be the dual of  $SH^*(M)$  (this reverses the situation for ordinary cohomology and homology). More precisely, we would like to consider  $SH_*(M)$  as coming from a chain complex with differential of degree  $+1$ , hence adjust the gradings so that

$$(12.4) \quad SH_*(M) \cong SH^{-*}(M)^\vee.$$

### Batalin-Vilkovisky algebras

DEFINITION 12.3. *The operad of framed little (two-dimensional) discs [75, 164] is the following operad  $\{FD_d\}_{d \geq 0}$  of topological spaces. Let  $D \subset \mathbb{C}$  be the closed unit disc. A point of  $FD_d$  is given by a  $d$ -tuple of embeddings*

$$(12.5) \quad \begin{aligned} \epsilon_1, \dots, \epsilon_d : D &\rightarrow D \setminus \partial D, \\ \epsilon_k(z) &= a_k z + b_k \quad (a_k \in \mathbb{C}^*, b_k \in \mathbb{C}), \end{aligned}$$

*which have pairwise disjoint images. The action of the symmetric group on  $FD_d$  permutes the embeddings, and the maps*

$$(12.6) \quad \circ_i : FD_q \times FD_p \longrightarrow FD_{p+q-1}$$

*are obtained by composing embeddings (sticking a big disc inside a smaller one).*

Each  $FD_d$  is an open complex manifold of (complex) dimension  $2d$ , homotopy equivalent to the “framed ordered configuration space”  $(S^1)^d \times \text{Conf}_d^{\text{ord}}(\mathbb{C})$ . An explicit description of the homology operad  $\{H_{-*}(FD_d; \mathbb{K})\}_{d \geq 0}$  was given in [75], the outcome being that algebras over it are identified with BV (Batalin-Vilkovisky) algebras. To make this explicit, let  $S^*$  be a graded vector space which is an algebra over the homology operad.  $FD_0$  is a point, and

the generator of  $H_0(FD_d; \mathbb{K}) \cong \mathbb{K}$  gives a distinguished element, the unit

$$(12.7) \quad 1 \in S^0.$$

Next,  $FD_1 \simeq S^1$ . We will always assume that  $[point] \in H_0(FD_1)$  acts as the identity on  $S$ . In the next degree, a (choice of) generator of  $H_1(FD_1)$  gives rise to the *Batalin-Vilkovisky* or loop rotation operator

$$(12.8) \quad \Delta : S^* \longrightarrow S^*[-1],$$

which satisfies

$$(12.9) \quad \begin{aligned} \Delta(1) &= 0 && \text{because } H_1(FD_0) = 0, \\ \Delta^2 &= 0 && \text{because } H_2(FD_1) = 0. \end{aligned}$$

Next,  $FD_2 \simeq (S^1)^3$ . We get a new generator in degree 0, the product

$$(12.10) \quad S^* \otimes S^* \longrightarrow S^*,$$

which is graded commutative, and for which (12.7) is a two-sided unit. All the higher homology of  $FD_2$  is generated by  $H_0(FD_2)$  and  $H_1(FD_1)$  under compositions (12.6). One noteworthy composed operation is the degree  $-1$  bracket

$$(12.11) \quad [b_2, b_1] = -(-1)^{|b_2|} \Delta(b_2 b_1) + (-1)^{|b_2|} (\Delta b_2) b_1 + b_2 (\Delta b_1),$$

which (as a consequence of the previously stated properties) is graded antisymmetric on  $S^*[1]$ , and satisfies  $[1, \cdot] = [\cdot, 1] = 0$ . From  $FD_3$ , we do not get any additional generators. However, in degree 0 we get an additional relation, namely associativity of (12.10). And in degree 1, we get the seven-term relation

$$(12.12) \quad \begin{aligned} \Delta(b_3 b_2 b_1) &= \Delta(b_3 b_2) b_1 + (-1)^{|b_3|} b_3 \Delta(b_2 b_1) + (-1)^{(|b_3|-1)|b_2|} b_2 \Delta(b_3 b_1) \\ &\quad - (\Delta b_3) b_2 b_1 - (-1)^{|b_3|} b_3 (\Delta b_2) b_1 - (-1)^{|b_3|+|b_2|} b_3 b_2 (\Delta b_1). \end{aligned}$$

In terms of (12.11), one can interpret (12.12) as saying that  $[b, \cdot]$  acts as a derivation of degree  $|b| - 1$  on the algebra  $S^*$ :

$$(12.13) \quad [b_3, b_2 b_1] = [b_3, b_2] b_1 + (-1)^{(|b_3|-1)|b_2|} b_2 [b_3, b_1].$$

Vanishing of  $\Delta^2$  and (12.12) also imply that the bracket satisfies the graded Jacobi identity on  $S^*[1]$ . The main theorem from [75] says that the generators and relations introduced above form a complete list.

This general discussion is relevant to us for the following reason, to be explained in more detail later:

**THEOREM 12.4.**  *$SH^*(M)$  carries a natural structure of an algebra over  $\{H_{-*}(FD_d)\}_{d \geq 0}$ , or equivalently of a BV algebra.*

While the unital commutative ring structure already played an important role in Lecture 6, the BV operator (12.8) is a new ingredient in our discussion, whose importance will increase in the future. Theorem 12.4 appears in [175]; a more detailed construction can be found in [157].

EXAMPLE 12.5. Consider the case of cotangent bundles  $M = T^*L$  (Example 6.9, and with the same assumptions on  $L$ ). Choose a complex volume form  $\eta_M$  whose restriction to the zero-section is real (this fixes  $\eta_M$  up to homotopy). Then, the isomorphism (6.7) will respect the integer gradings on both sides. In terms of that isomorphism, the BV algebra structure of symplectic cohomology has an interpretation in terms of string topology [50]. More precisely, the BV operator corresponds to the homomorphism induced by

$$(12.14) \quad \begin{aligned} S^1 \times \mathcal{L}L &\longrightarrow \mathcal{L}L, \\ (\tau, u) &\longmapsto u(\cdot - \tau). \end{aligned}$$

This can be proved using the same techniques as (6.7) itself (the possible exception is the approach in [202], which breaks circular symmetry, hence seems less suitable for this purpose). The more difficult part of the relationship, namely that the product corresponds to the string product, was proved in [2].

### The mirror symmetry viewpoint

Let's temporarily specialize to the simplest instance of Example 12.5, namely  $M = \mathbb{R} \times S^1 = T^*S^1$ . Write

$$(12.15) \quad SH^*(M) \cong \mathbb{K}[w, w^{-1}, \partial_w],$$

where the generator  $\partial_w$  has formal degree 1, and the whole expression is a commutative graded algebra (with unique relation  $w w^{-1} = 1$ ). We find it convenient to choose the degree one part of (12.15) in a slightly non-obvious way: namely, so that the decomposition

$$(12.16) \quad SH^*(M) \cong \bigoplus_k SH^*(M)^{(k)}$$

into summands parametrized by homotopy classes of loops, which means by  $k \in \pi_0(\mathcal{L}M) \cong \pi_1(M) \cong \mathbb{Z}$ , is given by eigenvalues of the Euler (or homogeneous rescaling) operator  $w\partial_w$ . Concretely, this means that

$$(12.17) \quad SH^*(M)^{(k)} \cong \mathbb{K}w^k \oplus \mathbb{K}w^{k+1}\partial_w.$$

With this taken into account, the BV operator satisfies

$$(12.18) \quad \begin{aligned} \Delta(w^k \partial_w) &= (k-1)w^{k-1}, \\ \Delta(w^k) &= 0. \end{aligned}$$

Geometrically (for  $k \neq 0$ ), this reflects the fact that the Reeb orbit representing  $w^{k-1}$  and  $w^k \partial_w$  is a  $|k-1|$ -fold multiple. The associated bracket (12.11) satisfies

$$(12.19) \quad \begin{aligned} [w^k \partial_w, w^l \partial_w] &= (l-k)w^{k-l-1} \partial_w, \\ [w^k \partial_w, w^l] &= l w^{k+l-1} \partial_w. \end{aligned}$$

Thinking of the mirror  $M^\vee = \mathbb{G}_m$  (as in Propositions 10.9 and 10.10), one finds that  $SH^*(M)$  is isomorphic to the Hochschild cohomology of that variety, which in this (smooth affine)



case is

$$(12.20) \quad HH^*(M^\vee, M^\vee) \cong H^0(M^\vee, \Lambda^*T(M^\vee)).$$

The product and bracket on  $SH^*(M)$  correspond to the standard product and bracket of polyvector fields. These can also be viewed as the Gerstenhaber algebra structure defined on (12.20) by identifying it with the Hochschild cohomology of coherent sheaves on  $M^\vee$  (in the general categorical sense of Lecture 9). The situation with (12.8) is a little more interesting: the corresponding operation on (12.20) is given by

$$(12.21) \quad H^0(M^\vee, \Lambda^*T(M^\vee)) \xrightarrow{\cong} H^0(M^\vee, \Omega_{M^\vee}^{n-*}) \xrightarrow{d} H^0(M^\vee, \Omega_{M^\vee}^{n+1-*}) \xrightarrow{\cong} H^0(M^\vee, \Lambda^{*-1}T(M^\vee)).$$

Here, the two isomorphisms are determined by the complex volume form

$$(12.22) \quad \eta_{M^\vee} = \frac{dw}{w},$$

and the remaining map is the de Rham differential. For instance, the action of (12.21) on vector fields can be written as

$$(12.23) \quad Z \mapsto \frac{d(i_Z \eta_{M^\vee})}{\eta_{M^\vee}} = \frac{L_Z \eta_{M^\vee}}{\eta_{M^\vee}}$$

and this indeed reproduces (12.18).

It seems plausible to expect the same relation between symplectic cohomology and Hochschild cohomology in a general context of local mirror symmetry (Lectures 3 and 11), but this conjecture has not been tested exhaustively so far. Maybe the most interesting point is that the BV operator on  $SH^*(M)$  contains information about the choice of complex volume form on the  $B$ -model (mirror) manifold  $M^\vee$ .

### Relation with the Fukaya category

There are canonical *open-closed string maps* [171]

$$(12.24) \quad SH^*(M) \longrightarrow HH^*(Fuk(M), Fuk(M)),$$

$$(12.25) \quad HH_{*-n}(Fuk(M), Fuk(M)) \longrightarrow SH_*(M).$$

The first of these is a map of (unital) Gerstenhaber algebras, which means that it is compatible with the ring structure and the bracket. The second map sends the Connes operator on Hochschild homology to the (dual of the) BV operator. One can put both structures under the same hat, as follows. As already mentioned in (the second proof of) Lemma 11.6,  $\mathcal{A} = Fuk(M)$  is weakly cyclic (in the terminology of Example 7.8). This induces an isomorphism

$$(12.26) \quad HH^*(\mathcal{A}, \mathcal{A}) \cong HH^{*-n}(\mathcal{A}, \mathcal{A}^\vee) \cong HH_{*-n}(\mathcal{A}, \mathcal{A})^\vee,$$

where the second isomorphism can be read off directly from the underlying chain complexes. With this isomorphism taken into account, (12.25) turns into the dual of (12.24). Equivalently, one can use (12.26) to equip Hochschild cohomology with a counterpart of the Connes

operator, following the algebro-geometric model of (12.21), see [199]; and then (12.24) becomes compatible with all the operations on symplectic cohomology.

EXAMPLE 12.6. Take  $M = \mathbb{R} \times S^1$  as before, and assume that  $\mathbb{K}$  is algebraically closed. As a consequence of (10.29) we have

$$(12.27) \quad HH^*(Fuk(M), Fuk(M)) \cong \prod_{a \in \mathbb{K}^*} HH^*(\Lambda, \Lambda).$$

$\Lambda$  is an exterior algebra in one variable, whose Hochschild cohomology can be written as

$$(12.28) \quad HH^*(\Lambda, \Lambda) \cong \mathbb{C}[[v, \partial_v]].$$

The map

$$(12.29) \quad SH^*(M) = \mathbb{K}[w, w^{-1}, \partial_w] \longrightarrow HH^*(Fuk(M), Fuk(M)) \cong \prod_{a \in \mathbb{K}^*} \mathbb{C}[[v, \partial_v]]$$

consists of taking the Taylor expansions of a Laurent polynomial around all possible points  $a \in \mathbb{K}^*$  at the same time. In particular, it is far from being an isomorphism.

EXAMPLE 12.7. Take  $M = T^*S^n$  for some  $n > 1$ . In that case, it is easy to see that  $Fuk(M)^{perf} \cong \mathcal{A}^{perf}$ , where  $\mathcal{A}$  is the subcategory having only the zero-section as an object (hence quasi-isomorphic to the algebra  $\mathbb{K}[\theta]/\theta^2$ , where  $|\theta| = n$ ). Using techniques from [5], one can prove that (12.24) is an isomorphism.

Open-closed string maps have a couple of straightforward applications. Let  $L \subset M$  be an object of  $Fuk(M)$  (which means, a closed Lagrangian submanifold which is exact, graded, *Spin*, and carries a flat  $\mathbb{K}$ -bundle  $\xi_L$ ). The first order term of (12.24) yields a homomorphism of unital rings  $SH^*(M) \rightarrow HF^*(L, L)$ , which fits into a commutative diagram

$$(12.30) \quad \begin{array}{ccc} SH^*(M) & \longrightarrow & HF^*(L, L) \\ \uparrow & & \uparrow \cong \\ H^*(M; \mathbb{K}) & \longrightarrow & H^*(L; Hom(\xi_L, \xi_L)). \end{array}$$

The left hand  $\uparrow$  is the map already mentioned in (6.10), and the bottom  $\rightarrow$  is the restriction map to  $L \subset M$  in cohomology, combined with the inclusion of the trivial line bundle  $\mathbb{K} \cdot id \subset Hom(\xi_L, \xi_L)$ . In particular, if a class  $x \in H^n(M; \mathbb{K})$  has the property that  $\langle x, [L] \rangle \neq 0$ , then the image of  $x$  in  $SH^n(M)$  is nonzero (one can give an alternative proof of this using Viterbo functoriality, but that is effectively a much more difficult tool). Dually, we get a map  $H_*(L; Hom(\xi_L, \xi_L)) \rightarrow SH_{-*}(M)$ , and in particular can use the fundamental homology class on the left hand side to define an element

$$(12.31) \quad [L]_{SH} \in SH_{-n}(M).$$

Under the map  $SH_*(M) \rightarrow H_{-*}(M; \mathbb{K})$  dual to (6.10), this becomes the ordinary fundamental class  $[L]$ , hence one should think of (12.31) as a refined homology class. Equivalently, it is simply the image of the Hochschild homology class of the object  $L$  under (12.25). The latter interpretation has the advantage that it can be generalized to objects of  $Fuk(M)^{perf}$ .

EXAMPLE 12.8. Suppose that  $M = \mathbb{R} \times S^1$ , and that  $L = \{0\} \times S^1$  equipped with a flat  $\mathbb{K}$ -vector bundle whose holonomy is  $A$ . If we use (12.15) and (12.22) to identify

$$(12.32) \quad SH_{-1}(M) \cong \mathbb{C}[[w, w^{-1}]],$$

then

$$(12.33) \quad \pm[L]_{SH} = \sum_{k \in \mathbb{Z}} \text{Tr}(A^{-k})w^k = \sum_i m_i \delta_{a_i}(w).$$

where the sign depends on the orientation. In the second equality, we are assuming that  $\mathbb{K}$  is algebraically closed, and that  $A$  has (generalized) eigenvalues  $a_i$  with multiplicity  $m_i$ . The formal Laurent expansion of the Dirac  $\delta$ -function at the point  $w = a$  is  $\delta_a(w) = \sum_{k \in \mathbb{Z}} (w/a)^k$ , whence the equality. The outcome matches expectations from mirror symmetry (compare with the discussion preceding Proposition 10.9). Taking the image under  $SH_{-1}(M) \rightarrow H_1(M)$  corresponds to retaining only the constant term of (12.33).

### Relation with the wrapped Fukaya category

For this version of the Fukaya category, one again has natural open-closed string maps:

$$(12.34) \quad SH^*(M) \longrightarrow HH^*(\mathcal{W}(M), \mathcal{W}(M)),$$

$$(12.35) \quad HH_{*-n}(\mathcal{W}(M), \mathcal{W}(M)) \longrightarrow SH^*(M).$$

The second map has a slightly different form than in (12.25), and that has nontrivial implications. For instance, if one starts with an object  $L$  of  $\mathcal{W}(M)$ , we again get a class

$$(12.36) \quad [L]_{SH}^\vee \in SH^n(M),$$

but that one no longer contains much interesting information: it is the image of the Poincaré dual of  $L$  in  $H^n(M; \mathbb{K})$  (which exists even if  $L$  is not compact, as long as it is properly embedded) under the map  $H^*(M; \mathbb{K}) \rightarrow SH^*(M)$ . However, that does not mean that the entire map (12.35) factors through  $H^*(M; \mathbb{K})$ .

REMARK 12.9. One can start more generally with an object of  $\mathcal{W}(M)^{perf}$ , and use (12.35) together with the Morita invariance of Hochschild homology to associate to this object a class in  $SH^n(M)$ . There seems to be no reason for such classes to come from ordinary cohomology.

One of the interesting aspects of (12.34), (12.35) is that one can compose the two maps to get a homomorphism

$$(12.37) \quad HH_{*-n}(\mathcal{W}(M), \mathcal{W}(M)) \longrightarrow HH^*(\mathcal{W}(M), \mathcal{W}(M)).$$

This map has an interpretation purely in terms of additional structures on  $\mathcal{W}(M)$ , which we will now explain. Interested readers should also consult [72], which in particular raises the question of whether (12.34), (12.35), and (12.37) are isomorphisms (for the latter map, this would be an “open Calabi-Yau” property of the wrapped Fukaya category, in the sense of [201, 76]).

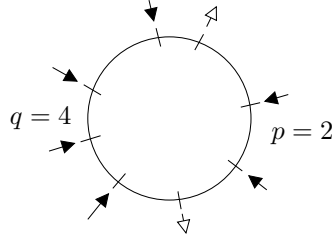


FIGURE 1.

### Diagonals

Let's temporarily return to the basic algebraic framework. Take an  $A_\infty$ -algebra  $\mathcal{A}$  over  $\mathbb{K}$  (for our application, we need the generalisation to  $A_\infty$ -categories, but that is straightforward). Define the *second order Hochschild cochain complex* [117, Section 3] to be

$$(12.38) \quad CC^{(2),*}(\mathcal{A}, \mathcal{A}) = \prod_{p, q \geq 0} \text{Hom}(\mathcal{A}^{\otimes p+q}, \mathcal{A}^{\otimes 2})[p+q].$$

The notation does not give a good intuitive picture. It is better to think of elements of  $CC^{(2),*}(\mathcal{A}, \mathcal{A})$  as operations with inputs and outputs arranged in a circle (Figure 1). The differential  $\partial$  on (12.38) is obtained by composing with the  $A_\infty$ -operations either on any input point or on one of the two output points. For instance, suppose that we start with  $\sigma = \sum_j \sigma_2^j \otimes \sigma_1^j \in \mathcal{A}^{\otimes 2}$ . Then the first terms of the cocycle equation  $\partial\sigma = 0$  are

$$(12.39) \quad \begin{cases} \sum_j (-1)^{|\sigma_1^j|} \mu_{\mathcal{A}}^1(\sigma_2^j) \otimes \sigma_1^j + \sigma_2^j \otimes \mu_{\mathcal{A}}^1(\sigma_1^j) = 0 & p = q = 0, \\ \sum_j (-1)^{|a|(|\sigma_1^j|+1)} \mu_{\mathcal{A}}^2(\sigma_2^j, a) \otimes \sigma_1^j = \sum_j (-1)^{(|a|+1)|\sigma_1^j|} \sigma_2^j \otimes \mu_{\mathcal{A}}^2(a, \sigma_1^j) & p = 0, q = 1, \\ \sum_j (-1)^{|\sigma_2^j|+|a|(|\sigma_2^j|+|\sigma_1^j|)} \mu_{\mathcal{A}}^2(a, \sigma_2^j) \otimes \sigma_1^j = (-1)^{|a|} \sigma_2^j \otimes \mu_{\mathcal{A}}^2(\sigma_1^j, a) & p = 1, q = 0, \\ \dots \end{cases}$$

The secondary Hochschild cohomology  $HH^{(2),*}(\mathcal{A}, \mathcal{A})$  is the cohomology of (12.38). Note that this carries an action of  $\mathbb{Z}/2$ , induced from one on the underlying chain complex (which exchanges  $p$  and  $q$ ). Projection to the  $p = q = 0$  term yields a map  $HH^{(2),*}(\mathcal{A}, \mathcal{A}) \rightarrow H(\mathcal{A})^{\otimes 2}$ . By (12.39), any class  $s = \sum_j s_2^j \otimes s_1^j$  in the image of that map must satisfy

$$(12.40) \quad \begin{cases} \sum_j (-1)^{|s_2^j| \cdot |x|} s_2^j x \otimes s_1^j = \sum_j (-1)^{|s_2^j| \cdot |x|} s_2^j \otimes x s_1^j, \\ \sum_j x s_2^j \otimes s_1^j = (-1)^{|s| \cdot |x|} \sum_j s_2^j \otimes s_1^j x \end{cases}$$

for all  $x \in H^*(\mathcal{A})$ . Finally, note that  $CC^{(2),*}(\mathcal{A}, \mathcal{A})$  carries an action of  $\mathbb{Z}/2$  (exchanging  $p$  and  $q$ ) which is compatible with the differential, hence induces an action on secondary Hochschild cohomology.

REMARK 12.10. *There is also an interpretation in more abstract categorical terms, along the lines of (9.29). In order to explain that in concise terms, suppose temporarily that  $\mathcal{A}$  is a dg algebra. One can introduce “ $\mathcal{A}$ -quadrmodules”, which are chain complexes with*

four mutually commuting actions of  $\mathcal{A}$  (a corresponding notion of multi- $A_\infty$ -module for  $A_\infty$ -algebras was introduced by Ma'u [128]). In particular,  $D = \mathcal{A} \otimes \mathcal{A}$  is a quadrimodule. We denote by  $D^\tau$  the same object but where the ordering of the four  $\mathcal{A}$ -actions is permuted cyclically. Then [117, Section 4]

$$(12.41) \quad HH^{(2),*}(\mathcal{A}, \mathcal{A}) \cong H^*(\text{hom}(D, D^\tau)),$$

where the right hand side takes place in a suitable derived category of quadrimodules.

Second order Hochschild cohomology comes with an analogue of the (classical) structure of  $HH_*(\mathcal{A}, Q)$  as a module over  $HH^*(\mathcal{A}, \mathcal{A})$ . Namely, for any  $\mathcal{A}$ -bimodule  $Q$ , there is a canonical map

$$(12.42) \quad HH^{(2),*}(\mathcal{A}, \mathcal{A}) \otimes HH_*(\mathcal{A}, Q) \longrightarrow HH^*(\mathcal{A}, Q).$$

An elementary piece of this goes as follows. Let  $s \in H(\mathcal{A})^{\otimes 2}$  be an element satisfying (12.39). This induces a map

$$(12.43) \quad \begin{aligned} H(Q) &\longrightarrow H(Q), \\ q &\longmapsto \sum_j (-1)^{|s_1^j||q|} s_2^j q s_1^j, \end{aligned}$$

which lands in the center  $Z(H(Q)) = \{q \in H(Q) : xq = (-1)^{|q||x|}qx \text{ for all } x \in H(\mathcal{A})\}$ , and it kills the commutator  $[H(\mathcal{A}), H(Q)]$ . In fact, if  $s$  comes from an element of  $HH^{(2),*}(\mathcal{A}, \mathcal{A})$ , the corresponding specialization of (12.42) and (12.43) fit into a commutative diagram

$$(12.44) \quad \begin{array}{ccc} HH_*(\mathcal{A}, Q) & \longrightarrow & HH^*(\mathcal{A}, Q) \\ \uparrow & & \downarrow \\ H(Q)/[H(\mathcal{A}), H(Q)] & \longrightarrow & Z(H(Q)) \end{array}$$

DEFINITION 12.11. A diagonal on  $\mathcal{A}$  of dimension  $n$  is an element of  $HH^{(2),n}(\mathcal{A}, \mathcal{A})$  which is invariant under the  $\mathbb{Z}/2$ -action.

THEOREM 12.12. The wrapped Fukaya category  $\mathcal{W}(M)$  has a canonical diagonal of degree  $n$ , which gives rise to (12.37).

We will not explain this in detail. It involves additional moduli spaces of Riemann surfaces, which are again punctured discs (as in the definition of the Fukaya category) but have two outputs. The simplest such moduli space, where the Riemann surface is a single infinite strip, gives rise to an element of  $HW^*(L_0, L_1) \otimes HW^*(L_1, L_0)$  of degree  $n$  for each pair of objects  $(L_0, L_1)$ , which is the categorical counterpart of  $s \in H(\mathcal{A})^{\otimes 2}$ .

REMARK 12.13. Our notion of diagonal is a first order approximation to that of pre-Calabi Yau structure in [117], which includes the  $A_\infty$ -structure, a cochain representative of a diagonal, and higher order terms (for the special case of a finite-dimensional  $\mathcal{A}$ , an equivalent notion of boundary  $A_\infty$ -algebra appears in [181]). It is plausible that wrapped Fukaya categories should have natural pre-Calabi Yau structures, but it seems that the required moduli spaces have not been constructed it.

### Two versions of the definition

In order to give some background to the rather formal discussion so far, we should review the definition of  $SH^*(M)$ . The starting point is the structure of  $M$  at infinity, which can be described by an embedding

$$(12.45) \quad N \times [0, \infty) \hookrightarrow M$$

where  $(N, \alpha_N)$  is a manifold with a contact one-form, and where the pullback of  $\theta_M$  by (12.45) is  $e^r \alpha_N$  ( $r \in [0, \infty)$  is the radial variable). In particular, the Hamiltonian vector field associated to the function  $e^r$  is  $(0, R_\alpha)$ , where  $R_\alpha$  is the Reeb vector field of  $(N, \alpha_N)$ . By perturbing the embedding (12.45) slightly, one can always achieve that all closed Reeb orbits (including multiples of the primitive ones) are transversally nondegenerate, and we will assume from now on that this is the case.

Choose a function  $h \in C^\infty([0, \infty), \mathbb{R})$  with the following properties:

$$(12.46) \quad \begin{cases} h'(0) > 0 \text{ is small,} \\ h''(r) > 0, \\ \lim_{r \rightarrow \infty} h'(r) = \infty. \end{cases}$$

Take a Morse function  $H$  on  $M$ , such that  $dH$  is small outside (12.45), and  $H(r, y) = h(e^r)$  at infinity. The associated time-dependent Hamiltonian vector field  $X$  then satisfies

$$(12.47) \quad X(r, y) = h'(e^r)(0, R_N).$$

By choosing  $h$  and  $H$  suitably, one can achieve that one-periodic orbits of  $X$  are either constant  $x(t) = p$ , where  $p$  is a critical point of  $H$ , or else of the form

$$(12.48) \quad x(t) = (r, y(\rho^{-1}t))$$

where  $y : \mathbb{R} \rightarrow N$  is a periodic Reeb orbit,  $\rho > 0$  the period (but not necessarily the minimal period), and  $r > 0$  the unique number such that  $h'(e^r) = \rho$ .

As usual in Hamiltonian Floer theory (see e.g. [162] for an introduction), we formally set up Morse theory on the free loop space  $\mathcal{L}M$  with respect to the action functional

$$(12.49) \quad A_H(x) = \int_{S^1} -x^* \theta_M + H_t(x(t)) dt.$$

Critical points are (parametrized) 1-periodic orbits of  $X$ . One can arrange that the orbits disjoint from (12.45) are nondegenerate. Those of the form (12.48) come in  $S^1$ -families (because of the parametrization) which are transversally nondegenerate. On each  $S^1$ -family, one can choose an auxiliary Morse function with two critical points. The resulting Morse-Bott type Floer cochain complex is of the form

$$(12.50) \quad CF^*(H) = \bigoplus_p \mathbb{K}[-i(p)] \oplus \bigoplus_y (\mathbb{K}[-i(y)] \oplus \mathbb{K}[-i(y) - 1]).$$

where the first sum is over critical points of  $M$ , and the second one is over periodic Reeb orbits on  $N$  (including multiple orbits). The degree  $i(\cdot)$  is a Conley-Zehnder index. The definition of the differential on (12.50) involves a mixture of pseudo-holomorphic cylinders

and Morse trajectories, in the manner of [31].  $SH^*(M)$  is then defined as the cohomology of (12.50) (this description, which partially fleshes out the one given in Lecture 6, is closely related to the points of view adopted in [38, 36]).

There is also an equivalent, but technically a little simpler, approach. Choosing an increasing sequence of numbers  $\lambda_k > 0$  such that  $\lim_k \lambda_k = \infty$ , and such that  $N$  has no closed Reeb orbit of period  $\lambda_k$ . Correspondingly, take Hamiltonians  $H_k$  such that  $H_t(r, y) = \lambda_k e^r$  over the cone (12.45). Note that these can now be time-dependent (on the rest of  $M$ ), with  $t \in \mathbb{R}/\mathbb{Z} = S^1$ . This breaks the  $S^1$ -symmetry, and one can therefore achieve that the one-periodic orbits are nondegenerate. The resulting Floer cochain complexes  $CF^*(H_k)$  are related by continuation maps  $CF^*(H_k) \rightarrow CF^*(H_{k+1})$ , and one defines

$$(12.51) \quad SH^*(M) = \varinjlim_k HF^*(H_k).$$

In either version of the Floer-theoretic framework, operations on  $SH^*(M)$  arise from Riemann surfaces  $S$  with marked points  $\Sigma \subset S$  and additional marked tangent directions at each of those points. There is an additional condition, namely that there should be a one-form  $\beta_S$  on  $S \setminus \Sigma$  such that:

- $d\beta_S$  is everywhere  $\leq 0$ , and vanishes near  $\Sigma$ .
- The integral  $\int \beta_S$  along a small loop around any point of  $\Sigma$  is nonzero.

The second condition divides  $\Sigma = \Sigma^{\text{in}} \cup \Sigma^{\text{out}}$ , depending on the sign of  $\int \beta_S$ . The associated operation will take as inputs elements of  $SH^*(M)$  at every input point, and take values in the tensor product of copies of  $SH^*(M)$  associated to the output points. Instead of a single Riemann surface, one can also use a family of such surfaces parametrized by a closed oriented manifold. The spaces of framed little discs appear naturally as moduli spaces of genus 0 Riemann surfaces with a single output point. More details can be found in [175, 157, 184].





## Part 3

# Circle actions



## LECTURE 13

### Equivariant modules **\*\*\*WARNING\*\*\***: CONTAINS AN ERROR (MARKED AS SUCH)

Our next topic is the equivariant counterpart of the basic theory from Lectures 7 and 8. Before we start with that, a few general remarks may be helpful. Given a discrete group  $G$  acting on an algebra  $A$ , a  $G$ -equivariant  $A$ -module is the same as a module over the semidirect product (also called skew group ring, or smash product) algebra  $A \rtimes G$ . This makes it straightforward to set up the theory of equivariant modules, even though its eventual complexity depends on that of the representation theory of  $G$  over the ground field  $\mathbb{K}$  of  $A$ ; and the same remains true in the context of  $A_\infty$ -algebras. We will see a bit of this theory in Lecture 14.

However, our primary interest lies in continuous symmetries, specifically actions of the algebraic group  $\mathbb{G}_m$ , and the ground field  $\mathbb{C}$  (so  $\mathbb{G}_m = \mathbb{C}^*$ ). A bit more generally, we will allow any reductive algebraic group  $G$  over  $\mathbb{C}$ , since that can be done without adding much complexity (readers interested in an exposition specifically geared towards  $\mathbb{G}_m$  might look at [182, Section 2]). The theory of (rational) representations of reductive groups is fairly close to the finite group case. However, the semidirect product trick no longer works, and in fact the theory of equivariant  $A_\infty$ -modules is a little quirky.

#### Background from representation theory

Let  $G$  be a linear algebraic group over  $\mathbb{C}$  (one possible textbook reference for the material which follows is [81]). A linear representation of  $G$  is called *rational* if it is the direct sum of finite-dimensional (algebraic) representations. For instance, the space of regular functions  $\mathbb{C}[G]$ , with  $g \in G$  acting by  $f \mapsto f(\cdot g)$ , is rational. Rational representations and equivariant linear maps between them form an abelian category, which is in fact equivalent to the category of comodules over  $\mathbb{C}[G]$  (with the coproduct dual to the group structure of  $G$ ). From this viewpoint,  $\mathbb{C}[G]$  is the cofree comodule, which means that

$$(13.1) \quad \text{Hom}(W, \mathbb{C}[G])^G \cong W^\vee$$

for any rational representation  $W$  (the map is given by composing with evaluation at the identity,  $\mathbb{C}[G] \rightarrow \mathbb{C}$ ; and the  $G$ -action on the right hand side of (13.1) corresponds to the action  $f \mapsto f(g^{-1}\cdot)$  on the left hand side). We should also recall the notion of character of a finite-dimensional representation,

$$(13.2) \quad \chi_W \in \mathbb{C}[G]^{class}.$$

This is defined by  $\chi_W(g) = \text{Tr}(g : W \rightarrow W)$ , and takes values in the ring of class functions (conjugation invariant regular functions, with the usual multiplication). Let  $\mathcal{R}^{fin}(G)$  be the abelian tensor category of finite-dimensional representations. Then, characters define a ring homomorphism

$$(13.3) \quad \chi : K_0(\mathcal{R}^{fin}(G)) \longrightarrow \mathbb{C}[G]^{class}.$$

From this point onwards, we will assume that  $G$  is reductive. Then, each rational representation has a canonical decomposition

$$(13.4) \quad W = \bigoplus_V W_V,$$

where the direct sum is over representatives of each isomorphism class of (finite-dimensional) irreducible representations, and  $W_V$  is a direct sum of copies of  $V$ . More explicitly,

$$(13.5) \quad W_V \cong (V^\vee \otimes W)^G \otimes V \cong \text{Hom}(V, W)^G \otimes V,$$

where the map from right to left is the contraction  $\phi \otimes v \mapsto \phi(v)$ . As an example, setting  $W = \mathbb{C}[G]$  and using (13.1) (with  $V$  replacing  $W$ ) gives us  $\mathbb{C}[G]_V \cong V^\vee$ , which yields a well-known algebraic analogue of the Peter-Weyl theorem [81, Theorem 4.2.7]:

$$(13.6) \quad \mathbb{C}[G] \cong \bigoplus_V V^\vee \otimes V.$$

Applying (13.4) to finite-dimensional representations shows that the category of such representations is semisimple. In fact, (13.3) yields an isomorphism

$$(13.7) \quad K_0(\mathcal{R}^{fin}(G)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \mathbb{C}[G]^{class}.$$

EXAMPLE 13.1. For  $G = \mathbb{G}_m = \mathbb{C}^*$ , the irreducible representations  $V$  are one-dimensional and labeled by integers. Hence, a rational representation is simply a vector space  $W$  with a decomposition as a direct sum of graded pieces. If we identify  $\mathbb{C}[G]^{class} = \mathbb{C}[G] \cong \mathbb{C}[t, t^{-1}]$  in the obvious way, then  $K_0(\mathcal{R}^{fin}(G)) = \mathbb{Z}[t, t^{-1}]$ .

We will also encounter representations which are not rational, but only products

$$(13.8) \quad W = \prod_{i \in I} W_i,$$

where the  $W_i$  are rational. For any irreducible  $V$ , set  $W_V = \prod_{i \in I} W_{i,V}$ . One gets a diagram of injective maps

$$(13.9) \quad \begin{array}{ccc} \bigoplus_V W_V & \longrightarrow & W \\ & \searrow \text{inclusion} & \downarrow \\ & & \prod_V W_V. \end{array}$$

The inclusion  $W_V \subset W$  can be characterized intrinsically, which means without reference to the product (13.8), as the image of the (injective) contraction  $\text{Hom}(V, W)^G \otimes V \rightarrow W$ . Unfortunately, it seems that there is no similar characterization of the projections  $W \rightarrow W_V$  which enter into the  $\downarrow$  map in (13.9). One way to add the necessary information is to equip (13.8) with the product of the discrete topologies on the  $W_i$  factors. Let's call the result a

pro-rational representation. Note that  $\bigoplus_V W_V$  is then dense in  $W$  (and so is the smaller subspace  $\bigoplus_{V,i} W_{i,V}$ ). In particular, for any  $V$  there is a unique idempotent endomorphism of  $W$  which is the identity on  $W_V$ , kills the subspaces associated to other finite-dimensional representations, and is continuous. This provides the desired description of the projections to  $W_V$ , at the cost of having to remember the topology.

EXAMPLE 13.2. Let  $W, U$  be rational representations, say  $W = \bigoplus_i W_i$  with  $W_i$  finite-dimensional. That makes

$$(13.10) \quad \text{Hom}(W, U) = \prod_i \text{Hom}(W_i, U) = \prod_i W_i^\vee \otimes U$$

into a pro-rational representation. The associated topology is that of pointwise convergence of maps  $W \rightarrow U$ .

Now suppose that  $C$  is a chain complex of pro-rational representations, with a differential which is equivariant and *continuous*. The differential preserves each  $C_V \subset C$ , and by continuity it also commutes with the projections  $C \rightarrow C_V$ . Then, (13.9) induces a diagram

$$(13.11) \quad \begin{array}{ccc} \bigoplus_V H^*(C_V) & \longrightarrow & H^*(C) \\ & \searrow \text{inclusion} & \downarrow \\ & & \prod_V H^*(C_V). \end{array}$$

In particular, specializing to the trivial representation, one finds that

$$(13.12) \quad H^*(C^G) \longrightarrow H^*(C)^G$$

is split-injective.

EXAMPLE 13.3. The  $\downarrow$  in (13.11) is not necessarily injective. For instance, consider

$$(13.13) \quad C = \left\{ \bigoplus_V V \xrightarrow{\text{inclusion}} \prod_V V \right\}.$$

This is a complex of pro-rational representations. In fact, the first term is rational, hence carries the discrete topology, which means that the differential is obviously continuous. We have  $H^*(C_V) = 0$  for any  $V$ , but  $H^*(C) \neq 0$ . Unfortunately, I do not know an example where (13.12) fails to be an isomorphism.

### Group actions on $A_\infty$ -categories

Throughout the rest of our discussion,  $\mathcal{A}$  will be an  $A_\infty$ -category with a  $G$ -action. The action is understood in a naive sense, meaning that each space  $\text{hom}_{\mathcal{A}}^k(X, Y)$  ( $X, Y \in \text{Ob}(\mathcal{A})$ ,  $k \in \mathbb{Z}$ ) is a rational representation of  $G$ , in a way which is strictly compatible with  $A_\infty$ -operations. We denote by  $\mathcal{A}^G$  the subcategory with the same objects, but where only invariant morphisms are allowed. Among the standard results that carry over to the equivariant context without problems, two are worth mentioning because they will be used later on: the theorem that says that any  $\mathcal{A}$  is quasi-isomorphic to a strictly unital one; and the Perturbation Lemma, which says that any  $\mathcal{A}$  is quasi-isomorphic to a minimal one (one with vanishing differential).

EXAMPLE 13.4. Let  $\mathcal{A}$  be an  $A_\infty$ -algebra such that  $\mu_{\mathcal{A}}^d = 0$  for all  $d \neq 2$ . Then, it carries an obvious action of the multiplicative group  $\mathbb{G}_m$ , namely the one that acts with weight  $d$  on  $\mathcal{A}^d$ . Of course, our assumption just says that  $\mathcal{A}$  is a graded algebra in the classical sense. It is therefore not surprising that in this case, the category of  $\mathbb{G}_m$ -equivariant  $A_\infty$ -modules (to be introduced soon) is essentially equivalent to the derived category of complexes of graded modules in the classical sense; see [182, Section 2].

There is a “converse” statement, stating that the presence of a suitable (infinitesimal) symmetry enforces formality. Let  $\mathcal{A}$  be a minimal  $A_\infty$ -algebra. Suppose that there is a Hochschild cocycle  $\phi \in CC^1(\mathcal{A}, \mathcal{A})$  such that  $\phi^0 \in \mathcal{A}^1$  vanishes, and such that  $\phi^1 \in \text{Hom}^0(\mathcal{A}, \mathcal{A})$  is the Euler derivation, meaning tht it is  $d$  times the identity on  $\mathcal{A}^k$  for all  $k \in \mathbb{Z}$ . Then,  $\mathcal{A}$  is necessarily formal. One proof involves the spectral sequence associated to the length filtration of  $HH^*(\mathcal{A}, \mathcal{A})$ . Its starting page is

$$(13.14) \quad E_1^{pq} = HH^p(\mathcal{A}, \mathcal{A}[q]),$$

where  $\mathcal{A}$  is the graded algebra of which  $\mathcal{A}$  is an  $A_\infty$ -deformation. Suppose that  $\mathcal{A}$  is not formal, and that  $m = [\mu_{\mathcal{A}^d}] \in HH^d(\mathcal{A}, \mathcal{A}[2-d])$  is the first nontrivial deformation class ( $d > 2$ ). In that case, the first nonvanishing differential on (13.14) is the Gerstenhaber bracket with  $m$ . If  $f \in HH^1(\mathcal{A}, \mathcal{A})$  is the class of the derivation  $\phi^1$ , then by definition

$$(13.15) \quad [m, f] = (d-2)m \neq 0.$$

But that would contradict the assumption that  $f$  survives to yield the leading order term of a class in  $HH^1(\mathcal{A}, \mathcal{A})$ .

EXAMPLE 13.5. Let  $M$  be a smooth projective variety,  $E$  the total space of its canonical bundle  $K_M$ , and  $i : M \rightarrow E$  the zero-section. Suppose that  $X$  is an object of  $D^b \text{Coh}(M)$ . There are canonical (up to quasi-isomorphism) proper differential graded algebras  $\mathcal{A}$  and  $\mathcal{B}$  such that

$$(13.16) \quad \begin{aligned} H^*(\mathcal{A}) &= \text{Hom}_{D^b \text{Coh}(M)}^*(X, X), \\ H^*(\mathcal{B}) &= \text{Hom}_{D^b \text{Coh}(E)}^*(i_* X, i_* X). \end{aligned}$$

One can arrange that  $\mathcal{A}$  is a subalgebra of  $\mathcal{B}$  (the induced map on cohomology is that given by  $i_*$ ). One can further arrange that  $\mathcal{B}$  carries a  $\mathbb{G}_m$ -action, corresponding to rotation of the fibres of the anticanonical bundle, which is trivial on  $\mathcal{A}$  and has weight 1 on a complementary subspace. If we identify that complementary subspace with  $\mathcal{B}/\mathcal{A}$ , the existence of this symmetry implies that the only nontrivial  $A_\infty$ -operations on  $\mathcal{B}$  are of the form

$$(13.17) \quad \mathcal{A}^{\otimes d} \longrightarrow \mathcal{A},$$

$$(13.18) \quad \mathcal{A}^{\otimes q} \otimes (\mathcal{B}/\mathcal{A}) \otimes \mathcal{A}^{\otimes p} \longrightarrow \mathcal{B}/\mathcal{A}.$$

The products (13.18) define the structure of  $\mathcal{B}/\mathcal{A}$  as an  $A_\infty$ -bimodule over  $\mathcal{A}$ , and then  $\mathcal{B}$  is the so-called trivial extension algebra formed from  $\mathcal{A}$  and that bimodule. In the particular case of the canonical bundle, we actually have

$$(13.19) \quad \mathcal{B}/\mathcal{A} \simeq \mathcal{A}^\vee[-\dim(E)].$$

Therefore,  $\mathcal{B}$  is entirely determined by  $\mathcal{A}$  (see [166, 24, 179] for more details).

We should point out that these considerations have geometric significance. Suppose for a moment that  $E$  is a general Calabi-Yau variety, containing  $M$  as a hypersurface. The normal bundle to  $M$  is then necessarily identified with  $K_M$ . However, it is not necessarily true that the formal neighbourhood is isomorphic to that of the zero-section inside the anticanonical bundle (the first example would be where  $M$  is a fibre of an elliptic fibration  $E \rightarrow \mathbb{C}$ ; then,  $K_M$  is trivial, but the  $j$ -invariant of the fibres usually varies, and that affects the structure of the formal neighbourhood of  $M$ ). Correspondingly, the isomorphism (13.19) of bimodules still holds, but fails to describe  $\mathcal{B}$  completely; the additional information contained in  $\mathcal{B}$  reflects the higher order infinitesimal geometry of the embedding  $M \subset E$ .

REMARK 13.6. It is legitimate to ask why we restrict our discussion to actions of  $G$  in a naive sense, rather than introducing a more “categorically appropriate” notion (as done in [57] for classical categories; the cochain level version would obviously be more complicated). The answer lies in our choice of intended applications. In simple algebraic situations (such as Example 13.4), the strict action on  $G$  can be written down directly. In algebro-geometric examples (such as Example 13.5) one can again get  $G$  to act strictly on the resulting  $A_\infty$ -algebras, even though that requires a little more care. Finally, in situations arising from symplectic topology, we will indeed encounter difficulties in constructing group actions: but those difficulties are primarily geometric ones, in which the lack of a more sophisticated algebraic formalism is only a secondary issue.

## Twisted complexes

One can define, in a relatively straightforward way, the category of equivariant twisted complexes  $\mathcal{A}^{eq-tw}$ . This is again an  $A_\infty$ -category with a  $G$ -action, and contains  $\mathcal{A}$  as a full  $A_\infty$ -subcategory. Objects of  $\mathcal{A}^{eq-tw}$  are of the form  $(C, \delta_C)$  as in (7.12), but where the  $W_i$  are finite-dimensional graded representations of  $G$ . The differential  $\delta_C$  should be  $G$ -invariant, and the filtration which is required to exist as part of the definition of a twisted complex must be compatible with the  $G$ -action. Morphism spaces are the same as in the non-equivariant case, and carry an obvious  $G$ -action.

**\*\*\*Warning\*\*\*.** The following Lemma is wrong. As an example, consider  $D^b \text{Coh}(\mathbb{C}P^1)$ , with the  $\mathbb{C}^*$ -action induced from  $\cdot$ . All objects are direct sums of their cohomology sheaves, and similarly for equivariant objects. However, the structure sheaf of a (general) point is not a direct summand of an equivariant sheaf. If one follows through the proof, it would tell one to write that structure sheaf first as  $\text{Cone}(\mathcal{O}(-1) \rightarrow \mathcal{O})$ , and then make the arrow equivariant by passing to  $\text{Cone}(\mathcal{O}(-1) \rightarrow \mathcal{O} \oplus \mathcal{O} \otimes V)$ , for a suitable character  $V$ . But that cone is isomorphic to  $\mathcal{O}(+1)$ , of which the structure sheaf of a point is not a direct summand. The error lies in the construction of (13.23). This affects some of the arguments in the following lecture – I will fix it at some point.

LEMMA 13.7. (**\*\*\*Incorrect\*\*\***) Every twisted complex is a homotopy retract of an equivariant twisted complex.

PROOF. As mentioned before, after applying a  $G$ -equivariant quasi-isomorphism, we may assume that  $\mathcal{A}$  is strictly unital. Our strategy of proof will be to show that if  $C_0, C_1$  are homotopy retracts of equivariant twisted complexes, and  $a \in \text{hom}_{\mathcal{A}^{eq-tw}}^0(C_0, C_1)$  is a closed degree 0 morphism between them, then the mapping cone  $\text{Cone}(a)$  is again a homotopy retract of an equivariant twisted complex (note that we do not assume that  $a$  is  $G$ -invariant; otherwise, the statement would be trivial). This clearly implies the desired result.

To keep the notation simple, let's first consider the special case where we start with two objects  $X_0, X_1$  of  $\mathcal{A}$  itself. For any irreducible representation  $V$ , consider the equivariant twisted complex (with zero differential)  $V \otimes V^\vee \otimes X_1$ , with the given action of  $G$  on  $V^\vee$ , and the trivial action on  $V$ . This comes with morphisms

$$(13.20) \quad X_1 \xrightarrow{\text{diagonal} \otimes e_{X_1}} V \otimes V^\vee \otimes X_1 \xrightarrow{\text{contraction} \otimes e_{X_1}} X_1$$

whose composition is  $\dim(V)$  times the identity. Because of the way in which we have defined  $V \otimes V^\vee \otimes X_1$ , these maps are generally not  $G$ -invariant. Instead, composition with the first map in (13.20) recovers the isomorphism

$$(13.21) \quad \text{hom}_{\mathcal{A}}(X_0, X_1)_V \cong \text{hom}_{\mathcal{A}^{tw}}(X_0, V^\vee \otimes X_1)^G \otimes V \cong \text{hom}_{\mathcal{A}}(X_0, V \otimes V^\vee \otimes X_1)^G.$$

Given a closed  $a \in \text{hom}_{\mathcal{A}}^0(X_0, X_1)$ , write it as  $a = a_{V_1} + \cdots + a_{V_r}$ , where each  $a_{V_i}$  belongs to the subspace associated to some irreducible representation  $V_i$ . Compose  $a_{V_i}$  with the first map in (13.20) for the corresponding representation to get a  $G$ -invariant morphism

$$(13.22) \quad X_0 \xrightarrow{\tilde{a} = \tilde{a}_{V_1} \oplus \cdots \oplus \tilde{a}_{V_r}} \bigoplus_i V_i \otimes V_i^\vee \otimes X_i.$$

If we compose this with the direct sum of the second maps in (13.20), the outcome is  $a$  times  $\dim(\bigoplus_i V_i)$ . After dividing by that constant, one gets the following diagram in  $\mathcal{A}^{tw}$ :

$$(13.23) \quad \begin{array}{ccccc} \text{Cone}(a) & \longrightarrow & \text{Cone}(\tilde{a}) & \longrightarrow & \text{Cone}(a). \\ & & \searrow & \nearrow & \\ & & & & \text{identity} \end{array}$$

Since  $\tilde{a}$  is invariant,  $\text{Cone}(\tilde{a})$  is an equivariant twisted complex, hence  $\text{Cone}(a)$  is a homotopy retract (a strict retract, in fact) of an equivariant twisted complex.

The same argument applies if, instead of  $(X_0, X_1)$ , one starts with equivariant twisted complexes and a morphism between them. Finally, suppose that we have two twisted complexes  $(C_0, C_1)$  which are homotopy retracts of equivariant twisted complexes  $(\tilde{C}_0, \tilde{C}_1)$ . Then, the cone of every morphism  $C_0 \rightarrow C_1$  is a homotopy retract of the cone of a morphism  $\tilde{C}_0 \rightarrow \tilde{C}_1$ , to which the previous statement applies.  $\square$

## Modules

An equivariant  $\mathcal{A}$ -module  $M$  associates to every object  $X$  in  $\mathcal{A}$  a graded vector space  $M(X)$  whose graded pieces are rational representations of  $G$ , together with  $A_\infty$ -module operations which are  $G$ -equivariant.



EXAMPLE 13.8. If  $Y$  is an object of  $\mathcal{A}$ , the Yoneda module  $Y^{yon}(X) = \text{hom}_{\mathcal{A}}(X, Y)$  is naturally equivariant.

EXAMPLE 13.9. Given an arbitrary  $\mathcal{A}$ -module  $M$ , one can define an equivariant module  $M^{orbit}$  as follows. The underlying vector space is

$$(13.24) \quad M^{orbit}(X) = \mathbb{C}[G] \otimes M(X),$$

with the  $G$ -action inherited from  $\mathbb{C}[G]$ . This means that if we think of elements as functions  $m : G \rightarrow M(X)$ , then the action is by  $(gm)(\cdot) = m(\cdot g)$ . In the same terms, the  $A_{\infty}$ -module structure is

$$(13.25) \quad \mu_{M^{orbit}}^{1;d}(m; a_d, \dots, a_1)(g) = \mu_M^{1;d}(m(g); g(a_d), \dots, g(a_1)).$$

Note that (13.25) is local in  $g \in G$ ; or to express it more formally,  $M^{orbit}(X)$  is a module over the algebra of functions  $\mathbb{C}[G]$  by pointwise multiplication, and this is compatible with its  $A_{\infty}$ -module structure (later, we will think of this additional structure as making  $M^{orbit}$  into a family of modules over the affine algebraic variety  $G$ ).

EXAMPLE 13.10. Suppose that  $M$  is already equivariant, and  $W$  is a rational representation. Then, one defines another equivariant module  $W \otimes M$  by setting  $(W \otimes M)(X) = W \otimes M(X)$  with the tensor product action of  $G$ , and the  $A_{\infty}$ -module structure

$$(13.26) \quad \mu_{W \otimes M}^{1;d}(w \otimes m; a_d, \dots, a_1) = w \otimes \mu_M^{1;d}(m; a_d, \dots, a_1).$$

In the special case  $W = \mathbb{C}[G]$ , the tensor product is isomorphic to  $M^{orbit}$ . This is not entirely trivial: even though these two modules associate the same space to any  $X$ , the  $G$ -actions are different. The isomorphism is set up in such a way that the image of  $m \in M^{orbit}(X)$  in  $(\mathbb{C}[G] \otimes M)(X)$  is the function  $g \mapsto g^{-1}m(g)$ .

We denote the dg category of equivariant  $\mathcal{A}$ -module by  $\mathcal{A}^{eq-mod}$ . This has the same morphisms as in the non-equivariant case, except that now  $\text{hom}_{\mathcal{A}^{eq-mod}}(M, N)$  carries a natural  $G$ -action. Restricting to invariant morphisms yields a subcategory  $\mathcal{A}^{eq-mod, G}$  (Example 13.9 is part of the right adjoint to the forgetful functor  $\mathcal{A}^{eq-mod, G} \rightarrow \mathcal{A}^{mod}$ ).

There is a hidden technical issue in this description. To simplify the notation in the following discussion, let's assume that  $\mathcal{A}$  is just an  $A_{\infty}$ -algebra, so that modules consist of a single graded vector space, equally denoted by  $M$ . The morphism spaces in  $\mathcal{A}^{eq-mod}$  are *not rational representations* of  $G$ . Instead,

$$(13.27) \quad \text{hom}_{\mathcal{A}^{eq-mod}}^k(M, N) = \text{hom}_{\mathcal{A}^{mod}}^k(M, N) = \prod_{d \geq 0} \text{Hom}^{k-d}(M \otimes \mathcal{A}^{\otimes d}, N)$$

can be made into a pro-rational representation (by writing  $M$  and  $\mathcal{A}$  as the direct sum of finite-dimensional representations, and  $\text{Hom}(\cdot, N)$  correspondingly as a direct product). More concretely, the associated topology is the topology of pointwise convergence (in the discrete topology) of each expression  $\phi^{1;d}(m; a_d, \dots, a_1)$ . In particular, the differential on (13.27) is continuous, as is the composition of morphisms. Applying the previous general

considerations, one has graded subspaces

$$(13.28) \quad \begin{aligned} \mathit{hom}_{\mathcal{A}^{eq-mod}}(M, N)_V &\cong V \otimes \mathit{Hom}(V, \mathit{hom}_{\mathcal{A}^{eq-mod}}(M, N))^G \\ &\cong V \otimes \mathit{hom}_{\mathcal{A}^{eq-mod, G}}(V \otimes M, N) \\ &\cong V \otimes \mathit{hom}_{\mathcal{A}^{eq-mod, G}}(M, V^\vee \otimes N), \end{aligned}$$

associated to irreducible representations  $V$ , a diagram

$$(13.29) \quad \begin{array}{ccc} \bigoplus_V V \otimes H^*(\mathit{hom}_{\mathcal{A}^{eq-mod, G}}(M, V^\vee \otimes N)) & \longrightarrow & H^*(\mathit{hom}_{\mathcal{A}^{eq-mod}}(M, N)) \\ & \searrow & \downarrow \\ & & \prod_V V \otimes H^*(\mathit{hom}_{\mathcal{A}^{eq-mod, G}}(M, V^\vee \otimes N)), \\ & \text{inclusion} & \nearrow \end{array}$$

and a split injection

$$(13.30) \quad H^*(\mathit{hom}_{\mathcal{A}^{eq-mod, G}}(M, N)) \hookrightarrow H^*(\mathit{hom}_{\mathcal{A}^{eq-mod}}(M, N))^G.$$

It will be important for us that the splitting is natural (under composition of morphisms). Of course, all of this also holds for general  $A_\infty$ -categories rather than algebras.

LEMMA 13.11. *Let  $\mathcal{A}$  be a proper  $A_\infty$ -category, and  $M, N$  equivariant  $\mathcal{A}$ -modules which are proper. Then the  $\downarrow$  map in (13.29) is an isomorphism, and so is (13.30).*

PROOF. Without loss of generality, we may assume that  $\mathcal{A}$  and the modules are minimal, hence consist of finite-dimensional graded vector spaces. In that case,  $\mathit{hom}_{\mathcal{A}^{eq-mod}}(M, N)$  is a direct product of finite-dimensional representations of  $G$ , hence isomorphic to the direct product of (13.28), and the result is obvious (readers wishing to avoid the initial use of Perturbation Lemma techniques may want to rely on the spectral sequence associated to the length filtration instead).  $\square$

### Perfect modules

Take an equivariant  $A_\infty$ -module  $N$ . One can extend this to an equivariant module over  $\mathcal{A}^{eq-tw}$ , in a canonical way. On the vector space level, this looks as follows:

$$(13.31) \quad C = \bigoplus_{i \in I} W_i \otimes X_i \implies N(C) = \bigoplus_{i \in I} N(X_i) \otimes W_i^\vee,$$

and then  $\delta_C$  enters into the definition of the  $A_\infty$ -module structure. In particular,  $N(C)$  is again a rational representation in each degree. Now let  $C^{yon}$  be the Yoneda module associated to an equivariant twisted complex. There is an equivariant quas-isomorphism

$$(13.32) \quad N(C) \longrightarrow \mathit{hom}_{\mathcal{A}^{eq-mod}}(C^{yon}, N),$$

By taking the mapping cone of (13.32) and applying (13.11) to it, one sees that the induced map on the subcomplexes associated to rational representations  $V$  are again quasi-isomorphisms. Hence, we get a commutative diagram

$$(13.33) \quad \begin{array}{ccc} \bigoplus_V H^*(N(C)_V) & \xrightarrow{\cong} & H^*(N(C)) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_V H^*(\text{hom}_{\mathcal{A}^{eq-mod}}(C^{yon}, N)_V) & \longrightarrow & H^*(\text{hom}_{\mathcal{A}^{eq-mod}}(C^{yon}, N)) \end{array}$$

which shows that the  $\rightarrow$  in (13.29) is an isomorphism if the first of the two modules involved comes from a twisted complex. We want to extend this property to a slightly larger class:

**DEFINITION 13.12.** *Let  $M$  be an equivariant  $\mathcal{A}$ -module. We say that  $M$  is equivariantly perfect if, in  $H^0(\mathcal{A}^{eq-mod})^G$ , it is a retract of the Yoneda module associated to an equivariant twisted complex. We write  $\mathcal{A}^{eq-perf} \subset \mathcal{A}^{eq-mod}$  for the full  $A_\infty$ -subcategory of equivariantly perfect modules.*

We have required the retractions to be  $G$ -invariant only on the cohomology level, but this is in fact not an issue. To see that, take  $C^{yon}$  as before, and let

$$(13.34) \quad [\pi] \in H^0(\text{hom}_{\mathcal{A}^{eq-mod}}(C^{yon}, C^{yon}))^G$$

an endomorphism which is idempotent and  $G$ -invariant (on cohomology). It follows from (13.33) that one can choose a representative  $\pi$  which is strictly  $G$ -invariant. Now suppose that  $M$  is the homotopy retract associated to  $[\pi]$ . This means that we have maps  $C \rightarrow M$  and  $M \rightarrow C$ , which are  $G$ -invariant on the level of cohomology, and whose composition is  $[\pi]$ . Using the split-injectivity of (13.30), one can apply a projection and obtain new maps which have  $G$ -invariant cochain level representatives, and whose composition is still  $[\pi]$  on the cohomology level. Applying (13.33) again, it follows that the cochain level product of those new representatives differs from  $\pi$  by the coboundary of a  $G$ -invariant cochain. Hence,  $M$  is in fact a homotopy retract of  $C$  in  $\mathcal{A}^{eq-mod,G}$ , which means that Definition 13.12 is not actually more general than the obvious stricter-looking alternative. This and our previous considerations of twisted complexes imply:

**LEMMA 13.13.** *Suppose that  $M$  is equivariantly perfect. Then the  $\rightarrow$  in (13.29) is an isomorphism. As a consequence,  $H^*(\text{hom}_{\mathcal{A}^{eq-mod}}(M, N))$  is a rational representation of  $G$ .  $\square$*

In particular, if we define  $\mathcal{A}^{eq-perf,G} \subset \mathcal{A}^{eq-mod,G}$  to be the full subcategory of equivariantly perfect modules (and their  $G$ -invariant morphisms), then  $H^*(\text{hom}_{\mathcal{A}^{eq-perf,G}}(M, N)) \cong H^*(\text{hom}_{\mathcal{A}^{eq-perf}}(M, N))^G$ .

Let's continue our earlier discussion in a slightly different direction. Our representative  $\pi$  can be taken to be the Yoneda image of a cocycle  $p \in \text{hom}_{\mathcal{A}^{eq-tw}}^0(C, C)^G$ . As in [176, Section 4b], one can extend  $p$  to an idempotent up to homotopy, which involves a series of higher order cochain, all of them again strictly  $G$ -invariant. This in turn allows the construction of an equivariant  $A_\infty$ -module  $M$  as in [176, Lemma 4.4], which in  $H^0(\mathcal{A}^{eq-mod,G})$  is the retract of  $C^{yon}$  associated to  $[\pi]$ . Hence:

LEMMA 13.14.  $H^0(\mathcal{A}^{eq-perf}, G)$  is Karoubi complete (admits homotopy retracts for all idempotent endomorphisms).  $\square$

PROPOSITION 13.15. Let  $M$  be an equivariant  $\mathcal{A}$ -module. Suppose that if we forget the equivariant structure,  $M$  is a perfect module. Then it is also an equivariantly perfect module.

PROOF. Suppose that (forgetting the equivariant structure)  $M$  is a homotopy retract of  $C^{yon}$ , the Yoneda module of a twisted complex  $C$ . By Lemma 13.7, we can assume without loss of generality that  $C$  is an equivariant twisted complex. Take the morphisms

$$(13.35) \quad \begin{aligned} \rho &\in \text{hom}_{\mathcal{A}^{eq-mod}}^0(C^{yon}, M), \\ \iota &\in \text{hom}_{\mathcal{A}^{eq-mod}}^0(M, C^{yon}) \end{aligned}$$

which express the fact that  $M$  is a retract. Under the  $\downarrow$  map in (13.29), these turn into

$$(13.36) \quad \begin{aligned} (\rho_V) &\in \prod_V \text{hom}_{\mathcal{A}^{eq-mod}, G}(V^\vee \otimes C^{yon}, M) \otimes V^\vee, \\ (\iota_V) &\in \prod_V \text{hom}_{\mathcal{A}^{eq-mod}, G}(M, V^\vee \otimes C^{yon}) \otimes V. \end{aligned}$$

By (13.33), only finitely many cohomology classes  $[\rho_V]$  can be nonzero. Denote the associated irreducible representations by  $(V_1, \dots, V_r)$ . Since  $[\rho] \cdot [\iota] = [e_M]$  on the cohomology level, it follows that

$$(13.37) \quad \sum_{i=1}^r [\rho_{V_i}] \cdot [\iota_{V_i}] = [e_M].$$

Here, the product of  $[\rho_{V_i}]$  and  $[\iota_{V_i}]$  is given by composition of morphisms together with the dual pairing of  $V$  and  $V^\vee$  (it is easy to see that no other  $\iota_V$  can contribute). Equivalently, form  $\tilde{C} = \bigoplus_i V_i \otimes V_i^\vee \otimes C$ , made into an equivariant twisted complex just as in the proof of Lemma 13.7, with its associated Yoneda module  $\tilde{C}^{yon}$ . We then have maps

$$(13.38) \quad \begin{aligned} \tilde{\rho} &= (\rho_{V_1}, \dots, \rho_{V_r}) \in \text{hom}_{\mathcal{A}^{eq-mod}, G}(\tilde{C}^{yon}, M), \\ \tilde{\iota} &= (\iota_{V_1}, \dots, \iota_{V_r}) \in \text{hom}_{\mathcal{A}^{eq-mod}, G}(M, \tilde{C}^{yon}) \end{aligned}$$

whose composition is again  $[e_M]$  on cohomology.  $\square$

### Equivariant Hochschild homology

Suppose, again for the sake of notational simplicity, that  $\mathcal{A}$  is an  $A_\infty$ -algebra with a  $G$ -action. We can apply the same idea as in (13.25) to  $A_\infty$ -bimodules, twisting the action by  $G$  on the right hand side only. In particular, by starting with the diagonal bimodule, one gets the *orbit bimodule*

$$(13.39) \quad \begin{aligned} O &= \mathcal{A} \otimes \mathbb{C}[G], \\ \mu_O^{s;1;r}(a'_s, \dots, a'_1; o; a_r, \dots, a_1)(g) \\ &= (-1)^{|a'_1| + \dots + |a'_s| + s - 1} \mu_{\mathcal{A}}^{r+1+s}(a'_s, \dots, a'_1, o(g), g(a_r), \dots, g(a_1)). \end{aligned}$$

This is a  $G$ -equivariant bimodule, with  $h \in G$  acting by

$$(13.40) \quad o(g) \longrightarrow h(o(h^{-1}gh)).$$

We define the equivariant Hochschild complex to be the invariant part of  $CC_*(\mathcal{A}, O)$  with respect to the action induced by (13.40):

$$(13.41) \quad CC_*^G(\mathcal{A}, \mathcal{A}) \stackrel{\text{def}}{=} CC_*(\mathcal{A}, O)^G = \left( \mathbb{C}[G] \otimes \bigoplus_{i \geq 1} \mathcal{A}^{\otimes i}[i-1] \right)^G$$

The expression in brackets is again a rational representation, hence if we define the equivariant Hochschild homology  $HH_*^G(\mathcal{A}, \mathcal{A})$  as the cohomology of (13.41), then

$$(13.42) \quad HH_*^G(\mathcal{A}, \mathcal{A}) \cong HH_*(\mathcal{A}, O)^G.$$

As in Example 13.9,  $O$  carries an additional structure of module over  $\mathbb{C}[G]$ , and (13.40) is compatible with the conjugation action of  $G$  on itself. Hence, (13.41) and its cohomology are naturally modules over  $\mathbb{C}[G]^{class}$ . In more geometric terms, the spaces  $HH_k(\mathcal{A}, O)$  can be thought of as quasi-coherent sheaves on  $G$  which are equivariant with respect to conjugation, and (13.42) are the invariant sections.

REMARK 13.16. *If  $G$  is a finite group, one has  $O = \bigoplus_{g \in G} g^* \mathcal{A}$ , where  $g^* \mathcal{A}$  is pullback of the diagonal bimodule by  $g$  acting on the right. One then finds that, in agreement with our original motivational remarks,*

$$(13.43) \quad HH_*^G(\mathcal{A}, \mathcal{A}) \cong HH_*(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G).$$

*The right hand side has been studied extensively as an algebraic analogue of orbifold cohomology, see for instance [172, Section 4] or [59].*

Given a equivariant  $A_\infty$ -functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ , one gets an induced equivariant bimodule map  $O_{\mathcal{A}} \rightarrow \mathcal{F}^* O_{\mathcal{B}}$  (where we have added the subscripts to distinguish between the two orbit bimodules involved), given by a formula similar to (13.39). Hence, there is an induced map between equivariant Hochschild complexes. Carrying over the discussion from Lecture 8, we note the following properties:

- (*Morita invariance*) The map  $HH_*^G(\mathcal{A}, \mathcal{A}) \rightarrow HH_*^G(\mathcal{A}^{eq-perf}, \mathcal{A}^{eq-perf})$  induced by the Yoneda embedding is an isomorphism.
- (*Künneth formula*) Suppose that  $G$  acts on  $\mathcal{A}$  and  $\mathcal{B}$ , and we form the tensor product  $\mathcal{A} \otimes \mathcal{B}$  in a way which is compatible with that action. Then there is a quasi-isomorphism of chain complexes of modules over the function algebra  $\mathbb{C}[G]$ ,

$$(13.44) \quad CC_*(\mathcal{A}, O_{\mathcal{A}}) \otimes_{\mathbb{C}[G]} CC_*(\mathcal{B}, O_{\mathcal{B}}) \longrightarrow CC_*(\mathcal{A} \otimes \mathcal{B}, O_{\mathcal{A} \otimes \mathcal{B}})$$

which is compatible with the action of  $G$  by conjugation. In particular, from that one gets an exterior product

$$(13.45) \quad HH_*^G(\mathcal{A}, \mathcal{A}) \otimes HH_*^G(\mathcal{B}, \mathcal{B}) \longrightarrow HH_*^G(\mathcal{A} \otimes \mathcal{B}, \mathcal{A} \otimes \mathcal{B}).$$

- (*Opposite property*) Consider  $\mathcal{A}^{opp}$  with the same action as  $\mathcal{A}$ . Passing to the opposite transforms the right pullback in (13.39) to the left pullback. Due to the

equivariance under conjugation, what one then gets is an isomorphism

$$(13.46) \quad HH_*^G(\mathcal{A}^{opp}, \mathcal{A}^{opp}) \cong HH_*^G(\mathcal{A}, \mathcal{A})$$

which is  $\mathbb{C}[G]^{class}$ -linear in a twisted way (one has to change the module structure by the automorphism  $g \mapsto g^{-1}$ ).

- (*Normalisation*) For  $\mathcal{A} = \mathbb{C}$  with the trivial action of  $G$ , it is straightforward to see (using the reduced version of the equivariant Hochschild complex) that

$$(13.47) \quad HH_*^G(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[G]^{class}.$$

Thanks to functoriality and Morita invariance, one can associate to any equivariantly perfect module  $P$  a class

$$(13.48) \quad [P]_{HH}^G \in HH_0^G(\mathcal{A}, \mathcal{A}).$$

In the special case of  $\mathcal{A} = \mathbb{C}$ , this means that to any complex of rational representations with finite-dimensional cohomology one can associate a class function, which unsurprisingly recovers (13.3). For any proper equivariant module one has a dual class,

$$(13.49) \quad [M]_{HH}^{\vee, G} \in Hom(HH_0^G(\mathcal{A}, \mathcal{A}), \mathbb{C}[G]^{class}),$$

and the analogue of (8.4), an “equivariant Cardy relation”, says that

$$(13.50) \quad \langle [M]_{HH}^{\vee, G}, [P]_{HH}^G \rangle = \chi(H^*(hom_{\mathcal{A}^{eq-mod}}(P, M))) \in \mathbb{C}[G]^{class}.$$

Supposing that  $\mathcal{A}$  is proper, one can also restrict to considering only perfect modules, which yields a pairing

$$(13.51) \quad (\cdot, \cdot)_{HH}^G : HH_0^G(\mathcal{A}, \mathcal{A}) \otimes HH_0^G(\mathcal{A}, \mathcal{A}) \longrightarrow \mathbb{C}[G]^{class}.$$

Since it involves passing to the opposite of  $\mathcal{A}$ , (13.51) is  $\mathbb{C}[G]^G$ -linear in the second variable, but linear only in the twisted sense in the first variable. This is compatible with what happens in (13.50) if one tensors  $M$  or  $P$  with finite-dimensional representations of  $G$ .

EXAMPLE 13.17. *Suppose that  $\mathcal{A}$  is a directed  $A_\infty$ -category with objects  $(X_1, \dots, X_m)$ , carrying an action of  $G$ . Then*

$$(13.52) \quad K_0(\mathcal{A}^{eq-tw, G}) \cong K_0(\mathcal{A}^{eq-perf, G}) \cong K_0(\mathcal{R}^{fin}(G))^m,$$

where the map from the right to the left carries  $(W_1, \dots, W_m)$  to the twisted complex  $\bigoplus_i W_i \otimes X_i$ . It is easy to see that

$$(13.53) \quad HH_*^G(\mathcal{A}, \mathcal{A}) \cong (\mathbb{C}[G]^{class})^m$$

(concentrated in degree 0). The map from (13.52) to (13.53) consists of  $m$  copies of (13.3). Starting from this and (13.50), one can give an explicit formula for (13.51) in terms of the characters of the morphism spaces in  $\mathcal{A}$ .

EXAMPLE 13.18. *Let  $\mathcal{A}$  be a linear graded algebra, carrying the action of  $G = \mathbb{G}_m$  which has weight  $j$  in degree  $j$ , as in Example 13.4. In that case,*

$$(13.54) \quad O = \mathcal{A} \otimes \mathbb{C}[G] = \mathcal{A}[t, t^{-1}] = \bigoplus_{k \in \mathbb{Z}} t^k \mathcal{A}.$$

The  $A_\infty$ -bimodule structure has only two nontrivial terms,

$$(13.55) \quad \begin{aligned} \mu_O^{1;1;0} : \mathcal{A} \otimes O &\longrightarrow O, \\ \mu_O^{0;1;1} : O \otimes \mathcal{A} &\longrightarrow O. \end{aligned}$$

Both of them are given by the multiplication in  $\mathcal{A}$  (up to signs due to general conventions). However, the left action of  $\mathcal{A}$  preserves each piece of (13.54), while the right action of  $\mathcal{A}^j$  maps the  $t^k$  piece to the  $t^{k+j}$  one.





LECTURE 14

**Making objects equivariant \*\*\*WARNING\*\*\*: PART OF  
THIS INHERITS ERRORS FROM THE PREVIOUS LECTURE**

Given a category with an action of a group  $G$ , one can ask when a given object  $Y$  can be made equivariant (in an appropriate sense). One significant example is Polishchuk’s result for derived categories of coherent sheaves (which was already mentioned in Lecture 1, as a generalisation of Proposition 1.8):

PROPOSITION 14.1 (Part of [153, Lemma 2.2]). *Let  $M$  be a smooth complex projective variety with an action of  $G$ , a connected reductive algebraic group such that  $\pi_1(G)$  is torsion-free. Let  $Y$  be an object of  $D^b \text{Coh}(M)$  such that*

$$(14.1) \quad \text{Hom}_{D^b \text{Coh}(M)}^*(Y, Y) = \begin{cases} 0 & * = 1, \\ \mathbb{C} & * = 0, \\ 0 & * < 0. \end{cases}$$

*Then  $Y$  is quasi-isomorphic to a bounded complex of  $G$ -equivariant coherent sheaves.*

Proofs of such results seem to naturally break into two steps. The first step is to make  $Y$  equivariant in a weak sense, which involves only cocycle identities that can be stated on the cohomological level. The second step is to replace weak equivariance by a (more desirable) stricter counterpart. For this second step, two alternative strategies have been pursued in the literature:

- One strategy involves obstructions which are cohomology classes of  $G$  with coefficients in the negative degree endomorphisms of  $Y$ . While this obstruction theory is very general, one then needs to impose suitable assumptions on  $G$  and  $Y$  which ensure that the obstructions vanish. A form of this is implicit in [153], and the fully developed idea would find a natural place in the theory of (higher) moduli stacks of objects [198]; there is also a more elementary formulation in [182], which the exposition here flows. One unfortunate aspect of this approach is that, in the case of algebraic groups, it ultimately encounters “convergence” issues, which apparently force one to impose additional assumptions.
- The other strategy, carried out in [65], is to impose restrictions on  $G$  from the start, which allow one to use a more elementary “averaging” process.

We will explore both approaches, since each is instructive in its own way. Ultimately, the more elementary approach seems to be more useful for our purpose. For expository reasons, we start by looking at discrete group actions, and then return to algebraic groups.

*Acknowledgments.* I would like to thank Bertrand Toën for explaining to me which form the obstruction theory should take.

### Discrete groups

Let  $\mathcal{A}$  be an  $A_\infty$ -category over a field  $\mathbb{K}$ , carrying an action of a (discrete) group  $G$ . Here, the notion of action is again understood in the naive sense, as an action on  $Ob(\mathcal{A})$  accompanied by maps between the morphism spaces, which strictly satisfy the relations of  $G$  and are compatible with the  $A_\infty$ -structure.

DEFINITION 14.2. *Suppose that  $\mathcal{A}$  is strictly unital (the  $G$ -action then automatically preserves the strict units). A strictly equivariant object of  $\mathcal{A}$  is an object  $Y$  together with morphisms*

$$(14.2) \quad \rho_Y^1(g) \in \text{hom}_{\mathcal{A}}^0(g(Y), Y), \quad g \in G,$$

such that

$$(14.3) \quad \begin{aligned} \mu_{\mathcal{A}}^1(\rho_Y^1(g)) &= 0, \\ \mu_{\mathcal{A}}^2(\rho_Y^1(g_2), g_2(\rho_Y^1(g_1))) &= \rho_Y^1(g_2g_1), \\ \rho_Y^1(e_G) &= e_Y. \end{aligned}$$

If  $Y$  is strictly fixed by  $G$ , then it becomes strictly equivariant by setting  $\rho_Y^1(g) = e_Y$  for each  $g \in G$ ; or one can also twist the action by a character  $\chi : G \rightarrow \mathbb{K}^\times$ , by setting  $\rho_Y^1(g) = \chi(g)e_Y$ . Strictly equivariant objects are a little more general than this example, but still not flexible enough for general use.

DEFINITION 14.3. *A weakly equivariant object of  $\mathcal{A}$  is an object  $Y$  together with morphisms*

$$(14.4) \quad \begin{aligned} \rho_Y^1(g) &\in \text{hom}_{\mathcal{A}}^0(g(Y), Y), \quad g \in G, \\ \rho_Y^2(g_2, g_1) &\in \text{hom}_{\mathcal{A}}^{-1}(g_2g_1(Y), Y), \quad g_1, g_2 \in G, \end{aligned}$$

such that the following relations hold:

$$(14.5) \quad \begin{aligned} \mu_{\mathcal{A}}^1(\rho_Y^1(g)) &= 0, \\ \mu_{\mathcal{A}}^1(\rho_Y^2(g_2, g_1)) + \mu_{\mathcal{A}}^2(\rho_Y^1(g_2), g_2(\rho_Y^1(g_1))) - \rho_Y^1(g_2g_1) &= 0, \end{aligned}$$

and  $\rho_Y^1(e_G)$  is cohomologous to the identity of  $Y$ .

This definition is of a quite different nature than the previous one: even though we have formulated it on the level of  $\mathcal{A}$ , its content is entirely cohomological, saying that the classes  $[\rho_Y^1(g)] \in \text{Hom}_{H(\mathcal{A})}^0(g(Y), Y)$  satisfy the condition

$$(14.6) \quad [\rho_Y^1(g_2g_1)] = [\rho_Y^1(g_2)] \cdot g_2([\rho_Y^1(g_1)]).$$

Similarly, if  $X, Y$  are two weakly equivariant objects, then  $\text{Hom}_{H(\mathcal{A})}^*(X, Y)$  carries an induced action of  $G$ , given by

$$(14.7) \quad [a] \mapsto [\rho_Y^1(g)] \cdot [g(a)] \cdot [\rho_X^1(g)]^{-1}.$$

Even though it was not strictly necessary so far, the chain level formulation in (14.5) is still useful, because it motivates an extension to higher homotopies:

DEFINITION 14.4. *A coherently equivariant object of  $\mathcal{A}$  is an object  $Y$  together with*

$$(14.8) \quad \rho_Y^r(g_r, \dots, g_1) \in \text{hom}_{\mathcal{A}}^{1-r}(g_r \cdots g_1(Y), Y), \quad g_1, \dots, g_r \in G$$

such that the following generalization of (14.5) holds:

$$(14.9) \quad \sum \mu_{\mathcal{A}}^d(\rho_Y^{r_d}(g_r, \dots, g_{r-d+1}), \dots, \\ g_r \cdots g_{r_1+r_2+1}(\rho_Y^{r_2}(g_{r_1+r_2}, \dots, g_{r_1+1})), g_r \cdots g_{r_1+1}(\rho_Y^{r_1}(g_{r_1}, \dots, g_1))) \\ + \sum_q (-1)^q \rho_Y^{r-1}(g_r, \dots, g_{q+1}g_q, \dots, g_1) = 0.$$

Here, the first sum is over all partitions of  $r$  ( $d \geq 1$ ,  $r_1 + \dots + r_d = r$ ). We also impose the same condition on  $\rho_Y^1(e_G)$  as in Definition 14.3.

PROPOSITION 14.5. *Let  $Y \in \text{Ob}(\mathcal{A})$  be weakly equivariant, and make  $\text{Hom}_{H(\mathcal{A})}^*(Y, Y)$  into a representation of  $G$  via (14.7). Suppose that the group cohomology with coefficients in that representation satisfies*

$$(14.10) \quad H^r(G, \text{Hom}_{H(\mathcal{A})}^{1-r}(Y, Y)) = 0 \quad \text{for } r \geq 2.$$

Then  $Y$  can be made coherently equivariant.

This is an obstruction theory exercise, which we will not reproduce (for some more details, see [182, Section 8c]).

EXAMPLE 14.6. *Consider the situation where the  $G$ -action on  $\mathcal{A}$  is trivial. Then, a coherently equivariant structure for  $Y$  is the same as an  $A_\infty$ -functor*

$$(14.11) \quad \mathcal{R} : \mathbb{K}[G] \longrightarrow \mathcal{A}$$

(the group ring  $\mathbb{K}[G]$  is considered as an  $A_\infty$ -category with a single object, which maps to  $Y$  under  $\mathcal{R}$ ). The standard obstruction theory for building  $A_\infty$ -functors involves Hochschild cohomology [176, Section 1g]; but as already noticed in [64, Equation (5.7)], for group rings Hochschild cohomology reduces to group cohomology.

The  $G$ -action on  $\mathcal{A}$  induces one on  $\mathcal{A}^{mod}$ , so in principle all the flavours of equivariance discussed above (strict, weak, coherent) could be carried over to modules. However, we will reserve the simpler name “equivariant  $A_\infty$ -module” for the naive notion as in Lecture 13, meaning that such a module consists of graded vector spaces  $M(X)$  together with linear maps  $M(X) \rightarrow M(g(X))$ , which strictly satisfy the group relations and are compatible with the  $A_\infty$ -module structure. In that sense, we have:

PROPOSITION 14.7. *Suppose that  $Y$  is coherently equivariant. Then its Yoneda module is quasi-isomorphic to an equivariant  $A_\infty$ -module.*

PROOF. This is done by an explicit construction, based on the standard projective bar resolution of  $G$ -modules. To motivate that construction, it is convenient to temporarily assume that  $\mathcal{A}$  is strictly unital, and to write down an “infinite twisted complex” of the form

$$(14.12) \quad C = \bigoplus g_r \cdots g_1(Y)[r-1]$$

where the sum is over  $r \geq 1$  and  $(g_1, \dots, g_r) \in G^r$ . Let’s denote the summands by

$$(14.13) \quad \frac{(g_r, \dots, g_1)}{g_r \cdots g_1(Y)[r-1]}.$$

The differential  $\delta_C$  combines multiplication on  $G$  and the morphisms (14.8). A piece of this “twisted complex” is represented (without signs) in the following diagram:

$$(14.14) \quad \begin{array}{ccccc} & & \frac{(g_3, g_2 g_1)}{g_3 g_2 g_1(Y)[1]} & \xrightarrow{g_3(\rho_Y^1(g_2 g_1))} & \frac{g_3}{g_3(Y)} \\ & \nearrow^{e_{g_3 g_2 g_1(Y)}} & & \nearrow^{g_3(\rho_Y^2(g_2, g_1))} & \nearrow^{g_3(\rho_Y^1(g_2))} \\ \frac{(g_3, g_2, g_1)}{g_3 g_2 g_1(Y)[2]} & \xrightarrow{g_3 g_2(\rho_Y^1(g_1))} & \frac{(g_3, g_2)}{g_3 g_2(Y)[1]} & & \\ & \searrow_{e_{g_3 g_2 g_1(Y)}} & & \searrow_{e_{g_3 g_2(Y)}} & \\ & & \frac{(g_3 g_2, g_1)}{g_3 g_2 g_1(Y)[1]} & \xrightarrow{g_3 g_2(\rho_Y^1(g_1))} & \frac{g_3 g_2}{g_3 g_2(Y)} \end{array}$$

The necessary Maurer-Cartan equation is a consequence of (14.9), as illustrated by the top part of (14.14). Moreover, again using (14.8), one defines a “morphism of twisted complexes”  $C \rightarrow Y$ .

The previous discussion is not entirely rigorous, since we have not specified the meaning of “infinite twisted complex”. However, one can certainly associate to  $(C, \delta_C)$  a Yoneda-type left  $\mathcal{A}$ -module  $N$ ; and at that point, one can in fact forget about the motivation and just take the definition of  $N$  as a starting point (this also allows one to drop the strict unitality assumptions). The underlying graded vector spaces are

$$(14.15) \quad N(X) = \prod \text{hom}_{\mathcal{A}}(g_r \cdots g_1(Y), X)[1-r],$$

with the same indexing set as before, and with differential written in parallel with (14.9) as

$$(14.16) \quad \begin{aligned} (\mu_N^{0;1}(f))^r(g_r, \dots, g_1) &= \sum (-1)^{|f|(r-r_d)} \mu_{\mathcal{A}}^d(f^{r_d}(g_r, \dots, g_{r-r_d+1}), \dots, \\ &g_r \cdots g_{r_1+r_2+1}(\rho_Y^{r_2}(g_{r_1+r_2}, \dots, g_{r_1+1})), g_r \cdots g_{r_1+1}(\rho_Y^{r_1}(g_{r_1}, \dots, g_1))) \\ &+ \sum_q (-1)^{|f|+q-1} f^{r-1}(g_r, \dots, g_{q+1}g_q, \dots, g_1). \end{aligned}$$

$N$  is an equivariant module, with the action  $(g \cdot f)^r(g_r, \dots, g_1) = f(g_r, \dots, g_1 g)$ . There is also a distinguished cocycle in  $N(Y)$ , which gives rise to a quasi-isomorphism from the left

Yoneda module of  $Y$  to  $N$  (the quasi-isomorphism property is proved by a simple filtration argument). By passing to  $\mathcal{A}^{opp}$ , one obtains the corresponding statement for right modules, which is the desired result.  $\square$

EXAMPLE 14.8. *Again, it is maybe instructive to think of the case when the action is trivial, so that the coherently equivariant structure is given by (14.11). Then*

$$(14.17) \quad N = \text{hom}_{\mathbb{C}[G]^{mod}}(\mathbb{C}[G], \mathcal{R}^* \mathcal{A}),$$

where we pull back the diagonal bimodule  $\mathcal{A}$  via (14.11) to an  $(\mathcal{A}, \mathbb{C}[G])$ -bimodule, and then take the space of  $A_\infty$ -module maps for the right action of  $\mathbb{C}[G]$ , leaving a left  $\mathcal{A}$ -module.

COROLLARY 14.9. *Suppose that  $Y$  is weakly equivariant, and that (14.10) holds. Then the Yoneda module of  $Y$  is quasi-isomorphic to an equivariant  $A_\infty$ -module.*  $\square$

This concludes our discussion of the obstruction theory approach. We now pass to the alternative and more elementary strategy. As before, we temporarily impose the condition that  $\mathcal{A}$  should be strictly unital. More importantly, we require that:

$$(14.18) \quad G \text{ is a finite group, and its order } |G| \text{ is coprime to } \text{char}(\mathbb{K}).$$

We need to quickly reconsider the definition of equivariant twisted complex over  $\mathcal{A}$ , to make sure that it is properly adjusted to the finite group case. Such a complex  $C$  is of the form (7.12) where the indexing set  $I$  carries an action of our group  $G$ , the associated objects satisfy  $X_{g(i)} = g(X_i)$ , and the vector spaces come with linear graded maps  $\lambda_i(g) : W_i \rightarrow W_{g(i)}$ , whose combined action

$$(14.19) \quad \bigoplus_i \lambda_i(g) \otimes e_{g(X_i)} : g(C) = \bigoplus_i W_i \otimes g(X_i) \longrightarrow C = \bigoplus_i W_{g(i)} \otimes X_{g(i)}$$

yields maps of twisted complexes satisfying the same conditions as in Definition 14.2. We will actually need only one particularly simple example: given any object  $Y$  of  $\mathcal{A}$ , the direct sum

$$(14.20) \quad Y^{orbit} = \bigoplus_{g \in G} g(Y)$$

(with vanishing differential) is obviously an equivariant twisted complex. In particular, it is a strictly equivariant object of the category  $\mathcal{A}^{tw}$  with respect to the induced  $G$ -action.

LEMMA 14.10. *Suppose that (14.18) holds, and let  $Y$  be weakly equivariant. Then  $Y$  is a homotopy retract of  $Y^{orbit}$ , in such a way that the associated (cohomology level) maps  $Y \rightarrow Y^{orbit}$  and  $Y^{orbit} \rightarrow Y$  are  $G$ -invariant with respect to (14.7).*

PROOF. Consider first the morphisms

$$(14.21) \quad \begin{aligned} & \bigoplus_{g \in G} g(\rho_Y^1(g^{-1})) : Y \longrightarrow Y^{orbit}, \\ & \bigoplus_{g \in G} \rho_Y^1(g) : Y^{orbit} \longrightarrow Y. \end{aligned}$$

Their composition

$$(14.22) \quad \sum_g \mu_{\mathcal{A}}^2(\rho_Y^1(g), g(\rho_Y^1(g^{-1}))) = \sum_g \mu_{\mathcal{A}}^1(\rho_Y^2(g, g^{-1})) + |G|\rho_Y^1(e)$$

is cohomologous to  $|G|$  times the identity (readers will recognize this idea from Lemma 13.7). By assumption (14.18), we can divide by  $|G|$ . Invariance of the maps (14.21) under (14.7) can be checked by hand.  $\square$

**PROPOSITION 14.11.** *In the same situation as Lemma 14.10, the Yoneda module associated to  $Y$  is quasi-isomorphic to an equivariant  $A_\infty$ -module.*

**PROOF.** Take the morphisms in (14.21), compose them in reverse order and divide by  $|G|$ , so as to get a cohomology level endomorphism of  $Y^{orbit}$  which is idempotent and  $G$ -invariant.

As mentioned in Lecture 7, the category  $\mathcal{A}^{mod}$  is closed under homotopy retracts. Similarly, given an equivariant module and a  $G$ -invariant idempotent endomorphism, one can find a  $G$ -invariant module which is its homotopy retract (under our assumptions on  $G$  and  $\mathbb{K}$ ). Indeed, we have already discussed this (for the more difficult case of algebraic groups) in Lemma 13.14. Applying this fact to the Yoneda module  $M^{orbit}$  of  $Y^{orbit}$  and the given idempotent yields the desired result.  $\square$

Clearly, (14.18) implies that all the cohomology groups in (14.10) vanish. Hence Proposition 14.11 is a special case of Corollary 14.9 (but the proof given above is more direct). Note also that in both approaches, we could have started with a weakly equivariant module instead of a weakly equivariant object of  $\mathcal{A}$  itself.

## Algebraic groups

We now return to the main subject of the discussion, which means actions of reductive groups  $G$  as in Lecture 13. This differs from the discrete group case in two aspects: first, the homological algebra of rational representations of  $G$  is built via comodules over the coalgebra of functions  $\mathbb{C}[G]$ , hence uses injective resolutions dual to the projective ones appearing in the discrete case. Secondly, actions of  $G$  on  $A_\infty$ -categories are trivial on objects by definition, which means that it makes more sense to argue throughout on the level of modules.

Recall from Example 13.9 that to any  $\mathcal{A}$ -module  $M$  one can associate its orbit module  $M^{orbit}$ . This is parallel to (14.20) in that it assembles all the pullbacks  $g^*M$ . To formulate this more rigorously: if we take any point  $g \in G$  and the corresponding evaluation homomorphism  $\mathbb{C}[G] \rightarrow \mathbb{C}$ , then the fibre at that point is

$$(14.23) \quad \mathbb{C} \otimes_{\mathbb{C}[G]} M^{orbit} \cong g^*M.$$

For any  $r \geq 1$ , we will also use the pullback of  $M^{orbit}$  under the total multiplication map  $G^r \rightarrow G$ ,  $(g_r, \dots, g_1) \mapsto g_r \cdots g_1$ , which we denote by

$$(14.24) \quad M^{orbit, (r)} = \mathbb{C}[G]^{\otimes r} \otimes_{\mathbb{C}[G]} M^{orbit}.$$

Explicitly:

$$(14.25) \quad \begin{aligned} M^{orbit,(r)}(X) &= \mathbb{C}[G]^{\otimes r} \otimes M(X), \\ \mu_{M^{orbit,(r)}}^{1;d}(m; a_d, \dots, a_1)(g_r, \dots, g_1) &= \mu_M^{1;d}(m(g_r, \dots, g_1); g_r \cdots g_1(a_d), \\ &\quad \dots, g_r \cdots g_1(a_1)). \end{aligned}$$

DEFINITION 14.12. *A strictly equivariant  $\mathcal{A}$ -module is an  $A_\infty$ -module  $M$  together with a homomorphism*

$$(14.26) \quad \rho_M^1 \in \text{hom}_{\mathcal{A}^{mod}}^0(M, M^{orbit})$$

*which pointwise over  $G$  satisfies conditions analogous to those in Definition 14.2:*

$$(14.27) \quad \begin{aligned} \mu_{\mathcal{A}^{mod}}^1(\rho_M^1(g)) &= 0, \\ \mu_{\mathcal{A}^{mod}}^2(g_1^* \rho_M^1(g_2), \rho_M^1(g_1)) &= \rho_M^1(g_2 g_1), \\ \rho_M^1(e_G) &= e_M. \end{aligned}$$

“Pointwise over  $G$ ” means after evaluation using (14.23) (one can also formulate this definition without reference to points, see [182, Equation 7.20]). Since we are considering pullbacks of modules rather than pushforward, the action of  $G$  is on the target object of (14.26) rather than the source as in (14.2). As a final point of comparison, no condition of strict unitality on  $\mathcal{A}$  is required, since we are actually working in  $\mathcal{A}^{mod}$  which is always strictly unital.

DEFINITION 14.13. *A weakly equivariant  $\mathcal{A}$ -module is an  $A_\infty$ -module  $M$  together with elements*

$$(14.28) \quad \begin{aligned} \rho_M^1 &\in \text{hom}_{\mathcal{A}^{mod}}^0(M, M^{orbit}), \\ \rho_M^2 &\in \text{hom}_{\mathcal{A}^{mod}}^{-1}(M, M^{orbit,(2)}) \end{aligned}$$

*which pointwise over  $G^2 = G \times G$  satisfies conditions analogous to those in Definition 14.3:*

$$(14.29) \quad \begin{aligned} \mu_{\mathcal{A}^{mod}}^1(\rho_M^1(g)) &= 0, \\ \mu_{\mathcal{A}^{mod}}^1(\rho_M^2(g_2, g_1)) + \mu_{\mathcal{A}^{mod}}^2(g_1^* \rho_M^1(g_2), \rho_M^1(g_1)) - \rho_M^1(g_2 g_1) &= 0, \\ [\rho_M^1(e_G)] &= [e_M] \in H^0(\text{hom}_{\mathcal{A}^{mod}}(M, M)). \end{aligned}$$

If  $M$  and  $N$  are weakly equivariant,  $H^*(\text{hom}_{\mathcal{A}^{mod}}(M, N))$  carries a  $G$ -action, defined as in (14.7).

LEMMA 14.14 (Compare [182, Lemma 7.9]). *If  $M$  and  $N$  are weakly equivariant, and  $M$  is perfect, then  $H^*(\text{hom}_{\mathcal{A}^{mod}}(M, N))$  is a rational  $G$ -module.*

PROOF. Consider the  $\mathbb{C}[G]$ -linear chain maps

$$(14.30) \quad \text{hom}_{\mathcal{A}^{mod}}(M, N \otimes \mathbb{C}[G]) \longrightarrow \text{hom}_{\mathcal{A}^{mod}}(M, N^{orbit}) \longleftarrow \text{hom}_{\mathcal{A}^{mod}}(M, N \otimes \mathbb{C}[G])$$

pointwise given by

$$(14.31) \quad \begin{aligned} \text{hom}_{\mathcal{A}^{mod}}(M, N) &\xrightarrow{\mu^2(\rho_N^1(g), \cdot)} \text{hom}_{\mathcal{A}^{mod}}(M, g^*N), \\ \text{hom}_{\mathcal{A}^{mod}}(M, g^*N) &\xleftarrow{\mu^2(\cdot, \rho_M^1(g))} \text{hom}_{\mathcal{A}^{mod}}(g^*M, g^*N) \cong \text{hom}_{\mathcal{A}^{mod}}(M, N). \end{aligned}$$

Both are quasi-isomorphisms. If  $M$  is perfect, the inclusion

$$(14.32) \quad \text{hom}_{\mathcal{A}^{mod}}(M, N) \otimes \mathbb{C}[G] \longrightarrow \text{hom}_{\mathcal{A}^{mod}}(M, N \otimes \mathbb{C}[G])$$

is also a quasi-isomorphism. Take the cohomology level map induced by (14.30) and restrict it to

$$(14.33) \quad H^*(\text{hom}_{\mathcal{A}^{mod}}(M, N)) \longrightarrow H^*(\text{hom}_{\mathcal{A}^{mod}}(M, N)) \otimes \mathbb{C}[G].$$

This gives  $H^*(\text{hom}_{\mathcal{A}^{mod}}(M, N))$  the structure of a  $\mathbb{C}[G]$ -comodule, which is the same as a rational representation. One checks easily that the associated  $G$ -action reproduces (14.7).  $\square$

**DEFINITION 14.15.** *A coherently equivariant  $\mathcal{A}$ -module is an  $A_\infty$ -module  $M$  together with elements*

$$(14.34) \quad \rho_M^r \in \text{hom}_{\mathcal{A}^{mod}}^{1-r}(M, M^{orbit, (r)})$$

for all  $r \geq 1$ , which pointwise over  $G^r$  satisfies conditions analogous to those in Definition 14.4; namely, the last part of (14.29) together with the following strengthening of the first two parts:

$$(14.35) \quad \begin{aligned} &\mu_{\mathcal{A}^{mod}}^1(\rho_M^r(g_r, \dots, g_1)) + \sum_i \mu_{\mathcal{A}^{mod}}^2(g_1^* \cdots g_i^* \rho_M^{r-i}(g_r, \dots, g_{i+1}), \rho_M^i(g_i, \dots, g_1)) \\ &+ \sum_q (-1)^q \rho_M^{r-1}(g_r, \dots, g_{q+1}g_q, \dots, g_1) = 0. \end{aligned}$$

The simpler form of (14.35) compared to (14.9) is due to the fact that  $\mathcal{A}^{mod}$  is a dg category.

**PROPOSITION 14.16** ([182, Lemma 8.3]). *Take a perfect  $A_\infty$ -module  $M$ . If  $M$  can be made weakly equivariant, then it can also be made coherently equivariant.*  $\square$

Even though this may not be analogous from the formulation, the result and proof are entirely analogous to Proposition 14.5. One encounters obstructions to equivariance lying in the cohomology of the algebraic group  $G$ , as in (14.10), but these vanish in our context because  $G$  is reductive and  $H^*(\text{hom}_{\mathcal{A}}(M, M))$  is a rational representation (by Lemma 13.13); see [89] for the required group cohomology background. In principle, the same idea could be applied to other linear algebraic groups, where the obstructions may be nonzero in general, but we have not tried to carry that out (and instead have included reductiveness in the basic setup of the theory of equivariant modules, in Lecture 13).

**PROPOSITION 14.17** ([182, Lemma 8.2]). *Let  $M$  be a coherently equivariant  $G$ -module such that for any  $X \in \text{Ob}(\mathcal{A})$ , the graded vector space  $M(X)$  is bounded below. Then  $M$  is quasi-isomorphic to an equivariant  $A_\infty$ -module.*



This is the analogue of Proposition 14.7, and is based on the same construction starting with the right module analogue of (14.15), which is

$$(14.36) \quad N(X) = \prod_{r=1}^{\infty} \mathbb{C}[G]^{\otimes r} \otimes M(X)[1-r]$$

and where  $G$  acts on the leftmost of the  $\mathbb{C}[G]$  factors. The technical wrinkle is that, because of the infinite product, (14.36) may not be a rational representation of  $G$  in general. The boundedness condition we have imposed ensures rationality, since it means that only finitely many  $r$  contribute to the direct product in each degree.

One can weaken the boundedness condition to a cohomological one, see [182, Corollary 8.4]. However, it remains a clear weakness of the obstruction theory approach (which one might interpret as a “convergence issue”). We will therefore now switch to the alternative and more direct strategy, in which this problem does not appear.

LEMMA 14.18. *Let  $M$  be a perfect  $A_{\infty}$ -module which is weakly equivariant. Then  $M$  is a homotopy retract of an equivariant module  $\tilde{N}$ , in such a way that the associated (cohomology level) maps  $M \rightarrow \tilde{N}$  and  $\tilde{N} \rightarrow M$  are  $G$ -invariant. Moreover,  $\tilde{N}$  is equivariantly perfect.*

This is roughly analogous to Lemma 14.10 (which applied to finite groups). However, the proof does not follow the same path, but instead uses the same idea as in Proposition 13.15.

PROOF. First, let’s start with an arbitrary perfect  $M$ . Lemma 13.7 shows that  $M$  is a homotopy retract of an equivariant module  $N$ , where the latter is also perfect (since it comes from an equivariant twisted complex). Denote the associated maps by

$$(14.37) \quad \begin{aligned} [\rho] &\in H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(N, M)), \\ [\iota] &\in H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(M, N)). \end{aligned}$$

At this point, add the assumption that  $M$  is weakly equivariant. From Lemma 14.14 we know that both cohomology groups in (14.37) are rational representations of  $G$ . Hence, only finitely many of the components

$$(14.38) \quad \begin{aligned} [\rho]_V &\in H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(N, M))_V \cong H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(V \otimes V^{\vee} \otimes N, M))^G, \\ [\iota]_{V^{\vee}} &\in H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(M, N))_{V^{\vee}} \cong H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(M, V \otimes V^{\vee} \otimes N))^G \end{aligned}$$

can be nonzero. Here, we have formed the equivariant module  $V \otimes V^{\vee} \otimes N$  by using the trivial action on  $V$  and the given action on  $V^{\vee}$  (as in Lemma 13.7).

Write  $V_1, \dots, V_r$  for the representations for which  $[\rho]_V$  or  $[\iota]_{V^{\vee}}$  are nonzero. Take  $\tilde{N} = \bigoplus_i V_i \otimes V_i^{\vee} \otimes N$ . Then, as in (13.38) but remaining on the cohomology level throughout, one constructs elements

$$(14.39) \quad \begin{aligned} [\tilde{\rho}] &\in H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\tilde{N}, M)), \\ [\tilde{\iota}] &\in H^0(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(M, \tilde{N})) \end{aligned}$$

which are  $G$ -invariant and provide the desired retraction.  $\square$

By arguing as in the proof of Proposition 14.11, one then concludes the following:

**COROLLARY 14.19.** *Let  $M$  be a perfect  $A_\infty$ -module which is weakly equivariant. Then  $M$  is quasi-isomorphic to an equivariant  $A_\infty$ -module, which is again perfect.  $\square$*

### Rigidity and equivariance

We now return to the first bullet point from the discussion at the start, which is how to obtain weak equivariance for a given object. The basic idea is to first establish equivariance on an infinitesimal level, and then to integrate that. This obviously works only for connected groups, and for fields like  $\mathbb{C}$  which permit exponentiation (indeed, we have already carried out an elementary example of such an argument, in Proposition 1.8).

Let  $\mathcal{A}$  be an  $A_\infty$ -category with an action of  $G$ . Any  $A_\infty$ -module  $M$  then comes with a canonical *Killing class*

$$(14.40) \quad Ki(M) \in \mathfrak{g}^\vee \otimes H^1(\text{hom}_{\mathcal{A}^{mod}}(M, M)),$$

where  $\mathfrak{g}$  is the Lie algebra. The natural cocycle representative of that class is

$$(14.41) \quad ki(M)^{1;d}(m; a_d, \dots, a_1) = - \sum_{j,k} \gamma_j^\vee \otimes \mu_M^{1;d}(m; a_d, \dots, \gamma_j(a_k), \dots, a_1),$$

where  $(\gamma_j)$ ,  $(\gamma_j^\vee)$  are dual bases of  $\mathfrak{g}$  and  $\mathfrak{g}^\vee$ , and  $\gamma_j(a_k)$  is the infinitesimal action on  $\mathcal{A}$ . Given two modules  $M$  and  $N$ , the composition with their deformation classes on either side yields the same map

$$(14.42) \quad Ki(N) \cdot = \cdot Ki(M) : H^*(\text{Hom}_{\mathcal{A}^{mod}}(M, N)) \longrightarrow \mathfrak{g}^\vee \otimes H^*(\text{Hom}_{\mathcal{A}^{mod}}(M, N)).$$

In terms introduced in Lecture 9, this reflects the more fundamental fact that (14.40) is the leading order term of a class in  $\mathfrak{g}^\vee \otimes HH^1(\mathcal{A}^{mod}, \mathcal{A}^{mod})$ . One consequence of (14.42) is that (14.40) is invariant under quasi-isomorphism.

If  $M$  is equivariant,  $ki(M)$  is the coboundary of

$$(14.43) \quad m \mapsto (-1)^{|m|} \sum_j \gamma_j^\vee \otimes \gamma_j(m).$$

Hence, we see that (14.40) is indeed an obstruction to finding an equivariant module quasi-isomorphic to  $M$ . In fact, even if  $M$  is only weakly equivariant, differentiation of  $\rho_M^1(g)$  at  $g = e_G$  still yields a cochain bounding  $ki(M)$ .

**PROPOSITION 14.20** ([182, Lemma 7.12]). *Let  $\mathcal{A}$  be an  $A_\infty$ -category over  $\mathbb{C}$ , with an action of  $G = \mathbb{G}_m$ . Suppose also that the morphism spaces  $H^*(\text{hom}_{\mathcal{A}}(X, Y))$  are finite-dimensional in each degree. Let  $M$  be a perfect module such that (14.40) vanishes, and such that*

$$(14.44) \quad H^0(\text{hom}_{\mathcal{A}^{mod}}(M, M)) \cong \mathbb{C}.$$

*Then  $M$  can be made weakly equivariant.*

The strategy of proof goes as follows. The weak version of properness we have imposed on  $\mathcal{A}$ , together with the fact that  $M$  is perfect, implies that  $H^0(\text{hom}_{\mathcal{A}^{mod}}(M, M^{orbit}))$  is finitely

generated as a module over the algebra of functions  $\mathbb{C}[G]$ . Vanishing of the infinitesimal obstruction implies vanishing of the Atiyah class for this module, which is therefore locally free (a standard fact about coherent sheaves admitting algebraic connections). Because of (14.44), the module has rank one, hence is trivial. A suitable choice of nowhere vanishing section then yields the element  $[\rho_M^1]$  providing the weak equivariance structure. We will not reproduce the details, in part because some of the relevant terminology will only be introduced later on (Lecture 20).

**COROLLARY 14.21.** *If  $A$  and  $M$  are as in Proposition 14.20, there is an equivariant  $A_\infty$ -module which is quasi-isomorphic to  $M$ .  $\square$*

This is the result we will actually use in applications. It is obtained by combining Corollary 14.19 with Proposition 14.20.

**REMARK 14.22.** *In principle, there is no reason to restrict oneself to the multiplicative group, but in a more general context one encounters an additional obstruction, which already appears in Proposition 14.1. Suppose that  $G$  is a connected reductive group. Suppose also that the group  $\text{Aut}(M) = H^0(\text{hom}_{\mathcal{A}^{\text{mod}}}(M, M))^\times$  is again reductive. Then  $M$  becomes equivariant not for the original  $G$ , but for a larger group  $\tilde{G}$  which fits into a short exact sequence*

$$(14.45) \quad 1 \rightarrow \text{Aut}(M) \longrightarrow \tilde{G} \longrightarrow G \rightarrow 1.$$

*The obstruction consists in the fact that (14.45) may not allow an algebraic splitting. If we assume that  $H^0(\text{hom}_{\mathcal{A}^{\text{mod}}}(M, M)) = \mathbb{C}$ , then  $\text{Aut}(M) = \mathbb{C}^*$  and the extension is central. But even then there are cases where no splitting exists, such as*

$$(14.46) \quad 1 \rightarrow \mathbb{C}^* \longrightarrow GL_2(\mathbb{C}) \longrightarrow PSL_2(\mathbb{C}) \rightarrow 1,$$

*which appears when one thinks of the action of  $PSL_2(\mathbb{C})$  on the projective line. The assumption on  $\pi_1(G)$  in Proposition 14.1 is exactly what allows one to rule out such difficulties.*



## LECTURE 15

### Spherical objects and simple singularities

\*\*\*WARNING\*\*\*: SAME ERROR PROPAGATES

We return to the fundamental idea of Lecture 1, namely that the equivariant Mukai pairing can be used to derive restrictions on suitable classes of objects. However, instead of applying this idea to exceptional objects as before, we now consider *spherical* ones [185]. In the context of Fukaya categories, such objects arise from Lagrangian spheres, and what we are looking for are restrictions on the homology classes of such spheres.

In order to apply this method, one must ensure two things: that a complete algebraic description of the Fukaya category is available; and that this description allows for a circle action of a suitable kind. While the origin of these actions is mysterious from the present viewpoint, one can readily check that they are present in a number of examples. Specifically, we will consider the Milnor fibres of simple (*ADE* type) singularities (a general background reference for the topology of isolated hypersurface singularities is [16]). The type *A* case has been studied extensively, and results similar to the ones here were obtained in [182]. Thanks to a certain amount of streamlining in the basic machinery, the *D* and *E* cases can now be treated in the same way (this is still by no means the most general achievable result; in fact, these kinds of questions can be useful as a yardstick of progress).

*Acknowledgments.* Remark 15.16, concerning more general plumbings of spheres, was explained to me by Mohammed Abouzaid.

#### Algebraic setup

Let  $\mathcal{A}$  be an  $A_\infty$ -category over  $\mathbb{C}$  which is proper and weakly cyclic of dimension  $n > 0$ . We recall (from Example 7.8) that the latter property means that there is a bimodule quasi-isomorphism

$$(15.1) \quad \mathcal{A}^\vee \cong \mathcal{A}[n].$$

Because of the relation (7.31) between  $\mathcal{A}^\vee$  and the Serre functor, this implies that

$$(15.2) \quad H^*(\mathrm{hom}_{\mathcal{A}^{\mathrm{perf}}}(P, Q)) \cong H^{n-*}(\mathrm{hom}_{\mathcal{A}^{\mathrm{perf}}}(Q, P))^\vee$$

for any perfect modules  $P, Q$ . Next, we also assume that  $\mathcal{A}$  is *simply-connected* in the following sense:

(15.3) One can associate to any object  $X$  a real number  $\sigma_X$ , such that the only part of  $H^*(\text{hom}_{\mathcal{A}}(X, Y))$  of degree  $\leq \sigma_Y - \sigma_X + 1$  consists of multiples of the identity endomorphisms.

LEMMA 15.1.  $HH_0(\mathcal{A}, \mathcal{A})$  is freely generated generated by the classes  $[e_X]$  of the identity endomorphisms of the objects of  $\mathcal{A}$ .

PROOF. Replace  $\mathcal{A}$  by a quasi-equivalent  $A_\infty$ -category which is minimal, and consider an expression in the reduced Hochschild complex (Remark 8.5)

$$(15.4) \quad a \otimes a_d \otimes \cdots \otimes a_1 \in CC_*^{\text{red}}(\mathcal{A}, \mathcal{A}).$$

This consists of a closed chain of morphisms  $a_1 \in \text{hom}_{\mathcal{A}}(X_0, X_1), \dots, a_d \in \text{hom}_{\mathcal{A}}(X_{d-1}, X_d), a \in \text{hom}_{\mathcal{A}}(X_d, X_0)$ . The degree of (15.4) in the Hochschild complex is

$$(15.5) \quad |a| + \sum_k |a_k| - d = (|a| + \sigma_{X_0} - \sigma_{X_d}) + \sum_k (|a_k| + \sigma_{X_k} - \sigma_{X_{k-1}}) - d \geq 0,$$

and equality can only hold if  $d = 0$  and  $a$  is homologous to a multiple of the identity endomorphism. On the other hand, each identity endomorphism  $e_X$  is a Hochschild cocycle.  $\square$

LEMMA 15.2. The pairing (8.10) on  $HH_0(\mathcal{A}, \mathcal{A})$  satisfies

$$(15.6) \quad (x_1, x_0)_{HH} = (-1)^n (x_0, x_1)_{HH}.$$

PROOF. By the previous Lemma, it is sufficient to show this when  $x_0$  and  $x_1$  are identity endomorphisms of objects  $X_0$  and  $X_1$ . But those Hochschild homology classes are also the classes  $[X_0]_{HH}$  and  $[X_1]_{HH}$  of those objects, in the general sense of (8.2). Then, the desired statement reduces to (15.2) by a special case of the Cardy relation (Lemma 8.1):

$$(15.7) \quad (x_1, x_0)_{HH} = \chi(H^*(\text{hom}_{\mathcal{A}}(X_1, X_0))) = \chi(H^{n-*}(\text{hom}_{\mathcal{A}}(X_0, X_1))) = (-1)^n (x_0, x_1)_{HH}.$$

$\square$

A perfect module  $S \in \text{Ob}(\mathcal{A}^{\text{perf}})$  is called *spherical* if its endomorphism ring is (nonzero and) as small as it can be given (15.2):

$$(15.8) \quad H^*(\text{hom}_{\mathcal{A}^{\text{perf}}}(S, S)) \cong \begin{cases} \mathbb{C} & * = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 15.3. Suppose that  $n$  is even. Then the class  $[S]_{HH}$  of any spherical object is nonzero. Moreover, if  $S_1, \dots, S_k$  are spherical objects such that  $H^*(\text{hom}_{\mathcal{A}^{\text{perf}}}(S_i, S_j)) = 0$  for all  $i \neq j$ , then the classes  $[S_i]_{HH}$  are linearly independent over  $\mathbb{R}$ .

PROOF. These are both straightforward consequences of the same Cardy relation we used before, which says that  $(S, S)_{HH} = 2$  and  $(S_i, S_j)_{HH} = 0$ .  $\square$

We will be interested in the corresponding question when  $n$  is odd. This is a priori much more difficult, and in fact we will be able to make progress on it only in the presence of suitable additional symmetries.

### Dilating circle actions

From now on, we suppose that  $\mathcal{A}$  carries an action of  $G = \mathbb{G}_m = \mathbb{C}^*$ . The diagonal bimodule and its dual are then naturally equivariant.

DEFINITION 15.4. *We say that the  $G$ -action is dilating of weight  $d > 0$  if there is an equivariant quasi-isomorphism*

$$(15.9) \quad \mathcal{A}^\vee \otimes V_d \cong \mathcal{A}[n],$$

where  $V_d$  is the one-dimensional representation of  $G$  with weight  $d$ .

The sign of  $d$  is not terribly important, since one can reverse it by passing to the inverse group action, but it will be crucial that  $d$  be nonzero. (15.9) implies that for any two equivariantly perfect modules, we have an isomorphism of  $G$ -representations

$$(15.10) \quad H^*(\text{hom}_{\mathcal{A}^{\text{perf}}}(P, Q)) \cong H^{n-*}(\text{hom}_{\mathcal{A}^{\text{perf}}}(Q, P))^\vee \otimes V_d.$$

Recall (from Lecture 13) that we have equivariant Hochschild homology  $HH_0^G(\mathcal{A}, \mathcal{A})$ , which is a module over  $\mathbb{C}[G] = \mathbb{C}[t, t^{-1}]$ , as well as the pairing (13.51), which takes on the form

$$(15.11) \quad (\cdot, \cdot)_{HH}^G : HH_0^G(\mathcal{A}, \mathcal{A}) \otimes HH_0^G(\mathcal{A}, \mathcal{A}) \longrightarrow \mathbb{C}[t, t^{-1}].$$

From (15.10) and the equivariant version of the Cardy relation (13.50), we obtain the following:

LEMMA 15.5. *If  $S$  is spherical and equivariant, the  $G$ -action on  $H^*(\text{hom}_{\mathcal{A}}(S, S))$  is trivial in degree 0 and has weight  $d$  in degree  $n$ . Hence,*

$$(15.12) \quad ([S]_{HH}^G, [S]_{HH}^G)_{HH}^G = 1 + (-1)^n t^d.$$

□

Along the same lines, we also have equivariant analogues of Lemma 15.1 and 15.2:

LEMMA 15.6. *The equivariant Hochschild homology  $HH_0^G(\mathcal{A}, \mathcal{A})$  is the free module over  $\mathbb{C}[G] \cong \mathbb{C}[t, t^{-1}]$  generated by the identity endomorphisms  $[e_X]$ .*

□

LEMMA 15.7. *The pairing (15.11) satisfies*

$$(15.13) \quad (y, x)_{HH}^G = (-1)^n t^d \left( (x, y)_{HH}^G \right)_{t \rightarrow t^{-1}}.$$

□

The three Lemmas above directly imply the desired analogue of Proposition 15.3:

PROPOSITION 15.8. *Suppose that  $\mathcal{A}$  is simply-connected in the sense of (15.3), has a dilating action of  $G$  with weight  $d > 0$ . We assume that  $n \geq 3$  is odd. Then the class  $[S]_{HH}$  of any spherical object is nonzero. Moreover, if  $S_1, \dots, S_k$  are spherical objects such that  $H^*(\text{hom}_{\mathcal{A}^{\text{perf}}}(S_i, S_j)) = 0$  for all  $i \neq j$ , then the classes  $[S_i]_{HH}$  are linearly independent over  $\mathbb{R}$ .*

PROOF. As discussed in Lecture 14, the assumption that  $S$  is spherical and  $n > 1$  means that one can always lift  $S$  to an equivariant module (which is then also equivariantly perfect). Write  $s_t = [S]_{HH}^G$  for the associated equivariant Hochschild homology class. There is a canonical forgetful map

$$(15.14) \quad HH_0^G(\mathcal{A}, \mathcal{A}) \otimes_{\mathbb{C}[t, t^{-1}]} \mathbb{C} \longrightarrow HH_0(\mathcal{A}, \mathcal{A}),$$

where the tensor product is with the simple module corresponding to the point  $t = 1$ . Under that map,  $s_t$  goes to the ordinary Hochschild homology class  $s = [S]_{HH}$ .

By Lemmas 15.1 and 15.6,  $HH_0^G(\mathcal{A}, \mathcal{A})$  is a free  $\mathbb{C}[t, t^{-1}]$ -module, and (15.14) is an isomorphism. If  $s = 0$ , we therefore could write  $s_t = (1 - t)q_t$ , which would imply

$$(15.15) \quad (s_t, s_t)_{HH}^G = (1 - t)(1 - t^{-1})(q_t, q_t)_{HH}^G = (1 - t)^2(-t^{-1})(q_t, q_t)_{HH}^G \in (1 - t)^2\mathbb{C}[t, t^{-1}].$$

But that contradicts Lemma 15.5, which shows that

$$(15.16) \quad (s_t, s_t)_{HH}^G = 1 - t^d = (1 - t)(1 + t + \dots + t^{d-1}).$$

To prove the second part, one argues along the same lines (by contradiction). If the classes in ordinary Hochschild homology are linearly dependent, we get a nontrivial relation in equivariant Hochschild homology of the form

$$(15.17) \quad a_1 s_{1,t} + \dots + a_k s_{k,t} = (1 - t)q_t$$

where  $s_{i,t} = [S_k]_{HH}^G$  and  $a_i \in \mathbb{R}$ . But then

$$(15.18) \quad (1 - t)(1 - t^{-1})(q_t, q_t)_{HH}^G = (a_1^2 + \dots + a_k^2)(1 - t^d),$$

which is again a contradiction. □

### Simple singularities

Let  $M$  be the Milnor fibre of a simple (ADE) type singularity, of complex dimension  $n$ . Concretely, these are the hypersurfaces

$$(15.19) \quad M = \{p(x_1, \dots, x_{n-1}, y, z) = 1\} \subset \mathbb{C}^{n+1},$$



where

$$(15.20) \quad p = x_1^2 + \cdots + x_{n-1}^2 + \begin{cases} y^2 + z^{m+1} & \text{in type } (A_m), m \geq 1, \\ yz^2 + z^{m-1} & \text{in type } (D_m), m \geq 4, \\ y^3 + z^4 & \text{in type } (E_6), \\ y^3 + yz^3 & \text{in type } (E_7), \\ y^3 + z^5 & \text{in type } (E_8). \end{cases}$$

We equip them with the restriction of the constant (exact) symplectic form on  $\mathbb{C}^{n+1}$ , and with the complex volume form  $\eta_M = \text{res}_M(dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy \wedge dz / (p-1))$ .

LEMMA 15.9. *Let  $\Gamma$  be the Dynkin diagram of type (ADE) corresponding to our singularity. Then  $M$  contains a collection of Lagrangian spheres  $S_v$ , indexed by the vertices of the diagram, which are in general position and satisfy*

$$(15.21) \quad S_v \cap S_w = \begin{cases} \text{one point} & \text{if } v, w \text{ are connected by an edge,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover, the classes  $[S_v]$  form a basis for  $H_n(M; \mathbb{Z})$ .

This is a classical result. For  $n = 1$ , one can use A'Campo's real Morsification method [11] to realize the required  $S_v$  as a distinguished basis of vanishing cycles. The higher-dimensional statement then follows from this and general stabilization results in singularity theory. Another consequence of the general theory (for weighted homogeneous singularities), spelled out in [169, Lemma 4.15], is:

LEMMA 15.10. *Let  $\phi$  be the composition of the Dehn twists along the spheres  $S_v$  (in a particular order, which we will not explain), considered as a graded symplectic automorphism of  $M$ . Then, some positive power  $\phi^b$  is isotopic to a shift inside the group of such automorphisms. More precisely,*

$$(15.22) \quad \phi^b \simeq [2a],$$

where

$$(15.23) \quad \frac{a}{b} = 1 - \frac{n-1}{2} - \begin{cases} \frac{1}{2} + \frac{1}{m+1} & \text{in type } (A_m), \\ \frac{m-2}{m-1} & \text{in type } (D_m), \\ \frac{7}{12} & \text{in type } (E_6), \\ \frac{5}{9} & \text{in type } (E_7), \\ \frac{8}{15} & \text{in type } (E_8). \end{cases}$$

Let  $\mathcal{A} \subset \text{Fuk}(M)$  be the full subcategory with objects  $S_v$ . With the exception of one case ( $n = 1$  and type  $(A_1)$ , which we have in fact considered before in Proposition 10.9), the fraction (15.23) is nonzero, which by [176, Corollary 5.8] says that the  $S_v$  are *split-generators* for the Fukaya category. We have not encountered this notion before: it means that the restriction

$$(15.24) \quad \text{Fuk}(M)^{\text{perf}} \longrightarrow \mathcal{A}^{\text{perf}}$$

is an equivalence. In particular,  $Fuk(M)$  then quasi-isomorphically embeds into  $\mathcal{A}^{perf}$ .

At least on the cohomological level,  $\mathcal{A}$  admits a simple description, which is a direct consequence of (15.21) and general facts about Floer cohomology (similar to Lemma 11.5). First, choose (arbitrarily) an orientation of the edges in the Dynkin diagram, and denote the result by  $\Gamma^\rightarrow$ . Associate to this a linear category  $A^\rightarrow$  over  $\mathbb{C}$  whose objects  $X_v$  correspond to vertices, and whose morphisms consist of the identity endomorphisms as well as morphisms associated to edges of  $\Gamma^\rightarrow$ . Compositions in  $A^\rightarrow$  are zero except insofar as the identity endomorphisms are involved. Then, form the trivial extension with respect to the dual of the diagonal bimodule (this is the same process as in Example 13.5):

$$(15.25) \quad A = A^\rightarrow \oplus (A^\rightarrow)^\vee[-n].$$

Concretely, the objects of  $A$  are still the  $X_v$ , and

$$(15.26) \quad Hom_A^*(X_v, X_w) = Hom_{A^\rightarrow}^*(X_v, X_w) \oplus Hom_{A^\rightarrow}^*(X_w, X_v)^\vee[-n].$$

The nontrivial compositions come from the algebra structure of  $A^\rightarrow$  and from the bimodule structure of  $(A^\rightarrow)^\vee$ . Then, the desired description is that, if the grading of the  $S_v$  is chosen appropriately,

$$(15.27) \quad H(\mathcal{A}) \cong A.$$

LEMMA 15.11. *Assume that  $n \geq 3$ . Then  $A$  is simply-connected in the sense of (15.3).*

PROOF. Assign to the vertices numbers  $\sigma_v \in \frac{1}{2}\mathbb{Z}$ , such that if there is an oriented edge from  $v$  to  $w$ , then  $\sigma_w = \sigma_v - n/2$  (this is always possible since  $\Gamma$  is a tree). Then by construction,  $Hom_A^*(X_v, X_w)$  is concentrated in degrees  $n/2 + \sigma_w - \sigma_v > 1 + \sigma_w - \sigma_v$  for  $v \neq w$ , while the non-identity part of  $Hom_A^*(X_v, X_v)$  is concentrated in degree  $n$ .  $\square$

LEMMA 15.12. *Assume that  $n \geq 3$ . Then  $A$  is intrinsically formal, and therefore  $\mathcal{A}$  itself is quasi-isomorphic to  $A$ .*

PROOF. This is the same kind of degree argument as in the previous Lemma. One shows that the bigraded Hochschild cohomology satisfies

$$(15.28) \quad HH^p(A, A[q]) = 0 \quad \text{for } pn/2 + q > n.$$

After specializing to  $p+q = 2$ , one sees that under our assumption that  $n \geq 3$ ,  $HH^d(A, A[2-d]) = 0$  for all  $d \geq 3$ . This allows one to apply Proposition 9.6 (alternatively, one can directly argue that any strictly unital  $A_\infty$ -structure must be formal, which corresponds to the vanishing of the reduced Hochschild cochain complex in the relevant degrees).  $\square$

REMARK 15.13. *Suppose that  $n = 2$ . Then intrinsic formality still applies in the  $(A_m)$  case, by a computation in [185]. It is not known whether the same holds in types  $(D_m)$  and  $(E_m)$ . In contrast, for  $n = 1$  intrinsic formality definitely fails, and in fact  $\mathcal{A}$  is not formal (except in the trivial  $(A_1)$  case) [121].*

At this point, we may just as well assume that  $\mathcal{A}$  is equal (and not just quasi-isomorphic) to the algebraically constructed model  $A$ . In particular, it then carries an action of the

multiplicative group (acting trivially in degree 0, and with weight 1 in degree  $n$ ), which is dilating with weight 1. Our previous algebraic results then yield the following:

PROPOSITION 15.14. *Suppose that  $n \geq 3$  is odd. Let  $S \subset M$  be a Lagrangian sphere. Then the class  $[S] \in H_n(M)$  is nonzero. Moreover, if  $S_1, \dots, S_k$  are pairwise disjoint such spheres, then their homology classes are linearly independent.*

Again, the corresponding statement for even  $n$  is elementary (because then,  $S \cdot S = \pm 2$  for any Lagrangian homology sphere, and (15.21) together with standard facts about Dynkin diagrams implies that the intersection pairing on  $H_n(M)$  is definite).

PROOF. Take the open-closed string map (12.25) and compose it with the standard map  $SH_*(M) \rightarrow H_{-*}(M; \mathbb{C})$ . The outcome is a map

$$(15.29) \quad HH_0(\text{Fuk}(M), \text{Fuk}(M)) \longrightarrow H_n(M; \mathbb{C}),$$

which takes the identity endomorphism of any object  $L$  to the ordinary homology class  $[L]$  (times the rank of the flat vector bundle  $\xi_L$ ; however, we will only use trivial flat line bundles). Because of the split-generation property of the  $S_v$ , and the Morita invariance of Hochschild homology, we have  $HH_0(\text{Fuk}(M), \text{Fuk}(M)) \cong HH_0(\mathcal{A}, \mathcal{A})$ . This, together with Lemma 15.1, also implies that (15.29) is an isomorphism. Under the quasi-isomorphic embedding  $\text{Fuk}(M) \hookrightarrow \mathcal{A}^{\text{perf}}$ ,  $S$  turns into a spherical object, which by Proposition 15.8 has a nonzero class in Hochschild homology. It follows that the class of  $S$  in  $HH_0(\text{Fuk}(M), \text{Fuk}(M))$  is nontrivial, and therefore so is its ordinary homology class.

For a collection of pairwise disjoint spheres, the same argument shows that  $[L_1], \dots, [L_k]$  are linearly independent over  $\mathbb{R}$ . But the “over  $\mathbb{R}$ ” restriction is irrelevant, since the homology classes lie in  $H_n(M; \mathbb{R}) \subset H_n(M; \mathbb{C})$  anyway.  $\square$

REMARK 15.15. *Because the proof takes place on the level of the Fukaya category, one could replace “sphere” by “rational homology sphere which is Spin”. Similarly, disjointness can be weakened to the vanishing of  $HF^*(S_i, S_j)$  for all  $i \neq j$  (which is certainly the case if the Lagrangian spheres are pairwise disjointable by isotopies). This has some amusing consequences: for instance, for  $n = 3$ ,  $M$  can't contain a Lagrangian submanifolds which is a lens space (other than a sphere). This is because one could equip that lens space with different flat line bundles, and then apply the previous argument to the resulting objects of the Fukaya category, which contradicts the fact that they all represent the same homology class.*

REMARK 15.16. *Suppose that we are given a finite tree  $\Gamma$ . Take cotangent bundles of spheres (one sphere  $S_v$  for each vertex) and plumb them together (along the edges). The result is a Weinstein manifold  $M$ , which is unique if the dimension is  $2n > 2$ . For a Dynkin diagram, this reproduces the ADE type Milnor fibres. Suppose from now on that  $n$  is odd and  $\geq 3$ . Following [10], Abouzaid suggested the following strategy for generalizing our results to this case:*

- *Consider the non-compact Lagrangian submanifolds  $L_v$  which are cotangent fibres of the various sphere components. Show that the wrapped Floer cohomology*

$HW^*(L_{v_0}, L_{v_1})$  is concentrated in degrees  $\leq 0$ , with the degree zero part consisting only of multiples the identity endomorphisms. This uses the same Maslov index computations as in [10].

- Show that every closed Lagrangian submanifold is split-generated by the  $L_v$ ;
- Then, a purely algebraic argument will imply that  $Fuk(M)$  is actually generated by the spheres  $S_v$ .

Then, the part of the argument concerning equivariance would go through as before (in fact, one could replace equivariant Hochschild homology by equivariant  $K$ -theory, since it would follow that the Grothendieck group  $K_0(Fuk(M))$  is the free abelian group generated by the vertices of  $\Gamma$ ).

LECTURE 16

**Suspension of Lefschetz fibrations \*\*\*WARNING\*\*\*:  
SAME ERROR PROPAGATES**

For  $ADE$  type Milnor fibres (15.19), we obtained the existence of  $\mathbb{C}^*$ -actions on the Fukaya category by an ad hoc computation of that category (essentially, as a consequence of formality, as in Example 13.4). Here, we want to explain a construction [179] which always gives rise to such actions. The construction itself is geometric, and closely related to local mirror symmetry, as discussed at the beginning of Lecture 11. Unfortunately, the additional symmetries themselves again appear only after a re-interpretation in purely algebraic terms. Still, given the greater level of generality, this is still a first step towards building some geometric intuition.

*Acknowledgments.* The idea of using Lefschetz fibrations where the Fukaya category of the fibre has a  $\mathbb{C}^*$ -action, which leads to Corollary 16.10, was suggested to the author by Mohammed Abouzaid.

**Algebraic suspension**

This discussion follows [179], with only slight changes in notation. Let  $\mathcal{B}$  be a  $A_\infty$ -category over a field  $\mathbb{K}$ , with a finite ordered set of objects  $(X_1, \dots, X_k)$ . We assume that each object is nonzero, meaning that  $hom_{\mathcal{B}}(X_i, X_i)$  is never acyclic. We will also assume, for technical simplicity, that  $\mathcal{B}$  is strictly unital, and that each space  $hom_{\mathcal{B}}(X_i, X_j)$  is finite-dimensional (a stricter version of properness).

We want to associate to  $\mathcal{B}$  two directed (Example 7.11)  $A_\infty$ -categories, called  $\mathcal{A}$  and  $\mathcal{C}$ . The first one is the directed subcategory  $\mathcal{A} \subset \mathcal{B}$ , which is defined by setting

$$(16.1) \quad hom_{\mathcal{A}}(X_i, X_j) = \begin{cases} hom_{\mathcal{B}}(X_i, X_j) & i < j, \\ \mathbb{K} \cdot e_{X_i} & i = j, \\ 0 & i > j; \end{cases}$$

The  $A_\infty$ -operations on  $\mathcal{A}$  are restrictions of those on  $\mathcal{B}$ . Of course, in general  $\mathcal{A}$  remembers only small part of the structure of  $\mathcal{B}$ , namely the  $A_\infty$ -operations where the inputs are in increasing order.

The second construction is similar, but has an intermediate step. Introduce the  $A_\infty$ -category  $Cl_2(\mathcal{B})$  which has twice the number of objects, denoted by

$$(16.2) \quad (X_1^-, \dots, X_k^-, X_1^+, \dots, X_k^+).$$

Both  $X_i^\pm$  are thought of as copies of  $X_i$ , but the  $-$  copy is shifted down by one, so that we have

$$(16.3) \quad \begin{aligned} \text{hom}_{Cl_2(\mathcal{B})}(X_i^-, X_j^-) &= \text{hom}_{\mathcal{B}}(X_i, X_j), \\ \text{hom}_{Cl_2(\mathcal{B})}(X_i^+, X_j^+) &= \text{hom}_{\mathcal{B}}(X_i, X_j), \\ \text{hom}_{Cl_2(\mathcal{B})}(X_i^-, X_j^+) &= \text{hom}_{\mathcal{B}}(X_i, X_j)[-1], \\ \text{hom}_{Cl_2(\mathcal{B})}(X_i^+, X_j^-) &= \text{hom}_{\mathcal{B}}(X_i, X_j)[1]. \end{aligned}$$

In particular,  $Cl_2(\mathcal{B})$  is quasi-equivalent to  $\mathcal{B}$  (the corresponding construction for algebras would be tensoring with the two-dimensional Clifford algebra, whence the notation). Then  $\mathcal{C} \subset Cl_2(\mathcal{B})$  is the directed subcategory, where the ordering of the objects is as in (16.2). One can schematically write

$$(16.4) \quad \mathcal{C} = \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{B}[-1] & \mathcal{A} \end{pmatrix} \subset Cl_2(\mathcal{B}) = \begin{pmatrix} \mathcal{B} & \mathcal{B}[1] \\ \mathcal{B}[-1] & \mathcal{B} \end{pmatrix}.$$

$\mathcal{C}$  retains a larger (but still finite, by directedness) piece of the  $A_\infty$ -structure of  $\mathcal{B}$  than  $\mathcal{A}$ . More precisely,  $\mathcal{C}$  depends on  $\mathcal{A}$  together with the structure of  $\mathcal{B}$  as an  $\mathcal{A}$ -bimodule.

Finally, we define another (non-directed)  $A_\infty$ -category  $\mathcal{B}^\sigma$ , with objects  $(X_1^\sigma, \dots, X_k^\sigma)$ . Namely, this is the full subcategory of  $\mathcal{C}^{tw}$  consisting of the twisted complexes

$$(16.5) \quad X_i^\sigma = \text{Cone}(e_{X_i} : X_i^-[-1] \rightarrow X_i^+) = \left( X_i^- \oplus X_i^+, \delta_{X_i^\sigma} = \begin{pmatrix} 0 & 0 \\ e_{X_i} & 0 \end{pmatrix} \right).$$

In addition to the data that enters into the construction of  $\mathcal{C}$ , which means  $\mathcal{A}$  and the  $\mathcal{A}$ -bimodule  $\mathcal{B}$ , this also the  $e_{X_i}$  as elements of  $\mathcal{B}$ . Of course, the quasi-isomorphism type of a mapping cone only depends on the cohomology class of the morphism involved, and also remains unchanged if one multiplies the morphism by a nonzero constant. Hence, if we make the following additional simplifying assumption:

$$(16.6) \quad \text{for } i = 1, \dots, k, \quad H^0(\text{hom}_{\mathcal{B}}(X_i, X_i)) = \mathbb{K}[e_{X_i}];$$

then  $\mathcal{A}$  and the quasi-isomorphism class of the  $\mathcal{A}$ -bimodule  $\mathcal{B}$  determine  $\mathcal{B}^\sigma$  up to quasi-isomorphism.

We call  $\mathcal{B}^\sigma$  the *suspension* of  $\mathcal{B}$ . If we consider it (or rather, the direct sum of its morphism spaces) as an  $A_\infty$ -algebra, it looks like (16.4) but with additional contributions to  $\mu_{\mathcal{B}^\sigma}^1$  coming from the differentials in (16.5). In the schematic notation from (16.4), those contributions are simply

$$(16.7) \quad \begin{pmatrix} a_- & 0 \\ b & a_+ \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ a_+ - a_- & 0 \end{pmatrix}.$$

Note that for  $i < j$ , the diagonal embedding

$$(16.8) \quad \text{hom}_{\mathcal{A}}(X_i, X_j) \hookrightarrow \text{hom}_{\mathcal{B}^\sigma}(X_i^\sigma, X_j^\sigma)$$

is a quasi-isomorphism, and compatible with  $A_\infty$ -composition maps. Hence, the directed subcategory  $\mathcal{A}^\sigma \subset \mathcal{B}^\sigma$  is quasi-isomorphic to  $\mathcal{A}$ .

LEMMA 16.1. *Use the quasi-isomorphism  $\mathcal{A} \rightarrow \mathcal{A}^\sigma$  to pull back the  $\mathcal{A}^\sigma$ -bimodule  $\mathcal{B}^\sigma/\mathcal{A}^\sigma$  to  $\mathcal{A}$ . The outcome is isomorphic to  $(\mathcal{B}/\mathcal{A})[-1]$ . Moreover, if we apply the same pullback to  $\mathcal{B}^\sigma$  itself, the outcome is quasi-isomorphic to  $\mathcal{A} \oplus (\mathcal{B}/\mathcal{A})[-1]$ .  $\square$*

LEMMA 16.2. *Assume that (16.6) holds, and that the  $\mathcal{A}$ -bimodule  $\mathcal{B}$  is quasi-isomorphic to  $\mathcal{A} \oplus (\mathcal{B}/\mathcal{A})[-1]$ . Then,  $\mathcal{B}^\sigma$  is quasi-isomorphic (as an  $A_\infty$ -category) to the trivial extension constructed from  $\mathcal{A}$  and the bimodule  $(\mathcal{B}/\mathcal{A})[-1]$ .*

PROOF. Since  $\mathcal{B}^\sigma$  depends only on  $\mathcal{A}$  and the structure of  $\mathcal{B}$  as an  $\mathcal{A}$ -bimodule, we may assume without loss of generality that  $\mathcal{B}$  is itself the trivial extension formed from  $\mathcal{A}$  and  $\mathcal{B}/\mathcal{A}$ . In that case, a quasi-isomorphic subcategory of  $\mathcal{B}^\sigma$  can be defined by allowing only morphisms, in the notation from (16.4), of the form

$$(16.9) \quad \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \quad \text{with } a \in \mathcal{A}, b \in \mathcal{B}/\mathcal{A},$$

and the desired result is then obvious.  $\square$

COROLLARY 16.3. *Suppose that  $H^{-1}(\text{hom}_{\mathcal{B}}(X_i, X_i)) = 0$  for all  $i$ . Then the double suspension  $\mathcal{B}^{\sigma\sigma}$  is quasi-isomorphic to the trivial extension formed from  $\mathcal{A}$  and  $(\mathcal{B}/\mathcal{A})[-2]$ .*

This is a direct consequence of the previous two Lemmas. The assumption on  $H^{-1}$  implies that (16.6) holds for  $\mathcal{B}^\sigma$ . In fact, one can remove that assumption entirely at the cost of a slightly more involved argument, for which we refer to [179]. From now on, we again restrict the discussion to  $\mathbb{K} = \mathbb{C}$ .

COROLLARY 16.4. *Suppose that we have a quasi-isomorphism of  $\mathcal{A}$ -bimodules  $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[-n]$ . Suppose also that  $H^{-1}(\text{hom}_{\mathcal{B}}(X_i, X_i)) = 0$  for all  $i$ . Then  $\mathcal{B}^{\sigma\sigma}$  is weakly cyclic of dimension  $n + 2$ . Moreover, it carries a  $\mathbb{C}^*$ -action which is dilating of weight 1 (Definition 15.4).*

By the previous Corollary,  $\mathcal{B}^\sigma$  is quasi-isomorphic to the trivial extension formed from  $\mathcal{A}$  and  $\mathcal{A}^\vee[-n - 2]$ . Equip it with the  $\mathbb{C}^*$ -action which acts trivially on  $\mathcal{A}$  and with weight 1 on  $\mathcal{A}^\vee[-n - 2]$ . The required isomorphism (15.9) is then obvious.

### Geometric suspension

The construction introduced above has a geometric meaning in terms of Lefschetz fibrations. In general, one starts with an exact Lefschetz fibration [176, Section 15]

$$(16.10) \quad \pi : E \longrightarrow D,$$

whose base  $D$  is a disc, and whose fibre is  $M$ . Choose a distinguished basis of vanishing cycles, which are Lagrangian spheres  $(S_1, \dots, S_k)$  in  $M$ . Denote by  $\mathcal{B}$  the associated full subcategory of  $\text{Fuk}(M)$ . One then constructs a new Lefschetz fibration

$$(16.11) \quad \pi^\sigma : E^\sigma \longrightarrow D,$$

whose fibre  $M^\sigma$  is the double cover of  $E$  branched along  $M$ . The total space of  $E^\sigma$  itself is essentially the product  $D \times E$  (“essentially” means up to deformation and rounding corners). There is a distinguished basis of vanishing cycles  $(S_1^\sigma, \dots, S_k^\sigma)$  in  $M^\sigma$  which corresponds to our original choice in  $M$ . Denote by  $\mathcal{B}^\sigma \subset \text{Fuk}(M^\sigma)$  the associated full subcategory. Then – as already implicit in our notation – one has:

PROPOSITION 16.5 ([179, Theorem 6.4]).  *$\mathcal{B}^\sigma$  is quasi-isomorphic to the previously defined algebraic suspension of  $\mathcal{B}$ .*

As a consequence, if we suspend twice, then the part of the Fukaya category of  $M^{\sigma\sigma}$  split-generated by vanishing cycles carries a  $\mathbb{C}^*$ -action which is dilating of weight 1. It is maybe instructive to see where instances of this process occur in more familiar geometric contexts.

### Singularity theory

Let  $M$  be the Milnor fibre of an isolated hypersurface singularity  $p(y) = 0$ . This naturally appears as a fibre of a Lefschetz fibration  $\pi$ , obtained by perturbing (Morsifying)  $p$ . Then,  $M^\sigma$  is the Milnor fibre of  $x^2 + p(y) = 0$ , and  $\pi^\sigma$  is the corresponding Morsification. Adding such quadratic terms is a well-known operation in singularity theory, where it behaves in a 4-periodic way with respect to the classical topological invariants.

Under suitable assumptions, one can show that the vanishing cycles split-generate the Fukaya category (a generalization of Lemma 15.10). The precise statement is as follows:

LEMMA 16.6. *Let  $p(y_1, \dots, y_{n+1})$  be a polynomial with an isolated singularity at  $0 \in p^{-1}(0)$ , and which is weighted homogeneous in the sense that there are rational numbers  $w_1, \dots, w_{n+1} > 0$  such that*

$$(16.12) \quad p(e^{iw_1 t} y_1, \dots, e^{iw_{n+1} t} y_{n+1}) = e^{it} p(y_1, \dots, y_{n+1}).$$

Set

$$(16.13) \quad w = w_1 + \dots + w_{n+1} - 1.$$

*Then, an appropriate composition of Dehn twists along the vanishing cycles is the shift by an integer multiple of  $w$ . Hence, if  $w \neq 0$ , the vanishing cycles split-generate the Fukaya category of the Milnor fibre.*  $\square$

Lemma 16.6 is the only place we use weighted homogeneity (it is an interesting question for what classes of other singularities such a split-generation result holds). Of course, adding quadratic terms preserves the class of weighted homogeneous singularities, but increases  $w$  by  $1/2$ . Hence, applying Proposition 16.5 and Corollary 16.4, we find that:

COROLLARY 16.7. *Suppose that  $p(y)$  is weighted homogeneous. Let  $M$  be the Milnor fibre of  $x_1^2 + x_2^2 + p(y)$ . Then  $\text{Fuk}(M)$  has a  $\mathbb{C}^*$ -action which is dilating with weight 1.*  $\square$

REMARK 16.8. *The ADE singularities in Lecture 15 are all at least twice suspended when the (complex) dimension of the fibre is 3 or higher. Hence, Corollary 16.7 strictly generalizes*



our previous construction of group actions for ADE Milnor fibres (for type  $(A_m)$ , complex dimension 2 is enough, which matches well with the more ad hoc approach considered in Remark 15.13).

### Application

As before, let  $\pi : E \rightarrow D$  be an exact Lefschetz fibration with fibre  $M$  (of dimension  $2n - 2$ ), and with zero first Chern class. Let  $(S_1, \dots, S_k)$  be a basis of vanishing cycles. Let  $\mathcal{B} \subset \text{Fuk}(M)$  the associated full  $A_\infty$ -subcategory, and  $\mathcal{A}$  the associated directed  $A_\infty$ -subcategory. The main construction in [176, Section 18] yields a cohomologically full and faithful embedding

$$(16.14) \quad \text{Fuk}(E) \hookrightarrow \mathcal{A}^{\text{perf}}.$$

PROPOSITION 16.9. *Suppose that  $\mathcal{B}$  carries a dilating action of  $G = \mathbb{C}^*$ , of weight 1. Take a Lagrangian submanifold  $S \subset E$  with  $H^1(S) = 0$  and which is Spin. Use the embedding (16.14) as well as the results from Lecture 14 to associate to it an equivariantly perfect module  $P$  over  $\mathcal{A}$ . Then, the  $G$ -action on  $H^n(\text{hom}_{\mathcal{A}^{\text{perf}}}(P, P))$  has weight 1.*

PROOF. After possibly rescaling the symplectic form, one can construct an exact symplectic embedding of  $E$  into  $M^\sigma$ . Moreover, the image of the resulting cohomologically full and faithful embedding

$$(16.15) \quad \text{Fuk}(E) \hookrightarrow \text{Fuk}(M^\sigma)$$

is split-generated by  $(S_1^\sigma, S_k^\sigma)$ . This follows from [176, Lemma 18.15]. Inspection of the construction shows that the resulting full and faithful embedding sits in a commutative (up to quasi-isomorphism of  $A_\infty$ -functors) diagram

$$(16.16) \quad \begin{array}{ccc} \text{Fuk}(E) & \hookrightarrow & \mathcal{A}^{\text{perf}} \\ & \searrow & \uparrow \\ & & (\mathcal{B}^\sigma)^{\text{perf}}. \end{array}$$

Here, the  $\rightarrow$  is (16.14), and the  $\uparrow$  is induced by projection  $\mathcal{B}^\sigma \rightarrow \mathcal{A}$  to the left upper corner of the matrix (16.4).

Let's temporarily forget about group actions, and just start with a bimodule quasi-isomorphism  $\mathcal{B} \cong \mathcal{B}^\vee[1 - n]$ . This induces a quasi-isomorphism  $\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[1 - n]$ . Now look at the short exact sequence of  $\mathcal{B}^\sigma$ -bimodules schematically denoted by

$$(16.17) \quad 0 \rightarrow \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{B}[-1] & \mathcal{A} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{B} & \mathcal{B}[1] \\ \mathcal{B}[-1] & \mathcal{B} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{B}/\mathcal{A} & \mathcal{B}[1] \\ 0 & \mathcal{B}/\mathcal{A} \end{pmatrix} \rightarrow 0.$$

The middle term carries a differential similar to (16.7), more precisely

$$(16.18) \quad \begin{pmatrix} b_{--} & b_{+-} \\ b_{-+} & b_{++} \end{pmatrix} \mapsto \begin{pmatrix} b_{+-} & 0 \\ b_{++} - b_{--} & b_{-+} \end{pmatrix},$$

which is clearly acyclic. The left term in (16.17) is the diagonal bimodule over  $\mathcal{B}^\sigma$ , while the right term is quasi-isomorphic to  $(\mathcal{B}^\sigma)^\vee[1-n]$ . Hence, the boundary map of the short exact sequence yields a quasi-isomorphism

$$(16.19) \quad (\mathcal{B}^\sigma)^\vee[-n] \cong \mathcal{B}^\sigma.$$

Now we return to the original context, where there is a  $G$ -action on  $\mathcal{B}$  that is dilating with weight 1. This induces an action on  $\mathcal{B}^\sigma$ , such that the previously mention projection  $\mathcal{B}^\sigma \rightarrow \mathcal{A}$  is equivariant. Moreover, by inspection of the construction of (16.19), one sees that the induced action is also dilating with weight 1. This implies the desired result.  $\square$

**COROLLARY 16.10.** *In the situation of Proposition 16.9, suppose that  $S \subset E$  is a Lagrangian sphere. Suppose that  $n \geq 3$  is odd. Then  $[S] \in H_n(E, M; \mathbb{Z})$ , where  $M$  is thought of as the fibre over a point of  $\partial D$ , is a primitive class (and in particular, nonzero).*

**PROOF.** The strategy is the same as in Proposition 15.8, but using  $K$ -theory instead of Hochschild homology, since that turns out to be simpler in this case. In fact, the directedness of  $\mathcal{A}$  implies that

$$(16.20) \quad K_0(\mathcal{A}^{perf}) \cong \mathbb{Z}^k,$$

$$(16.21) \quad K_0^G(\mathcal{A}^{eq-perf}) \cong \mathbb{Z}[t, t^{-1}]^k.$$

If one combines (16.20) with the map  $K_0(\text{Fuk}(E)) \rightarrow K_0(\mathcal{A}^{tw})$  induced by (16.14), the resulting homomorphism takes each closed Lagrangian submanifold  $S \subset E$  to the collection of intersection numbers  $(L \cdot \Delta_1, \dots, L \cdot \Delta_k)$ , where  $\Delta_i$  is the Lefschetz thimble corresponding to  $S_i$ . In particular, it fits into a diagram

$$(16.22) \quad \begin{array}{ccc} K_0(\text{Fuk}(E)) & \longrightarrow & \mathbb{Z}^k \\ \downarrow & & \uparrow \cong \\ H_n(E) & \longrightarrow & H_n(E, M). \end{array}$$

Take the equivariant Mukai pairing  $(\cdot, \cdot)^G$  on  $K_0^G(\mathcal{A}^{eq-perf})$ . Suppose that we have a Lagrangian sphere  $L \subset E$ , and let  $P$  be the associated object in  $\mathcal{A}^{eq-perf}$  as in Proposition 16.9. By that Proposition, we know that the equivariant  $K$ -theory class  $[P]^G$  satisfies

$$(16.23) \quad ([P]^G, [P]^G)^G = 1 - t.$$

As in Proposition 15.8, this implies that the reduction to  $t = 1$  (which means the ordinary  $K$ -theory class  $[P]$ ) is nontrivial. In fact, it implies that  $[P]$  is primitive: if we assume that  $[P] \equiv 0 \pmod{p}$  for some prime  $p$ , then  $[P]^G = (1-t)y + pz$ , which implies

$$(16.24) \quad ([P]^G, [P]^G)^G = (1-t)(1-t^{-1})(y, y)^G + p(1-t^{-1})(y, z)^G + p(1-t)(z, y)^G + p^2(z, z)^G.$$

This contradicts (16.23), as one can see for instance by taking the derivative at  $t = 1$ :

$$(16.25) \quad \begin{aligned} \partial_t([P]^G, [P]^G)^G \Big|_{t=1} &= p(y, z)^G - p(z, y)^G + p^2 \partial_t(z, z)^G \equiv 0 \pmod p, \\ \partial_t(1 - t) \Big|_{t=1} &\equiv -1 \pmod p. \end{aligned}$$

□

The same argument can be applied to yield independence results for the homology classes of disjoint Lagrangian submanifolds (again, in analogy with Proposition 15.8).

EXAMPLE 16.11. *As a concrete instance where Corollary 16.10 applies, one can look at the Milnor fibre of any isolated hypersurface singularity of the form*

$$(16.26) \quad x_1^k + x_2^2 + x_3^2 + p(x_4, \dots, x_{n+1}) = 0,$$

*since that is the total space of a Lefschetz fibration whose fibre is the Milnor fibre of  $x_2^2 + x_3^2 + p(x_4, \dots, x_{n+1}) = 0$ . In contrast with Corollary 16.7, we do not need weighted homogeneity, since Proposition 16.9 does not require the  $G$ -action to exist on the entire Fukaya category. This of course includes the ADE singularities from Lecture 15.*

### Local mirror symmetry

Suppose that  $W \in \mathbb{C}[y_1^\pm, \dots, y_{n-1}^\pm]$  is a generic Laurent polynomial, as in (11.1), but where the interior of the polytope  $P$  is assumed to contain the origin  $0 \in \mathbb{R}^{n-1}$ . This can itself be thought of as a Lefschetz fibration, and double suspension leads to the manifold  $M$  from (11.2). Therefore, it follows that a certain full subcategory  $\mathcal{B} \subset \text{Fuk}(M)$  carries a dilating action with weight 1. Here, we have adjusted the notation to agree with that in Lecture 11; in particular,  $M$  already represents the outcome of double suspension.

The mirror construction leads first of all to a smooth toric Calabi-Yau variety  $N$ . One can certainly find a complex volume form  $\eta_N$  and an action of  $\mathbb{C}^*$  (part of the torus action) which acts with weight 1 on that form. That would induce an action on the derived category of compactly supported coherent sheaves (or rather, the underlying  $A_\infty$ -category) which is dilating with weight 1. There are two issues with this naive interpretation:

- One can't choose that action so that it preserves the divisor  $F^{-1}(1)$ . Hence, the action does not restrict to one on the actual mirror  $M^\vee$  from (11.4).
- The volume form  $\eta_{M^\vee}$  appearing in mirror symmetry comes from a meromorphic volume form on  $N$  having a pole along  $F^{-1}(1)$ , and is not toric.

More careful inspection of the situation shows the following. First of all, one expects the image of the embedding

$$(16.27) \quad H^*(\mathcal{B}) \hookrightarrow H^*(\text{Fuk}(M)) \xrightarrow{\text{mirror}} D^b \text{Coh}_{\text{cpt}}(M^\vee)$$

to consist of complexes of sheaves supported on the fibre  $F^{-1}(0)$  of  $F : N \rightarrow \mathbb{C}$ . Therefore, what is needed is a circle action *in a formal neighbourhood of that fibre*, which does not

necessarily have to extend over the rest of  $M^\vee$ . Secondly, while our algebraic suspension construction provides a weak cyclic structure on  $\mathcal{B}$  (forgetting equivariance for the moment), it is by no means clear that this coincides with the natural (geometrically defined) structure on the Fukaya category. This means that, while the corresponding circle action on the mirror is expected to act with weight 1 on *some* complex volume form, that volume form does not necessarily have to be the one involved in the original mirror equivalence.

These observations help clarify the situation, but they also indicate that our algebraic approach to the construction of circle actions on Fukaya categories is somewhat less than natural.

## Part 4

# Infinitesimal symmetries



## Basic structures

By an “infinitesimal automorphism” of an  $A_\infty$ -category  $\mathcal{A}$ , we will mean a Hochschild cohomology class

$$(17.1) \quad [\beta] \in HH^1(\mathcal{A}, \mathcal{A}).$$

Such a class determines an infinitesimal deformation of each object  $X$ , which is an element

$$(17.2) \quad [\beta_X^0] \in H^1(\text{hom}_{\mathcal{A}}(X, X)).$$

Objects for which  $[\beta_X^0]$  vanishes can be considered as “stationary” under the infinitesimal automorphism. We call them (with a bit of structure added) *infinitesimally equivariant objects*, and explore their general properties. This follows the strategy adopted in Lecture 13 for  $A_\infty$ -categories with actions of a reductive algebraic group, even though the technical aspects are somewhat different, and closer to the behaviour of non-reductive groups. For instance, while we can talk about infinitesimally equivariant objects, the notion of an invariant morphism between such objects is not well-behaved.

In principle, one can take  $\mathcal{A} = \text{Fuk}(M)$  to be a Fukaya category, with any choice of (17.1), and apply the general algebraic theory to it. However, we are interested in the case where  $M$  is a Weinstein manifold with vanishing first Chern class, and in Hochschild cohomology classes of geometric origin, which means ones in the image of the open-closed string map

$$(17.3) \quad SH^*(M) \longrightarrow HH^*(\text{Fuk}(M), \text{Fuk}(M))$$

from Lecture 12. The resulting special case of the general theory has been developed in [184], which we follow closely both in this lecture and the next one.

### Infinitesimal equivariance

Take an  $A_\infty$ -category  $\mathcal{A}$  over some field  $\mathbb{K}$ , and a class as in (17.1), represented by a Hochschild cocycle  $\beta$ . The components of  $\beta$  are

$$(17.4) \quad \begin{aligned} & \beta_X^0 \in \text{hom}_{\mathcal{A}}^1(X, X) && \text{for any object } X, \\ & \beta_{X_0, X_1}^1 : \text{hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \text{hom}_{\mathcal{A}}(X_0, X_1) && \text{for any objects } X_0, X_1, \\ & \dots \end{aligned}$$

It is a consequence of the Hochschild cocycle condition  $\partial\beta = 0$  that  $\mu_{\mathcal{A}}^1(\beta_X^0) = 0$ . Next, if  $\beta_{X_0}^0$  and  $\beta_{X_1}^0$  both vanish, then  $\beta_{X_0, X_1}^1$  anti-commutes with  $\mu_{\mathcal{A}}^1$ . We want to work under a weaker version of the vanishing assumption:

DEFINITION 17.1. *An infinitesimally equivariant object of  $\mathcal{A}$  is a pair  $(X, \alpha_X)$  with  $X \in \text{Ob}(\mathcal{A})$  and  $\alpha_X \in \text{hom}_{\mathcal{A}}^0(X, X)$ , satisfying*

$$(17.5) \quad \mu_{\mathcal{A}}^1(\alpha_X) = \beta_X^0.$$

For brevity, we will often omit  $\alpha_X$  from the notation and speak of “the infinitesimally equivariant object  $X$ ”, but it is then always understood that a choice of  $\alpha_X$  has been made. Obviously, the obstruction for an object to carry such a structure is  $[\beta_X^0] \in H^1(\text{hom}_{\mathcal{A}}(X, X))$ . We consider two infinitesimally equivariant structures on  $X$  to be isomorphic if the associated  $\alpha_X$  differ by an element in the image of  $\mu_{\mathcal{A}}^1 : \text{hom}_{\mathcal{A}}^{-1}(X, X) \rightarrow \text{hom}_{\mathcal{A}}^0(X, X)$ . Hence, the essential freedom of choice is an affine space over  $H^0(\text{hom}_{\mathcal{A}}(X, X))$ .

Suppose that  $X_0, X_1$  are both infinitesimally equivariant. Then, the map

$$(17.6) \quad \begin{aligned} \phi^1 &= \phi_{X_0, X_1}^1 : \text{hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \text{hom}_{\mathcal{A}}(X_0, X_1), \\ \phi^1(a) &= \beta_{X_0, X_1}^1(a) + (-1)^{|a|} \mu_{\mathcal{A}}^2(\alpha_{X_1}, a) - \mu_{\mathcal{A}}^2(a, \alpha_{X_0}) \end{aligned}$$

satisfies

$$(17.7) \quad \mu_{\mathcal{A}}^1 \phi^1 + \phi^1 \mu_{\mathcal{A}}^1 = 0,$$

hence is a chain map with respect to the differential  $a \mapsto (-1)^{|a|} \mu_{\mathcal{A}}^1(a)$ . We denote by  $\Phi = \Phi_{X_0, X_1}$  the induced map on  $H^*(\text{hom}_{\mathcal{A}}(X_0, X_1))$ . These maps are derivations, which means that they satisfy the Leibniz rule with respect to composition of morphisms between infinitesimally equivariant objects. In particular, they kill identity endomorphisms:

$$(17.8) \quad \Phi_{X, X}([e_X]) = 0.$$

The derivation property again follows from the definition and the equation  $\partial\beta = 0$ , this time by an argument which involves the next term  $\beta^2$ .

REMARK 17.2. *One can explain (17.6) in a slightly more conceptual way, as follows. Let  $\tilde{\mathcal{A}}$  be the  $A_{\infty}$ -category whose objects are pairs  $(X, \alpha_X)$  consisting of an  $X \in \text{Ob}(\mathcal{A})$  and an arbitrary  $\alpha_X \in \text{hom}_{\mathcal{A}}^0(X, X)$ . The morphism spaces and  $A_{\infty}$ -compositions are defined exactly as in  $\mathcal{A}$ , which means ignoring the  $\alpha_X$ . Hence,  $\tilde{\mathcal{A}}$  is obviously quasi-equivalent to  $\mathcal{A}$  (all we have done is add many different copies of the same object). There is a canonical*

$$(17.9) \quad \begin{aligned} \tilde{\alpha} &\in CC^0(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}), \\ \tilde{\alpha}_X^0 &= \alpha_X, \\ \tilde{\alpha}^d &= 0 \quad \text{for } d > 0. \end{aligned}$$

*Our given  $\beta$  extends in an obvious way to a Hochschild cocycle  $\tilde{\beta}$  on  $\tilde{\mathcal{A}}$ . Define*

$$(17.10) \quad \tilde{\phi} = \tilde{\beta} - \partial\tilde{\alpha} \in CC^1(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}).$$

*Finally, let  $\mathcal{A}^{inf} \subset \tilde{\mathcal{A}}$  be the full subcategory of those objects for which the leading term  $\tilde{\phi}^0$  vanishes. By construction, these are the same as infinitesimally equivariant objects of  $\mathcal{A}$ . In addition to its  $A_{\infty}$ -structure,  $\mathcal{A}^{inf}$  carries a Hochschild cocycle  $\phi$ , obtained by restricting  $\tilde{\phi}$ . Since  $\tilde{\phi}^0$  vanishes by definition,  $\phi^1$  consists of chain maps, which in fact reproduce (17.6). One obtains the derivation property for these maps using  $\phi^2$ .*



Suppose now that  $\mathcal{A}$  is proper, and that  $\text{char}(\mathbb{K}) = 0$ . The “decategorification” of  $\Phi$  takes the form of a Mukai-type pairing between infinitesimally equivariant objects,

$$(17.11) \quad (X_0, X_1)_{inf} = \text{Str}(e^{u\Phi_{X_0, X_1}}) \in \mathbb{K}[[u]].$$

The reduction to  $u = 0$  reproduces the ordinary Mukai pairing, which means the Euler characteristic of  $H^*(\text{hom}_{\mathcal{A}}(X_0, X_1))$ . To reformulate (17.11) in even more basic terms, let’s assume temporarily that  $\mathbb{K} = \mathbb{C}$ . Consider the generalized eigenspace of  $\Phi_{X_0, X_1}$  associated to  $\lambda \in \mathbb{C}$ , and let  $\chi_\lambda \in \mathbb{Z}$  be its Euler characteristics. Then

$$(17.12) \quad (X_0, X_1)_{inf} = \sum_{\lambda} \chi_\lambda e^{u\lambda},$$

where only finitely many terms of the sum are nonzero. If we introduce another formal variable  $t = e^u$ , then this becomes

$$(17.13) \quad (X_0, X_1)_{inf} = \sum_{\lambda} \chi_\lambda t^\lambda,$$

which is similar to the expression for equivariant Mukai pairings in the presence of a circle action (except that here, all powers  $t^\lambda$  are allowed, and not just integer ones).

Instead of just having a pairing (17.11) on objects, one would like to realize it as a bilinear map on a suitable homology theory associated to  $(\mathcal{A}, [\beta])$ , in analogy with (8.10) and its equivariant version from Lecture 13. Before discussing that issue, we would like to introduce yet another reformulation, which is natural but turns out to be potentially misleading.

### A dead end

We return to the general situation (of an arbitrary coefficient field  $\mathbb{K}$ ). As discussed in Lecture 9, Hochschild cohomology describes the first order deformation theory of  $A_\infty$ -categories. More precisely (see Remark 9.5),  $HH^2(\mathcal{A}, \mathcal{A})$  classifies first order curved deformations of  $\mathcal{A}$  up to isomorphism. One can carry over that discussion to  $HH^k(\mathcal{A}, \mathcal{A})$  for any  $k$ , by arranging for the deformation parameter to have degree  $2 - k$ , including the case  $k = 1$  which is of interest here.

Hence, let  $\epsilon$  be a formal variable of degree 1, and take  $\mathbb{K}_\epsilon = \mathbb{K}[\epsilon]/\epsilon^2$  (which of course is a one-dimensional exterior algebra). Given a cocycle  $\beta \in CC^1(\mathcal{A}, \mathcal{A})$ , one defines a curved deformation  $\mathcal{A}_\epsilon$  of  $\mathcal{A}$  over  $\mathbb{K}_\epsilon$  by setting

$$(17.14) \quad \begin{aligned} Ob(\mathcal{A}_\epsilon) &= Ob(\mathcal{A}), \\ hom_{\mathcal{A}_\epsilon}(X_0, X_1) &= hom_{\mathcal{A}}(X_0, X_1) \otimes \mathbb{K}_\epsilon, \\ \mu_{\mathcal{A}_\epsilon}^0 &= \epsilon\beta^0, \\ \mu_{\mathcal{A}_\epsilon}^d &= \mu_{\mathcal{A}}^d + \epsilon\beta^d \quad \text{for } d > 0. \end{aligned}$$

In fact, the  $A_\infty$ -equations (with curvature) for  $\mathcal{A}_\epsilon$  are precisely the equations for  $\beta$  to be a Hochschild cocycle. It is therefore tempting to re-interpret the previous notions in this context. Suppose for simplicity that  $\mathcal{A}$  is strictly unital.

LEMMA 17.3. *Infinitesimally equivariant objects of  $\mathcal{A}$  correspond bijectively to curved strictly unital  $A_\infty$ -functors  $\mathbb{K}_\epsilon \rightarrow \mathcal{A}_\epsilon$ .*

Here,  $\mathbb{K}_\epsilon$  is considered as a special case of an  $A_\infty$ -category (with one object), and everything is supposed to be  $\mathbb{K}_\epsilon$ -linear. The statement is straightforward, once one has decoded the terminology involved. Generally, a curved  $A_\infty$ -functor  $\mathcal{X} : \mathbb{K}_\epsilon \rightarrow \mathcal{A}_\epsilon$  consists of an object  $X \in \text{Ob}(\mathcal{A}_\epsilon)$  together with elements

$$(17.15) \quad \begin{aligned} \mathcal{X}^0 &\in \text{hom}_{\mathcal{A}_\epsilon}^1(X, X) \quad \text{of order } O(\epsilon), \\ \mathcal{X}^d(1, \dots, 1) &\in \text{hom}_{\mathcal{A}_\epsilon}^{1-d}(X, X) \quad \text{for } d > 0, \end{aligned}$$

satisfying the  $A_\infty$ -functor equations (with curvature). However, in the strictly unital case,  $\mathcal{X}^1(1) = e_X$  is the identity, and  $\mathcal{F}^d(1, \dots, 1) = 0$  for all  $d > 1$ , hence all that remains is  $\mathcal{F}^0$ ; and if one writes that as  $\mathcal{F}^0 = -\epsilon\alpha_X$ , the  $A_\infty$ -functor equations reduce to (17.5). Going a little further, the  $A_\infty$ -category  $\text{fun}(\mathbb{K}_\epsilon, \mathcal{A}_\epsilon)$  of such  $A_\infty$ -functors can be identified with the first order deformation  $\mathcal{A}_\epsilon^{\text{inf}}$  of  $\mathcal{A}^{\text{inf}}$  associated to the Hochschild cocycle  $\phi$  (see Remark 17.2). In particular, if  $X_0$  and  $X_1$  are two infinitesimally equivariant objects, with their associated functors  $\mathcal{F}_0$  and  $\mathcal{F}_1$ , one has a long exact sequence

$$(17.16) \quad \cdots \rightarrow H^{*-1}(\text{hom}_{\mathcal{A}}(X_0, X_1)) \longrightarrow H^*(\text{hom}_{\text{fun}(\mathbb{K}_\epsilon, \mathcal{A}_\epsilon)}(\mathcal{X}_0, \mathcal{X}_1)) \longrightarrow H^*(\text{hom}_{\mathcal{A}}(X_0, X_1)) \rightarrow \cdots$$

whose boundary map is  $\Phi_{X_0, X_1}$ .

REMARK 17.4. *One can replace the strict unitality assumption by a suitable version of cohomological unitality. A version of Lemma 17.3 then still holds, but the correspondence is a bijection only on the level of quasi-isomorphism classes, and the proof is a little more involved.*

The point of view of the curved deformation  $\mathcal{A}_\epsilon$  can be useful when thinking of defining, say, infinitesimal equivariance for  $A_\infty$ -modules. In another apparently natural direction, one could consider the Hochschild homology of  $\mathcal{A}_\epsilon$ , defined in the usual way but over  $\mathbb{C}_\epsilon$ . This sits in a long exact sequence

$$(17.17) \quad \cdots \rightarrow HH_*(\mathcal{A}, \mathcal{A}) \longrightarrow HH_*(\mathcal{A}_\epsilon, \mathcal{A}_\epsilon) \longrightarrow HH_*(\mathcal{A}, \mathcal{A}) \rightarrow \cdots$$

whose boundary map is the Lie action of  $[\beta] \in HH^1(\mathcal{A}, \mathcal{A})$  on Hochschild homology. We have not mentioned this structure before, but its origin is fairly intuitive: just like actual automorphisms of  $\mathcal{A}$  act on Hochschild homology, there is a corresponding infinitesimal action, which means that  $HH_*(\mathcal{A}, \mathcal{A})$  is a representation of the graded Lie algebra  $HH^{*+1}(\mathcal{A}, \mathcal{A})$ .

Via Lemma 17.3 and the functoriality of Hochschild homology, every infinitesimally equivariant object gives rise to a class in  $HH_0(\mathcal{A}_\epsilon, \mathcal{A}_\epsilon)$ . Somewhat disappointingly, these classes are not sufficiently sensitive to the choice of  $\alpha_X$  to be of real interest (see the following Example). Therefore, it seems that  $HH_*(\mathcal{A}_\epsilon, \mathcal{A}_\epsilon)$  is not after all the correct homology theory for our purpose.

EXAMPLE 17.5. Consider the curved  $A_\infty$ -functor (in fact  $A_\infty$ -homomorphism, since both categories involved have a single object)

$$(17.18) \quad \begin{aligned} \mathcal{G} : \mathbb{K}_\epsilon &\longrightarrow \mathbb{K}_\epsilon, \\ \mathcal{G}^0 &= c\epsilon, \\ \mathcal{G}^1(1) &= 1, \quad \mathcal{G}^d = 0 \text{ for } d > 1. \end{aligned}$$

Here,  $c \in \mathbb{K}$  is some constant. This corresponds to making the unique object of the target category  $\mathbb{K}_\epsilon$  infinitesimally equivariant (with respect to  $\beta = 0$ ), but in a way which is twisted by the choice of  $c$ . There is an associated endomorphism  $\mathcal{G}_*$  of  $HH_*(\mathbb{K}_\epsilon, \mathbb{K}_\epsilon) \cong \mathbb{K}_\epsilon$ . In terms of (17.17), this is compatible with the identity endomorphisms of  $HH_*(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}$ . However, because of that fact and the grading, it follows that  $\mathcal{G}_*$  must be the identity, hence does not depend on  $c$ .

### Infinitesimally equivariant Hochschild homology

Let's again impose the assumption that  $\text{char}(\mathbb{K}) = 0$ . In that case, one can associate to  $\beta$  its infinitesimal orbit bimodule  $\mathcal{O}^{inf}$  as in (9.34). This is a bimodule over  $\mathcal{A}$ , which additionally is linear over  $\mathbb{K}[[u]]$ . In analogy with (13.41), we define the *infinitesimally equivariant Hochschild homology* to be

$$(17.19) \quad HH_*^{inf}(\mathcal{A}, \mathcal{A}) = H^*(\widehat{CC}_*(\mathcal{A}, \mathcal{O}^{inf})),$$

where one starts with the Hochschild chain complex with coefficients in  $\mathcal{O}^{inf}$  and then takes its  $u$ -adic completion. The  $u$ -adic filtration gives rise to a spectral sequence which starts with  $HH_*(\mathcal{A}, \mathcal{A})[[u]]$ , and whose first differential is given by  $[\beta]$  (this time in terms of the structure of Hochschild homology as a module over the Hochschild cohomology algebra; in particular, commutativity of that algebra ensures that this differential squares to zero). The edge homomorphism of that spectral sequence is the  $u = 0$  reduction map

$$(17.20) \quad HH_*^{inf}(\mathcal{A}, \mathcal{A}) \longrightarrow HH_*(\mathcal{A}, \mathcal{A}).$$

Again because of the functoriality of Hochschild homology, any infinitesimally equivariant object has a class

$$(17.21) \quad [X]_{HH}^{inf} \in HH_0^{inf}(\mathcal{A}, \mathcal{A}),$$

whose image under (17.20) is the ordinary Hochschild homology class.

EXAMPLE 17.6. Take  $\mathcal{A} = \mathbb{K}$  and  $\beta = 0$ , and make the unique object  $X$  infinitesimally equivariant by choosing some constant  $\alpha_X = c$ . The associated class is  $[X]_{HH}^{inf} = e^{cu} \in \mathbb{K}[[u]]$ . Unlike the corresponding situation in Example 17.5, this definitely depends on  $c$ .

CONJECTURE 17.7. Take a proper  $A_\infty$ -category  $\mathcal{A}$  defined over a field  $\mathbb{K}$  of characteristic 0, together with a class (17.1). Then, there is a canonical pairing

$$(17.22) \quad (\cdot, \cdot)_{HH}^{inf} : HH_0^{inf}(\mathcal{A}, \mathcal{A}) \otimes HH_0^{inf}(\mathcal{A}, \mathcal{A}) \longrightarrow \mathbb{K}[[u]]$$

satisfying

$$(17.23) \quad (-ux_0, x_1)_{HH}^{inf} = u(x_0, x_1)_{HH}^{inf} = (x_0, ux_1)_{HH}^{inf},$$

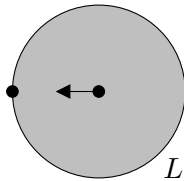


FIGURE 1.

and whose reduction to  $u = 0$  recovers (8.10). The appropriate version of the Cardy relation would say that, if  $X_0$  and  $X_1$  are infinitesimally equivariant objects, then the image of  $[X_0]_{HH}^{inf} \otimes [X_1]_{HH}^{inf}$  under (17.22) recovers (17.11).

This formulation of the Conjecture is probably not the definitive one, as one can already see by comparing it with the discussion in Lecture 8, where the natural pairing is between classes associated to perfect modules and “dual classes” associated to proper modules. Presumably, the same should apply to the infinitesimally equivariant case.

**Application to Fukaya categories**

Let  $M$  be as in Lecture 12. If we realize symplectic cohomology as the Floer cohomology  $HF^*(H)$  of a suitable Hamiltonian  $H$ , the chain level homomorphism underlying the open-closed string map (17.3) has components

$$(17.24) \quad \psi_L^{1;0} : CF^*(H) \longrightarrow CF^*(L, L),$$

$$(17.25) \quad \psi_{L_0, L_1}^{1;1} : CF^*(H) \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_1)[-1],$$

$$(17.26) \quad \dots$$

Like all operations involving Lagrangian and Hamiltonian Floer cohomology, one can think of these as being defined in terms of compact Riemann surfaces with marked boundary and interior points, and where each interior point comes with an additional preferred tangent direction (this is a direct relationship: moduli spaces of maps from those Riemann surfaces to  $M$  define the operations). Concretely, there is a single Riemann surface  $S$  underlying  $\psi^{1;0}$  (Figure 1). For  $\psi^{1;1}$ , we have a family of Riemann surfaces  $S_r$  parametrized by a closed interval,  $r \in [0, 1]$ , and which degenerates at the endpoints of that interval (Figure 2). Note the conventions for the tangent directions: in the case of  $S$ , the tangent direction at the interior marked point goes towards the boundary marked point. The same holds for the relevant component of  $S_r$  in the degenerate cases  $r = 0, 1$ . Finally, as we go from  $r = 0$  to  $r = 1$ , the tangent vector rotates by  $\pi$ .

From now on, fix some  $B \in SH^1(M)$  and a cocycle representative  $b \in CF^1(H)$ .

DEFINITION 17.8. *An infinitesimally equivariant Lagrangian submanifold is an object  $L$  of  $Fuk(M)$  together with  $\alpha_L \in CF^0(L, L)$ , satisfying  $\mu_{Fuk(M)}^1(\alpha_L) = \psi_L^{1;0}(b)$ .*

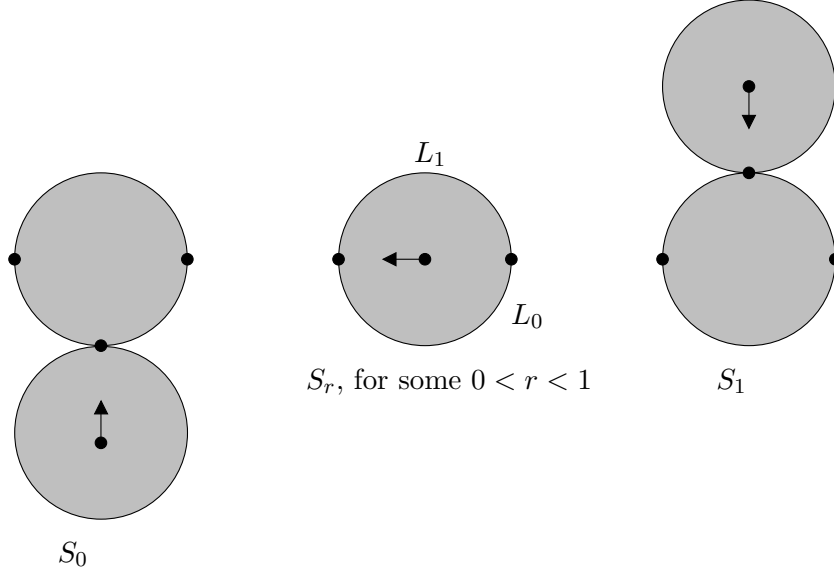


FIGURE 2.

This is actually a special case of Definition 17.1. Suppose for simplicity that  $L$  is connected, and that it carries a flat  $\mathbb{K}$ -line bundle (not a higher rank flat vector bundle). In view of the isomorphism  $HF^*(L, L) \cong H^*(L; \mathbb{K})$ , the obstruction to equivariance resides in  $H^1(L; \mathbb{K})$ ; and if that vanishes, then the effective freedom of choice is  $H^0(L; \mathbb{K}) \cong \mathbb{K}$ .

As before, given two infinitesimally equivariant objects, one gets an endomorphism

$$(17.27) \quad \begin{aligned} \phi &= \phi_{L_0, L_1} : CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_1), \\ \phi(a) &= \psi_{L_0, L_1}^{1;1}(b, a) + (-1)^{|a|} \mu_{Fuk(M)}^2(\alpha_{L_1}, a) - \mu_{Fuk(M)}^2(a, \alpha_{L_0}), \end{aligned}$$

and an induced endomorphism  $\Phi_{L_0, L_1}$  of  $HF^*(L_0, L_1)$ . We can then define, exactly as in (17.11) or (17.13), the equivalent expressions ( $u$ -intersection numbers or  $t$ -intersection numbers)

$$(17.28) \quad \begin{aligned} L_0 \cdot_u L_1 &= \text{Str}(e^{u\Phi_{L_0, L_1}}), \\ L_0 \cdot_t L_1 &= \sum_{\lambda} \chi_{\lambda} t^{\lambda}, \end{aligned}$$

where in the second equation (for  $\mathbb{K} = \mathbb{C}$  only)  $\chi_{\lambda}$  is the Euler characteristic of the generalized  $\lambda$ -eigenspace of  $\Phi_{L_0, L_1}$ . Because of the grading conventions used, reduction to  $u = 0$  (respectively  $t = 1$ ) yields the ordinary intersection number  $L_0 \cdot L_1$  up to a dimension-dependent sign  $(-1)^{n(n+1)/2}$ . Note that by definition

$$(17.29) \quad \sum_{\lambda} |\chi_{\lambda}| \leq \text{rank}(HF^*(L_0, L_1))$$

Hence, our improved intersection numbers give a lower bound for the number of essential intersection points (intersection points which can't be removed by a Hamiltonian isotopy).

This bound is stronger than that given by the ordinary intersection number  $L_0 \cdot L_1$ , and weaker than the total rank of Floer cohomology. It is quite well-behaved, as shown for instance by the following result:

PROPOSITION 17.9 ([184, Theorem 5.6]). *Suppose that  $\dim(M) = 2n > 2$ . Let  $S$  be a Lagrangian sphere (which is then automatically an object of the Fukaya category, unique up to a shift, and also automatically infinitesimally equivariant), and  $\tau_S$  the associated Dehn twist (as a graded symplectic automorphism, which acts on the Fukaya category). If  $L_1$  is infinitesimally equivariant, then there is a preferred way of making  $\tau_S(L_1)$  infinitesimally equivariant. Moreover, given that and another infinitesimally equivariant  $L_0$ , we have the  $t$ -Picard Lefschetz formula*

$$(17.30) \quad L_0 \cdot_t \tau_S(L_1) = L_0 \cdot_t L_1 - (L_0 \cdot_t S)(S \cdot_t L_1).$$

This is essentially a consequence of the compatibility of the maps  $\Phi$  with the long exact sequence [173]

$$(17.31) \quad \cdots \rightarrow HF^*(S, L_1) \otimes HF^*(L_0, S) \longrightarrow HF^*(L_0, L_1) \longrightarrow HF^*(L_0, \tau_S(L_1)) \rightarrow \cdots$$

It is interesting to consider the action of  $\tau_S^2$  on  $t$ -intersection numbers when  $M$  is of dimension  $2n$  with  $n$  even (which is when its action on standard homology is trivial).  $\Phi_{S,S}$  is a derivation of the algebra  $HF^*(S, S) \cong H^*(S; \mathbb{C})$ . Hence, it is zero in degree 0, but can in principle be any constant  $\lambda$  in degree  $n$ , and then

$$(17.32) \quad S \cdot_t S = 1 + t^\lambda.$$

Applying (17.30) yields

$$(17.33) \quad L_0 \cdot_t \tau_S^2(L_1) = L_0 \cdot_t L_1 + (S \cdot_t S - 2)(L_0 \cdot_t S)(S \cdot_t L_1).$$

In the case where  $\lambda \neq 0$  in (17.32), this shows that  $t$ -intersection numbers behave fundamentally differently from the ordinary topological ones (in particular, in dimensions  $2n = 4, 12$  where  $\tau_S^2$  is smoothly isotopic to the identity, they detect genuinely symplectic phenomena). The remaining case  $\lambda = 0$  is obviously somewhat less interesting.

Finally, one would like to have a geometric analogue of  $HH_*^{inf}(\mathcal{A}, \mathcal{A})$ , involving symplectic homology rather than Hochschild homology.

CONJECTURE 17.10. *Suppose that  $\text{char}(\mathbb{K}) = 0$ . Then there is a graded  $\mathbb{K}[[u]]$ -module  $SH_*^{inf}(M)$  depending on  $M$  and  $B$ , with the following properties. First of all, there is a spectral sequence  $SH_*(M)[[u]] \Rightarrow SH_*^{inf}(M)$ , where the first nontrivial boundary operator is the action of  $B$ . Secondly, each infinitesimally equivariant  $L$  defines a class  $[L]_{SH}^{inf} \in SH_{-n}^{inf}(M)$ . Finally, there is a pairing*

$$(17.34) \quad (\cdot, \cdot)_{SH}^{inf} : SH_*^{inf} \otimes SH_*^{inf} \longrightarrow \mathbb{K}[[u]]$$

*of degree  $2n$ , which satisfies the analogue of (17.23), and which sends  $[L_0]_{SH}^{inf} \otimes [L_1]_{SH}^{inf}$  to  $L_0 \cdot_u L_1$ .*

The edge homomorphism of our spectral sequence would yield a map  $SH_*^{inf}(M) \rightarrow SH_*(M) \rightarrow H_*(M; \mathbb{K})$ . With this in mind, Conjecture 17.10 can be considered as a first step towards

Desideratum 3.3 (expanding formally around  $t = 1$ , which is our  $u = 0$ ). Note that this is certainly not the final answer (in the same sense as holomorphic equivariant cohomology is not equivariant cohomology; see Lecture 2). In any case, even Conjecture 17.10 remains unproved at the moment. However, the first order approximation, using  $\mathbb{K}[u]/u^2$  instead of  $\mathbb{K}[[u]]$ , can be constructed using a slight variation of the argument in [180] (for any coefficient field  $\mathbb{K}$ , in fact), and has the desired properties.





LECTURE 18

## Dilations

The formalism from the previous lecture starts with an a priori arbitrary class  $B \in SH^1(M)$ . However, its ultimate behaviour depends strongly on the choice of  $B$ . For instance, if  $B$  comes from ordinary cohomology via the map  $H^*(M) \rightarrow SH^*(M)$ , the improved self-intersection numbers  $L \cdot_t L$  are trivial. By this, we mean that they do not depend on  $t$  at all, and reduce to the ordinary self-intersection numbers  $L \cdot L$ .

One condition which is useful in this context is the *dilation* property introduced in [184]. If  $B$  is a dilation, then  $L \cdot_t L$  is nontrivial for certain  $L$ , such as spheres and complex projective spaces. On the other hand, the existence of a dilation is quite a strong constraint, which will only be satisfied by a small class of Liouville manifolds.

### A topological version

Take an oriented manifold  $M^{2n}$  and a class  $B \in H^1(M; \mathbb{Z})$ , which is represented by an infinite cyclic covering  $\tilde{M} \rightarrow M$ . Denote the generator of the covering group by  $T : \tilde{M} \rightarrow \tilde{M}$ . Consider  $n$ -dimensional closed oriented submanifolds  $L \subset M$ . Let's say that such an  $L$  is infinitesimally  $B$ -equivariant if it comes with a lift  $\tilde{L}$  to  $\tilde{M}$ . We can then define improved intersection numbers by

$$(18.1) \quad L_0 \cdot_t L_1 = \sum_{k \in \mathbb{Z}} t^k T^k(\tilde{L}_0) \cdot \tilde{L}_1.$$

This may not be terribly interesting from our viewpoint, since clearly  $L \cdot_t L = L \cdot L$ , but it still has its uses, see e.g. [78, 30]. There is also a Poincaré dual formulation, which goes as follows. Let  $\Sigma$  be an oriented closed hypersurface in  $M$ , representing the Poincaré dual of  $B$ . Given a submanifold  $L \subset M$  which intersects  $\Sigma$  transversally, choose a locally constant function  $\alpha_L : L \setminus \Sigma \rightarrow \mathbb{Z}$  which jumps once when crossing  $\Sigma$  (in positive normal direction). Suppose that  $L_0, L_1$  are two such submanifolds, which intersect transversally, and such that  $L_0 \cap L_1$  is disjoint from  $\Sigma$ . One then defines  $L_0 \cdot_t L_1$  by counting each intersection point with its usual sign and a power  $t^{\alpha_{L_1}(x) - \alpha_{L_0}(x)}$ .

Suppose now that  $M$  is a Liouville manifold with vanishing first Chern class, with  $B$  as before. The composition

$$(18.2) \quad H^*(M; \mathbb{K}) \longrightarrow SH^*(M) \xrightarrow{[\psi_L^0]} HF^*(L, L) \xrightarrow{\cong} H^*(L; \mathbb{K})$$

is the ordinary restriction map. Hence, if  $L$  is a Lagrangian submanifold, the condition that  $L$  can be made infinitesimally equivariant with respect to the image of  $B$  in  $SH^1(M)$  is equivalent to  $B|L = 0$ . In fact, a cochain level realization of (18.2) shows that, given a suitable choice of representing hypersurface  $\Sigma$ , one can make  $L$  infinitesimally equivariant by choosing an  $\alpha_L$  as before. Up to chain homotopy, the resulting map (17.27) can be described as follows. The first term  $\psi_{L_0, L_1}^{1;1}(b, a)$  counts Floer trajectories crossing  $\Sigma$ , which means

$$(18.3) \quad \left\{ \begin{array}{l} u : \mathbb{R} \times [0, 1] \longrightarrow M, \\ u(s, 0) \in L_0, \quad u(s, 1) \in L_1, \\ \lim_{s \rightarrow -\infty} u(s, \cdot) = x_1, \\ \lim_{s \rightarrow +\infty} u(s, \cdot) = x_0, \\ \partial_s u + J_t(u) \partial_t u = 0, \\ \text{together with a choice of } t_0 \text{ such that } u(0, t_0) \in \Sigma. \end{array} \right.$$

However (assuming suitable transversality) there are no isolated points of this moduli space, since each Floer trajectory intersects  $\Sigma$  in a one-manifold. Hence, the only thing that counts are the two other terms in (17.27), which turn out to count constant holomorphic triangles, and contribute

$$(18.4) \quad x \longmapsto \alpha_{L_1}(x) - \alpha_{L_0}(x).$$

The associated intersection numbers (17.28) then reproduce the purely topological construction given above. This is relevant for us mainly as a warning note: one does *not* want to use classes in  $SH^1(M)$  which come from ordinary cohomology.

### Definition and first consequences

A class  $B \in SH^1(M)$  is called a *dilation* if its image under the BV operator is the unit for the ring structure of  $SH^*(M)$ :

$$(18.5) \quad \Delta B = 1.$$

In particular, such a class can't come from  $H^1(M)$ , since its image is annihilated by  $\Delta$ . The key observation is:

LEMMA 18.1 ([184, Section 4]). *Suppose that  $B$  is a dilation. Let  $L$  be an object of  $Fuk(M)$  which is infinitesimally equivariant with respect to  $B$ . Then, the action of  $\Phi_{L,L}$  on  $HF^n(L, L) \cong H^n(L; \mathbb{K})$  is  $+1$ .*

We will not give the proof, which in fact shows the following more general fact. If one starts with any class  $B \in SH^1(M)$  and builds the corresponding theory of infinitesimally equivariant submanifolds, the action of  $\Phi_{L,L}$  on  $HF^n(L, L)$  equals the product with the image of  $\Delta B$  under  $SH^0(L, L) \rightarrow HF^0(L, L)$ . In our case, that image is the identity element  $[e_L] \in HF^0(L, L)$ , whence the specific form of the result.

Poincaré duality in Floer cohomology says that product  $HF^*(L_1, L_0) \otimes HF^{n-*}(L_0, L_1) \rightarrow HF^n(L_0, L_0)$  is nondegenerate. If we use the resulting isomorphism

$$(18.6) \quad HF^*(L_1, L_0) \cong HF^{n-*}(L_0, L_1)^\vee,$$

the derivation property of the maps  $\Phi_{L_0, L_1}$  together with Lemma 18.1 shows that

$$(18.7) \quad \Phi_{L_1, L_0} = Id - \Phi_{L_0, L_1}^\vee,$$

which implies that

$$(18.8) \quad L_1 \cdot_t L_0 = (-1)^n t(L_0 \cdot_{t^{-1}} L_1).$$

In the special case of a  $\mathbb{K}$ -homology sphere  $S$ , Lemma 18.1 determines  $\Phi_{S, S}$  completely: it acts by 0 in degree 0, and by 1 in degree  $n$ . Hence,

$$(18.9) \quad S \cdot_t S = 1 + (-1)^n t,$$

which is welcome in view of (17.33).

### Cotangent bundles

For  $M = T^*L$  with  $L$  a closed oriented *Spin* manifold, we have the isomorphism (6.7). Under that isomorphism, the class  $1 \in SH^0(M)$  corresponds to the homology class of the constant loops  $L \subset \mathcal{L}L$ , and the meaning of the BV operator was discussed in Example 12.5. This means that the question of when  $M$  has a dilation can be answered in purely algebro-topological terms.

**EXAMPLE 18.2.** *If  $L$  is a  $K(\pi, 1)$ , then  $M = T^*L$  doesn't admit a dilation. By Viterbo functoriality (6.9), this implies that if some Liouville manifold does admit a dilation, it can't contain  $L$  as an exact Lagrangian submanifold. This certainly applies to Lagrangian tori. For instance, the manifold from Example 11.2 cannot possibly admit a dilation (the same applies to many other examples of local mirror symmetry).*

**EXAMPLE 18.3.** *Take  $L = S^2$ , and suppose that  $\text{char}(\mathbb{K}) \neq 2$  (we point out that in principle, the notion of dilation makes sense for an arbitrary coefficient field). Then, the evaluation map  $\mathcal{L}S^2 S^2 \rightarrow S^2$  induces an isomorphism  $H_2(\mathcal{L}S^2; \mathbb{K}) \rightarrow H_2(S^2; \mathbb{K})$ . Take a one-parameter family of loops which fill  $S^2$ , which means such that the associated map  $S^1 \times S^1 \rightarrow S^2$  has degree 1. Then, the class in  $H_1(\mathcal{L}S^2; \mathbb{K})$  of that family is a dilation.*

*On the other hand, take  $\text{char}(\mathbb{K}) = 2$ . Then*

$$(18.10) \quad H_1(\mathcal{L}S^2; \mathbb{K}) \cong \mathbb{K}, \quad H_2(\mathcal{L}S^2; \mathbb{K}) \cong \mathbb{K}^2.$$

*This is particularly easy to see from the point of view of Morse theory for the geodesic functional. Let  $\mathbb{R}P^3$  be the space of great circles, which is a critical manifold of Morse index 1. Any point on it defines the generator of  $H_1(\mathcal{L}S^2; \mathbb{K})$ . The first generator for  $H_2(\mathcal{L}S^2; \mathbb{K})$  is a loop in  $\mathbb{R}P^3$ , and the second generator consists of the constant loops  $[S^2]$ . With those conventions for (18.10), the BV operator is the diagonal map  $(1, 1) : \mathbb{K} \rightarrow \mathbb{K}^2$ . Hence, there is no dilation for this choice of coefficient field (see [135] for more complete computations).*

A simpler version of the argument above shows that  $T^*S^n$ ,  $n > 2$ , admits a dilation for any choice of coefficients  $\mathbb{K}$ .

It would be interesting to have an example of a simply-connected *formal* manifold whose cotangent bundle does not admit a dilation (in characteristic zero).

### Complements of smooth ample divisors

Let  $\bar{M}$  be a smooth complex projective variety of dimension  $n$ . Suppose that there is a smooth ample divisor  $D$  such that

$$(18.11) \quad K_{\bar{M}} \cong \mathcal{O}_{\bar{M}}(-mD) \quad \text{for some } m \in \mathbb{Z}.$$

We will consider

$$(18.12) \quad M = \bar{M} \setminus D,$$

which is an affine variety with a natural (up to multiplication with a constant) trivialization of  $K_M$ . We will be interested in one of the simplest Gromov-Witten invariants of  $\bar{M}$ . Namely, let  $\tilde{\mathcal{M}}_{0,2}(\bar{M}; 1)$  be the space of genus zero stable maps with two marked points which have intersection number 1 with  $D$ . After a generic small perturbation of the almost complex structure, this space will be a smooth oriented manifold, which comes with evaluation maps  $ev_1, ev_2 : \tilde{\mathcal{M}}_{0,2}(\bar{M}; 1) \rightarrow \bar{M}$ . We write the resulting Gromov-Witten invariant as an endomorphism of cohomology,

$$(18.13) \quad \begin{aligned} \bar{\Gamma} : H^*(\bar{M}; \mathbb{K}) &\longrightarrow H^{*+2-2m}(\bar{M}; \mathbb{K}), \\ \bar{\Gamma}(x) &= ev_{2,*}(ev_1^*x). \end{aligned}$$

We then further combine  $\bar{\Gamma}$  with the inclusion maps  $i_* : H^*(D; \mathbb{K}) \rightarrow H^{*+2}(\bar{M}; \mathbb{K})$  and restriction maps  $H^*(\bar{M}; \mathbb{K}) \rightarrow H^*(M; \mathbb{K})$  to get

$$(18.14) \quad \Gamma : H^*(D; \mathbb{K}) \longrightarrow H^{*+4-2m}(M; \mathbb{K}).$$

Let's now turn to symplectic topology. The contact manifold describing the structure of  $M$  at infinity is the circle bundle  $p : N \rightarrow D$  associated to the normal bundle of  $D$ . The cohomology of  $N$  sits in a short exact sequence

$$(18.15) \quad 0 \rightarrow H^*(D; \mathbb{K})/im(\kappa) \xrightarrow{p^*} H^*(N; \mathbb{K}) \xrightarrow{p_*} ker(\kappa)[-1] \rightarrow 0,$$

where  $\kappa : H^*(D; \mathbb{K}) \rightarrow H^{*+2}(D; \mathbb{K})$  is the cup product with the Chern class of the normal bundle. One can choose the contact one-form so that the Reeb flow on  $N$  is fibrewise rotation, hence  $2\pi$ -periodic. Let's think of the symplectic cohomology of  $M$  as the direct limit of Floer cohomologies  $HF^*(H_k)$ , as in (12.51), where the flow of  $H_k$  on the conical end of  $M$  equals  $(2k+1)\pi$  times the Reeb flow. The first approximation  $HF^*(H_0) \cong H^*(M; \mathbb{K})$  is ordinary cohomology. The next one includes the primitive Reeb orbits, hence sits in a long exact sequence

$$(18.16) \quad \dots \rightarrow H^{*+2m-3}(N; \mathbb{K}) \xrightarrow{d} H^*(M; \mathbb{K}) \longrightarrow HF^*(H_1) \rightarrow \dots$$

The degree shift in the cohomology of  $N$  is a consequence of (18.11) and a Conley-Zehnder index computation. Our aim is to give a description of  $d$  in terms of the Gromov-Witten invariant (18.13), and then a conjectural partial description of the BV operator  $\Delta$  on  $HF^*(H_1)$  in the same terms.

LEMMA 18.4 (Diogo). *Assume that  $m > 0$ . Then, the map  $d$  in (18.16) is the composition*

$$(18.17) \quad H^*(N; \mathbb{K}) \xrightarrow{p_*} H^{*-1}(D; \mathbb{K}) \xrightarrow{\Gamma} H^{*+3-2m}(M; \mathbb{K}).$$

This is proved in [58] (it seems likely that the method used there would apply to all values of  $m$ ). An earlier related conjecture, formulated in terms of contact homology rather than symplectic homology, can be found in [101].

LEMMA 18.5. *The BV operator  $\Delta$  on  $HF^*(H_1)$  fits into a commutative diagram whose rows are given by (18.16)*

$$(18.18) \quad \begin{array}{ccccccc} \dots & \longrightarrow & H^{*+2m-3}(N; \mathbb{K}) & \xrightarrow{d} & H^*(M; \mathbb{K}) & \longrightarrow & HF^*(H_1) \longrightarrow \dots \\ & & \downarrow p^*p_* & & \downarrow 0 & & \downarrow \Delta \\ \dots & \longrightarrow & H^{*+2m-4}(N; \mathbb{K}) & \xrightarrow{d} & H^{*-1}(M; \mathbb{K}) & \longrightarrow & HF^{*-1}(H_1) \longrightarrow \dots \end{array}$$

□

This is straightforward from a Morse-Bott formulation of Floer cohomology. However, it does not determine the part of  $\Delta$  which is responsible for the possible existence of dilations. To get a better description, note (again roughly following [101]) that we can use  $\Gamma$  to define a secondary operation

$$(18.19) \quad \begin{aligned} H^*(N; \mathbb{K}) \supset \ker(p^*p_*) &\xrightarrow{p_*} \ker(p^*) = \text{im}(\kappa) \xleftarrow{\kappa} H^{*-3}(\bar{D}; \mathbb{K}) / \ker(\kappa) \\ &= H^{*-3}(\bar{D}; \mathbb{K}) / \text{im}(p_*) \xrightarrow{\Gamma} H^{*+1-2m}(M; \mathbb{K}) / \text{im}(d). \end{aligned}$$

CONJECTURE 18.6. *The following diagram commutes:*

$$(18.20) \quad \begin{array}{ccc} \ker(p^*p_*) \cap \ker(d) & \longrightarrow & \frac{HF^{*+2-2m}(H_1)}{\text{im}(H^*(M; \mathbb{K}) \rightarrow HF^*(H_1))} \\ \downarrow & & \downarrow \Delta \\ H^{*+1-2m}(M; \mathbb{K}) / \text{im}(d) & \longrightarrow & HF^{*-1-2m}(H_1). \end{array}$$

Here, the top  $\rightarrow$  partially inverts one of the maps in (18.16); the bottom  $\rightarrow$  is also taken from that sequence, in a more straightforward way;  $\Delta$  descends to the quotient because of (18.18); and the left  $\downarrow$  is (18.19) restricted to  $\ker(d)$ .

EXAMPLE 18.7. *Take  $\bar{M} = \mathbb{C}P^n$ , and  $D = \mathbb{C}P^{n-1}$  a hyperplane. In this case  $m = n + 1$ ,  $M \cong \mathbb{C}^n$ , and  $N \cong S^{2n-1}$ . Because there is one line through every pair of points on  $\bar{M}$ , the endomorphism (18.14) maps  $H^{2n}(D; \mathbb{K})$  nontrivially to  $H^0(M; \mathbb{K})$ . As a consequence of that and Lemma 18.4,  $d : H^{2n+1}(N; \mathbb{K}) \rightarrow H^0(M; \mathbb{K})$  is nonzero. This means that the identity in  $H^0(M; \mathbb{K})$  maps trivially to symplectic cohomology, hence that  $SH^*(\mathbb{C}^n) = 0$ , as already*

stated in Example 6.8 (the argument we're giving here is by no means the simplest proof of that fact).

EXAMPLE 18.8. Take  $\bar{M} = \mathbb{C}P^1 \times \mathbb{C}P^1$ , and  $D$  the diagonal. In this case  $m = 2$ ,  $M \cong T^*S^2$ , and  $N \cong \mathbb{R}P^3$ . The relevant Gromov-Witten map is

$$(18.21) \quad \begin{aligned} \Gamma : H^*(D; \mathbb{K}) &\longrightarrow H^*(M; \mathbb{K}), \\ \Gamma(1) &= 2, \\ \Gamma([\textit{point}]) &= 0. \end{aligned}$$

The first expression  $\Gamma(1)$  counts the number of lines in  $\bar{M}$  going through a fixed point of  $M$ , which is 2 because of the two rulings. For  $\Gamma([\textit{point}])$  one fixes a point on  $D$  and adds up the homology classes of the two curves in the rulings going through that point. The sum is the homology class of the diagonal, which vanishes when restricted to  $M$ .

From (18.21) we see that  $d : H^3(N; \mathbb{K}) \rightarrow H^2(M; \mathbb{K})$  vanishes. Suppose now that  $\text{char}(\mathbb{K}) \neq 2$ , so that  $p^*p_* : H^3(N; \mathbb{K}) \rightarrow H^2(N; \mathbb{K})$  vanishes and  $\Gamma(1)$  is nonzero. Then, the secondary operation (18.19) yields

$$(18.22) \quad H^3(N; \mathbb{K}) \xrightarrow[\cong]{p_*} H^2(D; \mathbb{K}) \xleftarrow[\cong]{\kappa} H^0(D; \mathbb{K}) \xrightarrow[\cong]{\Gamma} H^0(M; \mathbb{K}).$$

From Conjecture 18.6, one would then conclude that  $M$  has a dilation (we already established that as a fact in Example 18.3, by entirely different methods).

EXAMPLE 18.9. Let  $\bar{M}$  be a cubic threefold in  $\mathbb{C}P^4$ , and  $D$  a hyperplane section. This is another case with  $m = 2$ , and we take  $\mathbb{K} = \mathbb{Q}$  for simplicity.  $\bar{M}$  is known to contain a two-parameter family of lines (this goes back to Fano; see [14] for an exposition). However,

$$(18.23) \quad p^*p_* : H^3(N; \mathbb{Q}) \longrightarrow H^2(N; \mathbb{Q})$$

is injective. Hence, Lemma 18.5 shows that no element in  $HF^*(H_1)$  can satisfy  $\Delta B = 1$  in that group; of course, it is still possible that a dilation might appear at the next step in the direct limit leading to  $SH^*(M)$ .

### Fibration methods

So far, we have only seen one positive example of a dilation, namely  $T^*S^n$  for  $n \geq 2$ . By the Künneth formula [139], it follows that products  $T^*S^n \times F$ , where  $F$  is any Liouville manifold with vanishing first Chern class, also admit dilations. One can generalize this idea as follows

THEOREM 18.10 ([184, Proposition 7.3]). *Take an exact symplectic Lefschetz fibration  $\pi : M \rightarrow \mathbb{C}$ , where the fibre  $F$  is a Liouville manifold of dimension  $2n \geq 4$ , and  $c_1(F) = 0$  (it then follows that  $M$  has the same structure). If  $F$  admits a dilation, then so does  $M$ .*

SKETCH OF PROOF. One defines a “vertical” or “fibrewise” version of symplectic cohomology  $SH_{\text{vert}}^*(M)$ , which is somewhat easier to compute. Concretely, it sits in a long exact

sequence

$$(18.24) \quad \cdots \rightarrow H^*(M, \{\operatorname{re}(\pi) \gg 0\}; \mathbb{K}) \longrightarrow SH_{\operatorname{vert}}^*(M) \longrightarrow SH^*(F) \rightarrow \cdots$$

The cohomology group on the left has one generator for each critical point of  $\pi$ , and that generator sits in degree  $n + 1 \geq 3$ . Hence, the map  $SH_{\operatorname{vert}}^*(M) \rightarrow SH_{\operatorname{vert}}^*(F)$  is an isomorphism in degrees  $* = 0, 1$ . That map is a homomorphism of BV algebras. On the other hand, there is another homomorphism of BV algebras

$$(18.25) \quad SH_{\operatorname{vert}}^*(M) \longrightarrow SH^*(M).$$

One transfers the given dilation of  $F$  through those two maps. □

COROLLARY 18.11. *The Milnor fibre of the  $(A_m)$  singularity,*

$$(18.26) \quad M = \{x_1^2 + \cdots + x_n^2 + x_{n+1}^{m+1} = 1\} \subset \mathbb{C}^{n+1},$$

*admits a dilation provided that  $n \geq 3$  (and if  $n = 3$ , the coefficient field  $\mathbb{K}$  has to be of characteristic  $\neq 2$ ).*

This is a direct consequence of Theorem 18.10, since the projection  $x_{n+1} : M \rightarrow \mathbb{C}$  is a Lefschetz fibration with fibre  $T^*S^{n-1}$ . By an iterated version of the same argument, one shows:

COROLLARY 18.12. *The Milnor fibre of any isolated hypersurface singularity  $p(x, y) = 0$ , where  $p(x, y) = x_1^2 + \cdots + x_{n+1}^2 + q(y_1, \dots, y_{m+1})$  for  $n + 1 \geq 3$ , admits a dilation (if  $n = 3$ , one has the same restriction on  $\operatorname{char}(\mathbb{K})$  as before). □*

In the language of Lecture 16, these singularities are at least triply suspended. It is useful to compare this result with the ones obtained there by purely algebraic means. Corollary 16.7 ensured the existence of  $\mathbb{C}^*$ -actions on the Milnor fibres of doubly suspended weighted homogeneous singularities (weighted homogeneity was necessary in order to ensure that the our algebraic description of the Milnor fibre was complete, something that is not an issue when using geometric methods). That  $\mathbb{C}^*$ -action was dilating, but with respect to an isomorphism (15.9) which was chosen artificially. In contrast, the dilation condition (18.5) refers to the geometrically given operator  $\Delta$ , or in other words to the canonical cyclic structure of the Fukaya category (this stricter condition explains the need to suspend three times rather than twice). The final difference is that dilations only provide infinitesimal symmetries, whereas the algebraic construction provided ones that integrate to circle actions. We should mention that, in spite of this fairly clear intuitive picture, the relationship between the two approaches has not been studied rigorously.





## Quasi-dilations

The dilation property has a straightforward interpretation in terms of mirror symmetry, which is the first point to be explained in this lecture. However, that interpretation also shows that in most cases where mirror symmetry applies, one can't hope to find an actual dilation. However, there is a slightly more general property (the quasi-dilation property) which leads to similar practical consequences for improved intersection numbers, and which does apply to many of the basic examples of local mirror symmetry.

*Acknowledgments.* The present lecture contains one result (Corollary 19.6) which is unpublished joint work of the author and Jake Solomon.

### Mirror symmetry motivation

Let  $M^\vee$  be a smooth affine algebraic variety of dimension  $n$ , which comes with a complex volume form  $\eta_{M^\vee}$ . In Lecture 12, we discussed the Hochschild cohomology (12.20) and its BV operator  $\Delta$  (12.21) (in the case  $M^\vee = \mathbb{C}^*$ , but the same formulae hold in general). In particular, given a vector field  $B \in H^0(M^\vee, TM^\vee) = HH^1(M^\vee, M^\vee)$ , we have

$$(19.1) \quad \Delta Z = 1 \Leftrightarrow \frac{d(i_Z \eta_{M^\vee})}{\eta_{M^\vee}} = 1 \Leftrightarrow d(i_Z \eta_{M^\vee}) = \eta_{M^\vee} \Leftrightarrow L_Z \eta_{M^\vee} = \eta_{M^\vee}.$$

Hence, the dilation condition is the mirror dual to having a vector field which expands  $\eta_{M^\vee}$  (this explains the terminology).

A little more generally, we can look for vector fields  $Z$  which expand some other volume form, say  $f\eta_{M^\vee}$  for an invertible function  $f$ . The generalization of (19.1), read backwards, is

$$(19.2) \quad L_Z(f\eta_{M^\vee}) = f\eta_{M^\vee} \Leftrightarrow d(i_Z f\eta_{M^\vee}) = f\eta_{M^\vee} \Leftrightarrow \Delta(fZ) = f.$$

In terms of the Lie bracket (12.11) on  $HH^{*+1}(M^\vee, M^\vee)$  which is part of the BV algebra structure, this can be written as

$$(19.3) \quad \Delta Z = 1 - f^{-1}[f, Z].$$

**EXAMPLE 19.1.** *Let's return to the original example of  $M^\vee = \mathbb{C}^*$ . There can be no solutions to (19.1), because  $\eta_{M^\vee} = dw/w$  is not exact even as a  $C^\infty$  differential form (the same will hold in all cases where the SYZ form of mirror symmetry [190] applies, since by definition  $\eta_{M^\vee}$  integrates nontrivially over any closed special Lagrangian submanifold).*

In contrast, there are many solutions of (19.2):

$$(19.4) \quad \begin{cases} f(w) = w^k & \text{for some nonzero integer } k, \\ Z = (\frac{1}{k}w + cw^{1-k})\partial_w & \text{where } c \in \mathbb{C} \text{ is an arbitrary constant.} \end{cases}$$

The choice  $c = 0$  may seem most natural, since then the vector field can be integrated to an action of  $G = \mathbb{C}^*$ . However, that action has no stationary points, hence there are no  $G$ -equivariant torsion sheaves.

### Definition

Let  $M$  be as in Lecture 18. A pair  $(B, h) \in SH^1(M) \times SH^0(M)$  is called a *quasi-dilation* if  $h$  is invertible with respect to the commutative ring structure on  $SH^0(M)$ , and the analogue of (19.2) holds:

$$(19.5) \quad \Delta(hB) = h.$$

Obviously, for  $h = 1$  this reduces to the notion of dilation.

LEMMA 19.2. *Suppose that  $(B, h)$  is a dilation. Let  $L$  be an object of  $Fuk(M)$  which is infinitesimally equivariant with respect to  $B$ . Then, the action of  $\Phi_{L,L}$  on  $HF^n(L, L) \cong H^n(L; \mathbb{K})$  is  $+1$ .*

This generalizes Lemma 18.1. The first part of proof is the same: the action of  $\Phi_{L,L}$  on  $HF^n(L, L)$  is multiplication with the image of

$$(19.6) \quad \Delta B = 1 - h^{-1}[h, B] \in SH^0(M)$$

under the cohomology level map induced by (17.24), which one can also think of as a composition

$$(19.7) \quad SH^*(M) \longrightarrow HH^*(Fuk(M), Fuk(M)) \longrightarrow HF^*(L, L).$$

The first part of (19.7) is the open-closed string map (12.24), which is compatible with both product and Lie bracket. The second part is the projection map, which exists for any  $A_\infty$ -category  $\mathcal{A}$  and has the form:

$$(19.8) \quad \begin{aligned} HH^*(\mathcal{A}, \mathcal{A}) &\longrightarrow H^*(\text{hom}_{\mathcal{A}}(X, X)), \\ [\beta] &\longmapsto [\beta^0]. \end{aligned}$$

Because  $L$  is infinitesimally  $B$ -equivariant, the image of  $B$  under (19.7) must vanish. The image of  $h$  is an invertible element of  $L$ , hence (if one assumes  $L$  to be connected, which one can do without loss of generality) a multiple of the identity. Lemma 19.2 is a consequence of this and the following elementary algebraic fact:

LEMMA 19.3. *Fix an object  $X$  of  $\mathcal{A}$ . Take two classes in  $HH^*(\mathcal{A}, \mathcal{A})$  whose images in  $H^*(\text{hom}_{\mathcal{A}}(X, X))/\mathbb{K} \cdot [e_X]$  vanish. Then, the Gerstenhaber bracket of those two classes maps to zero in  $H^*(\text{hom}_{\mathcal{A}}(X, X))$ .*

PROOF. Without loss of generality, we may assume that  $\mathcal{A}$  is minimal and strictly unital, and use the reduced version of the Hochschild cochain complex. Given any two Hochschild cocycles  $(\beta_2, \beta_1)$ , we have

$$(19.9) \quad [\beta_2, \beta_1]^0 = \beta_2^1(\beta_1^0) - (-1)^{(|\beta_2|-1)(|\beta_1|-1)}\beta_1^0(\beta_2^0) \in \text{hom}_{\mathcal{A}}(X, X).$$

By assumption,  $\beta_1^0$  and  $\beta_2^0$  are multiples of the identity endomorphism; and  $\beta_2^1(e_X) = \beta_1^1(e_X) = 0$ , because they are reduced Hochschild cochains. (Of course, there is a version of the argument which works throughout with cohomologically unital categories and the full Hochschild complex, but the formulae become more complicated in that case.)  $\square$

As a consequence, the refined intersection numbers associated to quasi-dilations will have the same properties as for dilations, in particular (18.8) and (18.9).

EXAMPLE 19.4. *Let's consider the mirror to Example 19.1, meaning  $M = T^*S^1$  with coefficient field  $\mathbb{K} = \mathbb{C}$ . A description its symplectic cohomology which is compatible with mirror symmetry was already given in (12.15), and this implies that  $M$  admits a quasi-dilation. For concreteness, let's consider only the case corresponding to  $k = 1$  in (19.4). Then, the invertible class  $h$  lies in  $SH^*(M)^{(1)}$ , which means that it corresponds to a simple Reeb orbit winding once around the circle. For the simplest choice of quasi-dilation, the class  $B$  lies in  $H^1(M; \mathbb{C}) \cong SH^1(M)^{(0)}$ , and in particular satisfies  $\Delta B = 0$ , while  $hB \in SH^1(M)^{(1)}$ .*

*Take the object of  $\text{Fuk}(M)$  corresponding to the zero-section  $L \subset M$  equipped with a flat line bundle with arbitrary holonomy  $a$ . As an instance of our previous general discussion (18.2), the map*

$$(19.10) \quad H^1(M; \mathbb{C}) \cong H^1(SH^1(M)^{(0)}) \rightarrow HF^1(L, L) \cong H^1(L; \mathbb{K})$$

*is nontrivial, hence  $L$  is never infinitesimally equivariant.*

*More generally, we can add any element  $c \in SH^1(M)^{(-1)} \cong \mathbb{C}$  to  $B$ , and that corresponds to using a general value of  $c$  in (19.4). The map  $SH^1(M)^{(-1)} \rightarrow HF^1(L, L)$  is multiplication by  $a^{-1}$ ; this is the same computation as in (12.33). The consequence is that if we choose a nonzero  $c$ , there will be a unique value of  $a$  for which  $L$  becomes infinitesimally equivariant.*

### Examples

The most basic example is the Milnor fibre of the  $(A_m)$  type singularity of complex dimension  $n = 2$ . As in Corollary 18.11 we consider this as the total space of a Lefschetz fibration  $\pi : M \rightarrow \mathbb{C}$ , but where now the fibre is a conic  $F = \mathbb{C}^* = T^*S^1$ . We will use the “fibrewise” version of symplectic cohomology  $SH_{\text{vert}}^*(M)$  as in (18.24), together with a suitably adapted open-closed string formalism. Namely, let  $\Delta \subset M$  be the Lefschetz thimble associated to an embedded path  $\gamma : [0, \infty) \rightarrow \mathbb{C}$ , where  $\gamma(0)$  is a critical value, and  $\text{re}(\gamma(t)) \rightarrow \infty$  as  $t \rightarrow \infty$ . Let  $V \subset F$  be the associated vanishing cycle, which is just the zero-section  $S^1 \subset T^*S^1$  (equipped with the *Spin* structure that bounds a *Spin* structure on the disc  $\Delta$ ). One can

construct a diagram of open-closed string maps of the form

$$(19.11) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H^*(M, \{\operatorname{re}(\pi) \gg 0\}; \mathbb{K}) & \longrightarrow & SH_{\operatorname{vert}}^*(M) & \longrightarrow & SH^*(F) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H_c^*(\Delta; \mathbb{K}) & \longrightarrow & H^*(\Delta; \mathbb{K}) & \longrightarrow & H^*(V; \mathbb{K}) \longrightarrow \cdots \end{array}$$

The top row is (18.24), and the bottom row is the standard topological diagram involving  $\Delta$  and its “boundary at infinity”  $V$ . The left hand  $\downarrow$  is the standard restriction map on cohomology, while the right hand  $\downarrow$  is  $[\psi_V^0] : SH^*(V) \rightarrow HF^*(V, V) \cong H^*(V; \mathbb{K})$ .

Now take a basis of vanishing paths  $\gamma_i$ , with associated  $\Delta_i$  and  $V_i$  (of course, all the  $V_i$  are the same up to isotopy). Combining the relevant diagrams (19.11) yields

$$(19.12) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H^*(M, \{\operatorname{re}(\pi) \gg 0\}; \mathbb{K}) & \longrightarrow & SH_{\operatorname{vert}}^*(M) & \longrightarrow & SH^*(F) \longrightarrow \cdots \\ & & \downarrow \cong & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \bigoplus_i H_c^*(\Delta_i; \mathbb{K}) & \longrightarrow & \bigoplus_i H^*(\Delta_i; \mathbb{K}) & \longrightarrow & \bigoplus_i H^*(V_i; \mathbb{K}) \longrightarrow \cdots \end{array}$$

where the left hand  $\downarrow$  is now an isomorphism. Diagram-chasing shows:

LEMMA 19.5. *A class  $B \in SH^1(F)$  can be lifted to  $SH_{\operatorname{vert}}^1(M)$  if and only if its image in  $H^1(V; \mathbb{K}) \cong HF^1(V, V)$  vanishes.  $\square$*

Now, take  $B$  to be a quasi-dilation which makes  $V$  infinitesimally equivariant. This exists by our previous discussion, and in fact there are infinitely many possible choices (corresponding to different  $k$  in (19.4); the constant  $c$  on the other hand is uniquely determined by our condition). Since the map  $SH_{\operatorname{vert}}^0(M) \rightarrow SH^0(F)$  is an isomorphism of rings, it follows that the lift of  $B$  to  $SH_{\operatorname{vert}}^1(M)$  is again a quasi-dilation.

COROLLARY 19.6 (Seidel-Solomon). *The  $(A_m)$  type Milnor fibre (18.26) in dimension  $n = 2$  admits a quasi-dilation.  $\square$*

By the same principle as Corollary 18.12, this implies:

COROLLARY 19.7. *The Milnor fibre of any isolated hypersurface singularity  $p(x, y) = 0$ , where  $p(x, y) = x_1^2 + x_2^2 + q(y_1, \dots, y_{m+1})$ , admits a quasi-dilation.  $\square$*

As already pointed out in [184, Section 7], there is a generalization of (18.24) to Lefschetz fibrations with base  $\mathbb{C}^*$ . This becomes particularly simple if we assume that the monodromy around 0 is trivial, in which case the long exact sequence takes on the form

$$(19.13) \quad \cdots \rightarrow H^*(M, \{|\pi| \ll 1\}; \mathbb{K}) \longrightarrow SH_{\operatorname{vert}}^*(M) \longrightarrow SH^*(F) \otimes H^*(S^1; \mathbb{K}) \rightarrow \cdots$$

The left-hand group is again generated by Lefschetz thimbles for vanishing paths running towards 0, and one can generalize (19.11) accordingly. Then, the same argument as before yields:

COROLLARY 19.8. *The manifold (11.7), and its generalizations from Example 11.4, admit quasi-dilations.*  $\square$



## Part 5

# Families of objects





LECTURE 20

## Basic notions



LECTURE 21

**Elliptic curves and mapping tori**



LECTURE 22

**Analytic and formal geometry**



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