Radial mating-of-trees and reversibility of whole plane SLE_κ for $\kappa \geq 8$

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Outline

1. Radial and whole plane SLE processes
2. Liouville Quantum Gravity surfaces
3. Chordal and radial mating-of-trees
4. Reversibility of whole plane SLE_κ via radial mating-of-trees

The radial SLE_κ processes

- Fix $\kappa > 0$, and let $\{B_t\}_{t\geq 0}$ be the standard Brownian motion.
- The radial SLE_κ curve η from 1 to 0 in the unit disk $\mathbb{D}$ can be characterized by

$$\frac{dg_t(z)}{dt} = g_t(z)\frac{U_t + g_t(z)}{U_t - g_t(z)}; \quad g_0(z) = z \tag{1}$$

  where $U_t = e^{i\sqrt{\kappa}B_t}$ and g_t is the conformal map from $\mathbb{D}\backslash\eta([0,t])$ to $\mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$.
- Can be extended to other domains via conformal mapping.

Whole plane SLE_κ processes

- Let $\kappa > 0$ and $(B_t)_{t\in\mathbb{R}}$ be a standard two-sided Brownian motion.
- The whole plane SLE_κ curve η from 0 to ∞ in $\mathbb{C}$ can be characterized by

$$\frac{dg_t(z)}{dt} = g_t(z)\frac{U_t + g_t(z)}{U_t - g_t(z)}; \quad g_{-\infty}(z) = z \tag{2}$$

where $U_t = e^{i\sqrt{\kappa}B_t}$ and g_t is the conformal map from $\mathbb{C}\backslash\eta((-\infty, t])$ to $\mathbb{C}\backslash\mathbb{D}$ with $g_t(\infty) = \infty$ and $g_t'(\infty) > 0$.

Radial and whole plane SLE_κ processes

- Whole plane SLE_κ can be viewed as bi-infinite version of radial SLE_κ: If η is a whole plane SLE_κ, then $1/g_s(\eta([s, s+t]))_{t\geq 0}$ is a radial SLE_κ.
- The Lowener pair of radial SLE_κ in $\mathbb{C}\backslash\varepsilon\mathbb{D}$ from ε to ∞ converges (in local uniform topology) to that of whole plane SLE_κ.

The reversibility of whole plane SLE_κ processes

- For $\kappa \in (0, 4]$, Zhan'10 proved that whole plane SLE_κ processes are reversible, where a description of the time reversal of radial SLE_κ was also given.
- For $\kappa \in (0, 8]$, Miller-Sheffield'13 have proved the reversibility of whole plane SLE_κ via SLE/GFF coupling.
- For $\kappa > 8$, reversibility does not hold for chordal SLE_κ, yet the reversibility of whole plane SLE_κ has been conjectured in Viklund-Wang'20 by studying the Loewner energy.

Liouville quantum gravity (LQG) surfaces

- Let $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and ϕ be a variant of the Gaussian free field on some domain D.
- Area measure: $\mu_\phi(d^2z) =$ " $e^{\gamma\phi(z)}d^2z$" and length measure: $\nu_\phi(dx) =$ " $e^{\frac{\gamma}{2}\phi(x)}dx$".
- Typical LQG surfaces (Duplantier-Miller-Sheffield '14): quantum wedges, quantum cones, quantum disks, quantum spheres with a weight parameter $W > 0$.
- Two canonical perspectives: scaling limits of random planar maps/Liouville conformal field theory.

LQG surfaces via LCFT

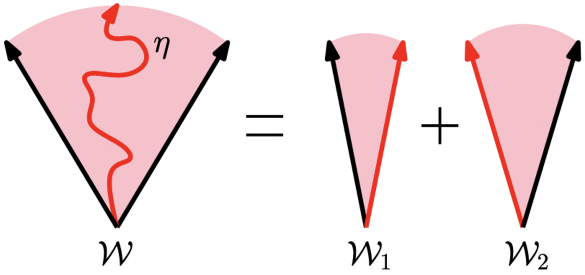
- Start with the GFF on $\mathbb{D}$ with average on $\partial\mathbb{D}$ being 0.
- Liouville field on $\mathbb{D}$: sample $(h, \mathbf{c})$ from $P_{\mathbb{D}} \times [e^{-Qc}dc]$, and set $\phi(z) = h(z) + \mathbf{c}$. Let $\mathrm{LF}_{\mathbb{D}}$ be the law of ϕ [David-Kupiainen-Rhodes-Vargas '14].
- Let $\beta_j, \alpha_k \in \mathbb{R}$, $z_k \in \mathbb{D}$ and $x_j \in \partial\mathbb{D}$. Liouville field with insertions: $\mathrm{LF}_{\mathbb{D}}^{(\alpha_k, z_k),(\beta_j, x_j)}(d\phi) = \prod_j e^{\frac{\beta_j}{2}\phi(x_j)} e^{\alpha_k \phi(z_k)} \mathrm{LF}_{\mathbb{D}}(d\phi)$.
- The quantum wedges/cones/disks/spheres can be viewed as *uniform embedding* of Liouville fields with two insertions. (Ang-Holden-Sun'21).

SLE/LQG couplings: $\kappa < 4$ case

Let $\kappa = \gamma^2 \in (0,4)$.

Theorem (Duplantier-Miller-Sheffield '14)

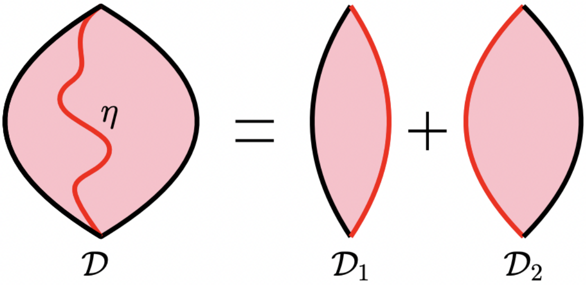
$$\mathcal{M}^{\text{wedge}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2) \\ = \mathcal{M}^{\text{wedge}}(W^L) \times \mathcal{M}^{\text{wedge}}(W^R). \tag{3}$$



SLE/LQG couplings: $\kappa < 4$ case

Theorem (Ang-Holden-Sun '20)

$$\mathcal{M}_2^{\mathrm{disk}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2) = c \int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{disk}}(W^L; \ell), \mathcal{M}_2^{\mathrm{disk}}(W^R; \ell) d\ell. \tag{4}$$

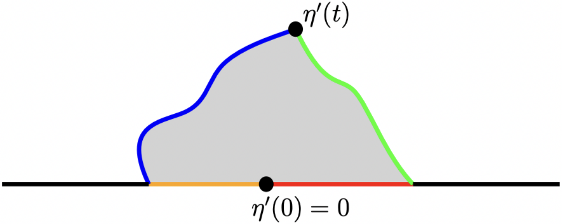


$\kappa \geq 8$ case: (Duplantier-Miller-Sheffield '14)

- Let $\gamma \in (0, \sqrt{2}]$ and $\kappa = 16/\gamma^2 \geq 8$.
- Let $(\mathbb{H}, \phi, 0, \infty)$ be a weight $2 - \frac{\gamma^2}{2}$ quantum wedge decorated with an independent space-filling chordal SLE_κ processes η' from 0 to ∞. Parameterize η' by quantum area.
- Let (X_t, Y_t) be the change in boundary length. Then $(X_t, Y_t)_{t\geq 0}$ evolves as planar Brownian motions with correlation $-\cos(4\pi/\kappa)$.
- $(X_t, Y_t)_{t\geq 0}$ a.s. determines the pair (ϕ, η').

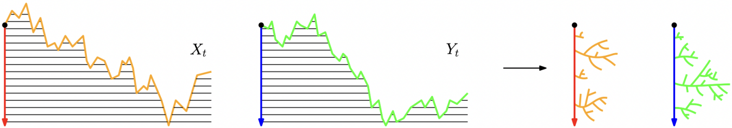
Chordal mating-of-trees

$\gamma \in (0, \sqrt{2}]$:



- X_t = blue − orange; Y_t = green − red;
- $-\inf_{0\le s\le t} X_s$ = orange; $-\inf_{0\le s\le t} Y_s$ = red.

Mating of Continuum Random Trees



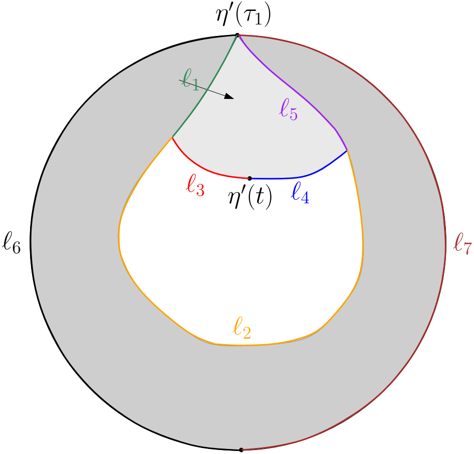
- Given the Brownian motion (X_t, Y_t), identifying points on the same horizontal segment gives a pair of correlated continuum random trees.
- The chordal mating-of-trees theorem suggests that there is a way to glue the two trees to obtain an SLE_κ decorated weight $2 - \frac{\gamma^2}{2}$ quantum wedge.

Radial Mating-of-trees

Theorem (Ang-Y.23')

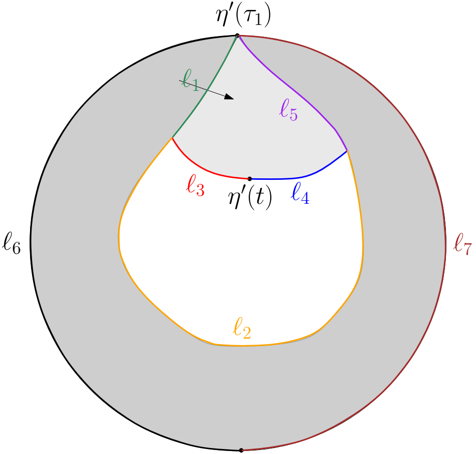
Let $\gamma \in (0, \sqrt{2}]$ *and* $\kappa = 16/\gamma^2$. *Consider a disk* $(\mathbb{D}, \phi, 0, 1)$ *sampled from* $\mathrm{LF}_{\mathbb{D}}^{(\frac{\gamma}{4}+\frac{2}{\gamma},0),(\frac{3\gamma}{2},1)}$ *conditioned on having unit boundary length. Sample an independent radial* SLE_κ *process* η' *in* $\mathbb{D}$ *from 1 to 0 parameterized by quantum area. Then the boundary length process* $(X_t, Y_t)_{t\geq 0}$ *associated with* (ϕ, η') *evolves as the correlated Brownian motion as in the chordal case and stopped at time* $\tau = \inf\{t > 0 : X_t + Y_t + 1 = 0\}$. *Moreover, the pair* (ϕ, η') *is measurable w.r.t.* $(X_t, Y_t)_{0\leq t\leq \tau}$.

Boundary length process



$X_{\tau_1} = -\ell_6$; $Y_{\tau_1} = \ell_1 + \ell_2 + \ell_5 - \ell_7$.
$X_t - X_{\tau_1} = \ell_4 - \ell_5$; $Y_t - Y_{\tau_1} = \ell_3 - \ell_1$.

Boundary length process



Boundary length of $(\mathbb{D}, \phi) = \ell_6 + \ell_7 = 1$;
Boundary length of $\eta'([t, \tau)) = \ell_2 + \ell_3 + \ell_4 = 1 + X_t + Y_t$.

Reverse SLE processes

The chordal reverse $\mathrm{SLE}_\kappa(\underline{\rho})$ process with force points $z_1, ..., z_n \in \bar{\mathbb{H}}$ is characterized by

$$\begin{aligned} d\tilde{W}_t &= \sum_{j=1}^{n} \mathrm{Re}(\frac{-\rho_j}{\tilde{g}_t(z_j) - \tilde{W}_t})dt + \sqrt{\kappa}dB_t; \\ d\tilde{g}_t(z_j) &= -\frac{2}{\tilde{g}_t(z_j) - \tilde{W}_t}dt, \quad \tilde{g}_0(z) = z \end{aligned} \tag{5}$$

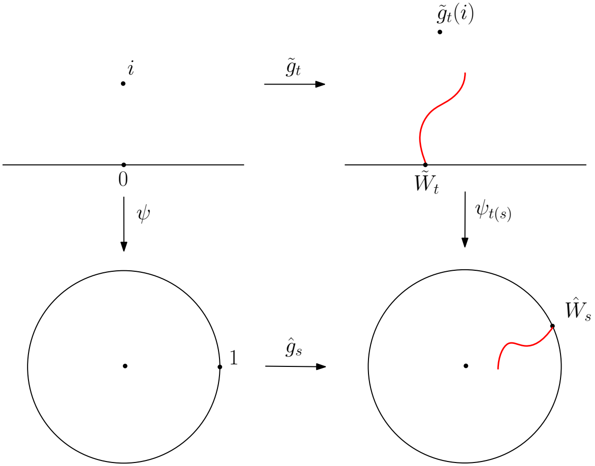
where $\tilde{g}_t$ maps $\mathbb{H}$ to $\mathbb{H}\backslash\eta([0,t])$ and fixes ∞.
The radial reverse SLE_κ in $\mathbb{D}$ is defined by

$$\frac{d\tilde{g}_t(z)}{dt} = -\tilde{g}_t(z)\frac{e^{i\sqrt{\kappa}B_t} + \tilde{g}_t(z)}{e^{i\sqrt{\kappa}B_t} - \tilde{g}_t(z)}; \quad g_0(z) = z, \tag{6}$$

where $\tilde{g}_t$ maps $\mathbb{D}$ to $\mathbb{D}\backslash\eta([0,t])$ and fixes 0.

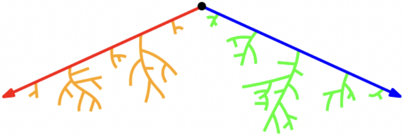
Coordinate change for reverse SLE processes

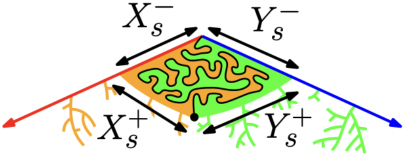
When conformally mapped to $\mathbb{D}$, reverse chordal $\mathrm{SLE}_\kappa(\kappa+6)$ agrees with reverse radial SLE_κ.



The mating-of-trees cells

(Duplantier-Miller-Sheffield '14) Consider the pairing of two Brownian CRTs. Run until the total quantum area is s. Denote the law of resulting surface by $\mathcal{P}_s$.



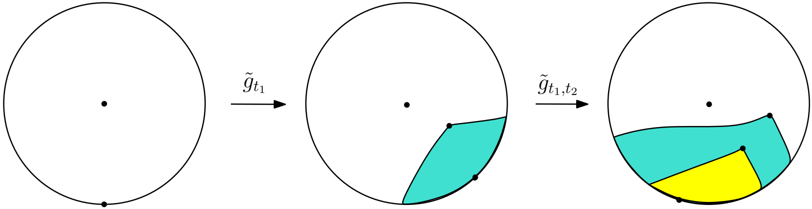


This also agrees with running an SLE_κ on weight $2 - \frac{\gamma^2}{2}$ quantum wedge and stop when the quantum area is s.

Proof Outline

Step 1. The capacity zipper.

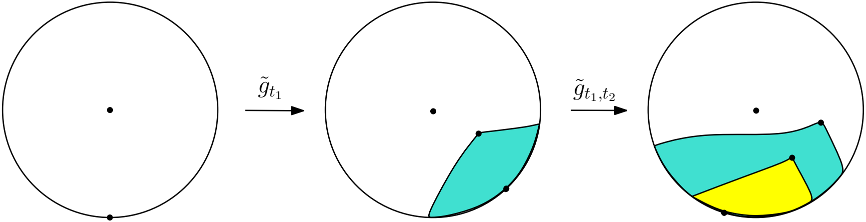
- Using the martingale in [Theorem 5.5, DMS14], one can construct a process $(\phi_t, \eta_t)_{t\geq 0}$, where (ϕ_t, η_t) has the law $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},\eta_t(0))} \times \mathrm{rrSLE}_\kappa^t$.
- $\phi_{t_1} = \phi_{t_2} \circ \tilde{g}_{t_1,t_2} + Q \log |\tilde{g}'_{t_1,t_2}|$.



Proof Outline

Step 2. The quantum zipper.

- For each fixed capacity time T, centered reverse SLE_κ stopped at time T agrees with forward SLE_κ stopped at time T.
- For each $A > 0$, let τ be the first time when the quantum area of $\eta_t([0, t])$ is A. Then (ϕ_τ, η_τ) (when centered) has law $\mathrm{LF}_{\mathbb{D}}^{(\frac{3\gamma}{2},1),(Q+\frac{\gamma}{4},0)} \times \text{radial } \mathrm{SLE}_\kappa^\tau$.
- Can be understood as the conformal welding of partially mated CRT with LF.



Proof Outline

Step 3. The Brownian motions.

- By comparing to the chordal case via the coordinate change, before wrapping around, the parts discovered by the forward radial SLE_κ curve is a quantum cell.
- Then the boundary length process (X_t, Y_t) becomes Brownian motion weighted by $(1 + X_t + Y_t)$ before wrapping around time.
- Weighting by field circle average near 0 produces $\mathrm{LF}_{\mathbb{D}}^{(\frac{3\gamma}{2},1),(Q-\frac{\gamma}{4},0)} \times$ radial SLE_κ, and there is no weighting on the BM.
- Using self-similarity to recursively extend to all time $t > 0$.

Mating-of-trees on weight $\frac{\gamma^2}{2}$ quantum sphere

- Using GFF tail estimates, disks from $\mathrm{LF}_{\mathbb{D}}^{(\frac{3\gamma}{2},1),(Q-\frac{\gamma}{4},0)}$ conditioned on having quantum area 1 and boundary length ε converges weakly to weight $\frac{\gamma^2}{2}$ quantum sphere with area 1 as $\varepsilon \to 0$.
- Under suitable embedding, one deduce that for a weight $\frac{\gamma^2}{2}$ quantum sphere decorated with whole plane SLE_κ, the boundary length $(X_t + Y_t)$ evolves as a Brownian excursion from 0 to 0, with duration being the quantum area of the sphere.
- Then there is a matrix Λ determined by κ, such that $(X_t, Y_t) = \Lambda(W^1, W^2)$, where W^1 is a Brownian excursion and W^2 is a Brownian motion independent of W^1.

Reversibility of whole plane SLE_κ

- By considering the chordal mating-of-trees on weight $4 - \gamma^2$ quantum sphere decorated with SLE_κ loop, the mating-of-trees cells are reversible.
- The pair (ϕ, η') can be decomposed into a welding of countably many cells, each of which being reversible.
- This allows us to deduce the reversibility of the whole plane SLE_κ via the reversibility of the boundary length process (X_t, Y_t).

Thanks for listening!