Conformal welding of LQG surfaces and multiple SLE

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Outline

1. LQG, LCFT and SLE
2. Multiple SLE and partition functions
3. Multiple SLE and conformal welding: $\kappa \in (0, 4)$ case
   - Relation with $\kappa \in (0, 4)$ multiple SLE
   - Relation with imaginary geometry
   - Relation with SLE Green's function
4. Multiple SLE and conformal welding: $\kappa \in (4, 8)$ case

The Gaussian Free Field

- The GFF on the upper half plane $\mathbb{H}$: The Gaussian random field on $\mathbb{H}$ with mean 0 and covariance

$$\mathrm{Cov}(h(z), h(w)) = G_{\mathbb{H}}(z, w)$$

where $G_{\mathbb{H}}(z, w)$ is the Green's function

$$G_{\mathbb{H}}(z, w) = -\log|z - w| - \log|z - \bar{w}| + 2\log|z|_+ + 2\log|w|_+$$

with $|z|_+ = \max\{|z|, 1\}$.
- h is a well-defined generalized function.

Liouville quantum gravity (LQG)

- Let $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and ϕ be a variant of the GFF, e.g., $\phi = h + f$ where f is a continuous function.
- Area measure: $\mu_\phi(d^2z) =$ " $e^{\gamma\phi(z)}d^2z$" $= \lim_{\varepsilon\to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma\phi_\varepsilon(z)}d^2z$.
- Length measure: $\nu_\phi(dx) =$ " $e^{\frac{\gamma}{2}\phi(x)}dx$" $= \lim_{\varepsilon\to 0} \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}\phi_\varepsilon(x)}$" dx.

Liouville conformal field theory on $\mathbb{H}$

- Start with the GFF h on $\mathbb{H}$.
- Sample $(h, \mathbf{c})$ from $P_{\mathbb{H}} \times [e^{-Qc}dc]$, and set $\phi(z) = h(z) - 2Q \log |z|_+ + \mathbf{c}$. Let $\mathrm{LF}_{\mathbb{H}}$ be the law of ϕ [David-Kupiainen-Rhodes-Vargas '14].
- Let $\beta_j \in \mathbb{R}$ and $x_j \in \partial\mathbb{H}$. Liouville field with boundary insertions: $\mathrm{LF}_{\mathbb{H}}^{(\beta_j, x_j)}(d\phi) = \prod_j e^{\frac{\beta_j}{2}\phi(x_j)} \mathrm{LF}_{\mathbb{H}}(d\phi)$.

LQG surfaces

- Let $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.
- Say $(D_1, \phi_1) \sim_\gamma (D_2, \phi_2)$, if there exists $f : D_1 \to D_2$ conformal with $\phi_2 = \phi_1 \circ f^{-1} + Q \log |(f^{-1})'|$.
- A quantum surface is an equivalence class over the relation $\sim_\gamma$.

Quantum disks

- Let $W > 0$ be the *weight* parameter. Let $\beta = \gamma + \frac{2-W}{\gamma}$.
- Weight W (thick) quantum disks: $W > \frac{\gamma^2}{2}$, and near each marked point z_0 the field looks like $h - \beta \log|\cdot - z_0|$. Can be viewed as *uniform embedding of* $\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,\infty)}$ (Ang-Holden-Sun'21).
- Thick-thin duality: Weight $W \in (0, \frac{\gamma^2}{2})$ quantum disk is a Poissonian chain of weight $\gamma^2 - W$ quantum disks.
- Special weight $W = 2$: the two marked points can be resampled from the boundary length measure, which defines $\mathrm{QD}_{0,2}$.
- $\mathrm{QD}_{0,n}$: starting from $\mathrm{QD}_{0,2}$ and sample $n - 2$ marked points from the boundary length measure.

Quantum triangles

- Let $W_1, W_2, W_3 > 0$ be the *weight* parameters, and $\beta_j = \gamma + \frac{2-W_j}{\gamma}$.
- Weight (W_1, W_2, W_3) (thick) quantum triangles: $(\mathbb{H}, \phi, 0, \infty, 1)/\sim_\gamma$ with ϕ sampled from $\frac{1}{(Q-\beta_1)(Q-\beta_2)(Q-\beta_3)} \mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,\infty),(\beta_3,1)}$.
- Thick-thin duality: when $W_1 < \frac{\gamma^2}{2}$, a weight (W_1, W_2, W_3) quantum triangle is the concatenation of a weight $(\gamma^2 - W_1, W_2, W_3)$ quantum triangle (core) with a weight W_1 quantum disk. Similar extension to the case where one or more $W_j < \frac{\gamma^2}{2}$.
- Special limiting argument to define $\frac{\gamma^2}{2}$ weights.

The SLE_κ processes

- Fix $\kappa > 0$, and let $\{B_t\}_{t \geq 0}$ be the standard Brownian motion.
- The SLE_κ curve η from 0 to ∞ on the upper half plane $\mathbb{H}$ can be characterized by

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - W_t}; \quad g_0(z) = z \tag{1}$$

where $W_t = \sqrt{\kappa} B_t$ and g_t is the conformal map from $\mathbb{H} \backslash \eta([0, t])$ to $\mathbb{H}$ with $\lim_{|z| \to \infty} |g_t(z) - z| = 0$.
- The definition is extended to other domains via *conformal invariance.*

$\mathrm{SLE}_\kappa(\underline{\rho})$ processes

- Fix the *weights* $\rho^{0,L}, ..., \rho^{k,L}; \rho^{0,R}, ..., \rho^{\ell,R} \in \mathbb{R}$ and the *force points* $x^{k,L} < ... < x^{0,L} = 0^- < 0^+ = x^{0,R} < ... < x^{\ell,R}$.
- The $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η from 0 to ∞ on the upper half plane $\mathbb{H}$ with force points $\underline{x}$ can be characterized by the Loewner equation (1) with

$$dW_t = \sum_{q\in\{L,R\}} \sum_i \frac{\rho^{i,q}}{W_t - g_t(x^{i,q})} dt + \sqrt{\kappa} dB_t \tag{2}$$

$\widetilde{\mathrm{SLE}}_\kappa(\rho_-;\rho_+,\rho_1;\alpha)$ processes

- Let η be an $\mathrm{SLE}_\kappa(\rho_-;\rho_+,\rho_1)$ process with force points $0^-;0^+,1$.
- Let D_η be the connected component of $\mathbb{H}\backslash\eta$ containing 1, and σ_η,ξ_η be the first and the last point on ∂D_η traced by η.
- Consider the conformal map $\psi_\eta : D_\eta \to \mathbb{H}$ sending $(\sigma_\eta, 1, \xi_\eta)$ to $(0,1,\infty)$.
- Define $\widetilde{\mathrm{SLE}}_\kappa(\rho_-;\rho_+,\rho_1;\alpha)$ by

$$\frac{d\,\widetilde{\mathrm{SLE}}_\kappa(\rho_-;\rho_+,\rho_1;\alpha)}{d\,\mathrm{SLE}_\kappa(\rho_-;\rho_+,\rho_1)}(\eta) = |\psi_\eta'(1)|^\alpha. \tag{3}$$

- Such processes have close relation with hypergeometric SLE processes and time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes (Y.'22).

SLE pure partition function

Let $b = \frac{6-\kappa}{2\kappa}$ be the boundary scaling exponent, and α be a link pattern. The pure partition function $\mathcal{Z}_\alpha$ satisfies the following:

- PDE: $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2b}{(x_j-x_i)^2}\right)\right]\mathcal{Z}_\alpha(\mathbb{H}; x_1, \ldots, x_{2N}) = 0$;
- Conformal covariance: for $f : \mathbb{H} \to \mathbb{H}$ conformal, $\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = \prod f'(x_i)^b \mathcal{Z}_\alpha(f(x_1), \ldots, f(x_{2N}))$;
- Asymptotic: $\lim_{x_j, x_{j+1}\to\xi}(x_{j+1} - x_j)^{2b}\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = \mathcal{Z}_{\alpha\setminus\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N})$ if $\{j, j+1\} \in \alpha$ and else 0.

Existence and uniqueness

- Uniqueness (Flores-Kleban '15): For $\kappa \in (0, 8)$, functions satisfying the three properties are essentially unique.
- Exact solution for $N = 1, 2$.
- Existence: $\kappa \in (0, 8) \backslash \mathbb{Q}$ (Kytölä-Peltola'16): Coulumb gas techniques;
  $\kappa \in (0, 4]$ (Peltola-Wu'19; Beffara-Peltola-Wu'21): global multiple SLE;
  $\kappa \in (0, 6]$ (Wu'20): hypergeometric SLE.

Characterizations of multiple SLE_κ

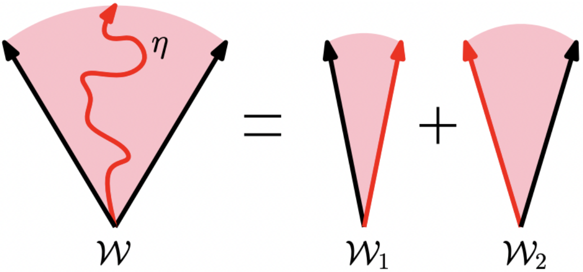
- Local construction via Loewner flow (e.g. Dubédat'07, Graham'07, Kytölä-Peltola'16);
- Global construction by weighting the law of N independent SLE_κ curves for $\kappa \in (0, 4]$ (e.g. Kozdron-Lawler'06, Peltola-Wu'19);
- Recursive construction by weighting the law of SLE_κ by pure partition functions for $\kappa \in (0, 6]$ or $\kappa \in (6, 8)$, $N = 2$ (Wu'20);
- Resampling property: given $N - 1$ curves, the conditional law of the remaining curve is the SLE_κ. ($\kappa \in (0, 8)$ for $N = 2$ (Miller-Werner'18) and $\kappa \in (0, 4]$ for $N \geq 3$ (Beffara-Peltola-Wu'18)).

Conformal welding of quantum wedges

Let $\kappa = \gamma^2 \in (0, 4)$.

Theorem (Duplantier-Miller-Sheffield '14)

$$\mathcal{M}^{\text{wedge}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2) \\ = \mathcal{M}^{\text{wedge}}(W^L) \times \mathcal{M}^{\text{wedge}}(W^R). \tag{4}$$

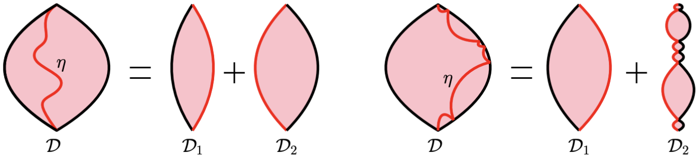


Conformal welding of quantum disks

Theorem (Ang-Holden-Sun '20)

Let $\kappa = \gamma^2 \in (0, 4)$.

$$\mathcal{M}_2^{\mathrm{disk}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2)$$
$$= c \int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{disk}}(W^L; \ell), \mathcal{M}_2^{\mathrm{disk}}(W^R; \ell) d\ell. \quad (5)$$

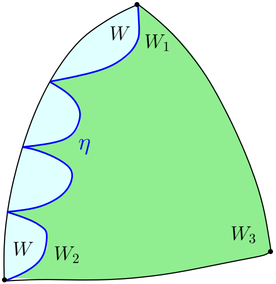


Conformal welding of quantum triangles

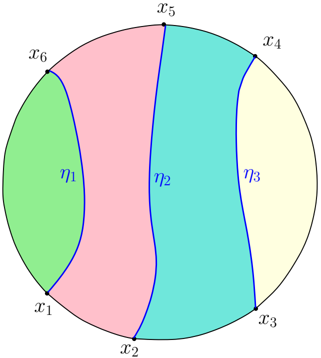
Theorem (Ang-Sun-Y.' 22)

$$\mathrm{QT}(W+W_1, W+W_2, W_3) \otimes \widetilde{\mathrm{SLE}}_\kappa(W-2; W_2-2, W_1-W_2; \alpha) = c\int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{disk}}(W;\ell), \mathrm{QT}(W_1, W_2, W_3;\ell))d\ell. \quad (6)$$

where $\alpha = \frac{W_3+W_2-W_1-2}{4\kappa}(W_3+W_1+2-W_2-\kappa)$.



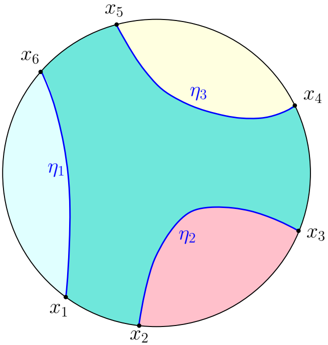
Conformal welding of LQG disks by link pattern



$\alpha = \{\{1,6\},\{2,5\},\{3,4\}\}$, with $\mathrm{Weld}_\alpha(QD)$ written as

$$\int_{\mathbb{R}_+^3} \mathrm{Weld}(\mathrm{QD}_{0,2}(\ell_1), \mathrm{QD}_{0,4}(\ell_1,\ell_2), \mathrm{QD}_{0,4}(\ell_2,\ell_3), \mathrm{QD}_{0,2}(\ell_3))\, d\ell_1\, d\ell_2\, d\ell_3.$$

Conformal welding of LQG disks by link pattern



$\alpha = \{\{1,6\},\{2,3\},\{4,5\}\}$, with $\mathrm{Weld}_\alpha(QD)$ written as

$$\int_{\mathbb{R}_+^3} \mathrm{Weld}(\mathrm{QD}_{0,2}(\ell_1), \mathrm{QD}_{0,2}(\ell_2), \mathrm{QD}_{0,2}(\ell_3), \mathrm{QD}_{0,6}(\ell_1,\ell_2,\ell_3))d\ell_1\, d\ell_2\, d\ell_3.$$

Conformal welding of LQG disks by link pattern

Theorem (Ang-Sun-Y. '23+)

Let $\gamma \in (0,2)$, $\kappa = \gamma^2$ and $\beta = \gamma - \frac{2}{\gamma}$. Let $N \geq 2$ and $\alpha \in \mathrm{LP}_N$ be a link pattern. Then there exists a constant $c \in (0,\infty)$ such that

$$\int_{0<y_1<\ldots<y_{2N-3}<1} \Big[\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,1),(\beta,\infty),(\beta,y_1),\ldots,(\beta,y_{2N-3})} \times \mathrm{mSLE}_{\kappa,\alpha}(\mathbb{H},0,y_1,\ldots,y_{2N-3},1,\infty)\Big] dy_1 \ldots dy_{2N-3} = c\,\mathrm{Weld}_\alpha(\mathrm{QD}) \quad (7)$$

where the left hand side is understood as the law of a curve-decorated quantum surface.

Random modulus = partition function

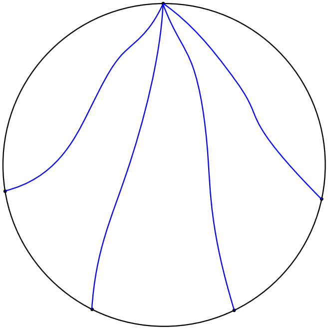
- The above theorem implies that the random location of the marked points under conformal welding is encoded by multiple SLE pure partition function.
- This implication also works for other settings.

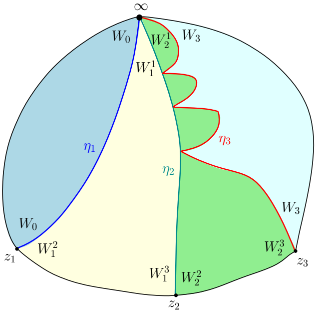
Imaginary Geometry flow lines

- Let h be a GFF on $\mathbb{H}$ with piecewise boundary conditions and $\kappa \in (0, 4)$.
- (Miller-Sheffield'12) Heuristically, $\eta(t)$ is a flow line of angle θ if

$$\eta'(t) = e^{i(\frac{h(\eta(t))}{\chi} + \theta)} \text{ for } t > 0, \text{ where } \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}. \tag{8}$$

- Such η are $\mathrm{SLE}_\kappa(\underline{\rho})$ processes.





Conformal welding and Imaginary geometry

Theorem (Ang-Sun-Y '23+)

Let $W_0, W_n > 0$, and $W_1^1, W_1^2, W_1^3, ..., W_{n-1}^1, W_{n-1}^2, W_{n-1}^3 > 0$, such that for each $1 \leq j \leq n-1$, $W_j^1 + 2 = W_j^2 + W_j^3$. Also assume that for every $0 \leq i < j \leq n$, $W_i^3 + \sum_{i<k<j} W_k^1 + W_j^2 > \frac{\gamma^2}{2}$. The conformal welding of $\mathcal{M}_2^{\mathrm{disk}}(W_0)$, $\mathrm{QT}(W_1^1, W_1^2, W_1^3)$,...,$\mathrm{QT}(W_{n-1}^1, W_{n-1}^2, W_{n-1}^3)$,$\mathcal{M}_2^{\mathrm{disk}}(W_n)$ in the previous picture is given by

$$c \cdot \int_{0<x_2<...<x_{n-1}<1} \prod_{1\leq i<j\leq n} (x_j - x_i)^{\frac{\rho_i\rho_j}{2\kappa}} \mathrm{LF}_{\mathbb{H}}^{(\beta_j,x_j)_{1\leq j\leq n},(\beta_\infty,\infty)}(d\phi) \times \mathrm{IG}_{\underline{x},\underline{\lambda},\underline{\theta}}(d\eta_1...d\eta_n)\, dx_2...dx_{n-1} \tag{9}$$

where $x_1 = 0$, $x_n = 1$, and $\mathrm{IG}_{\underline{x},\underline{\lambda},\underline{\theta}}$ denote the flow lines of the Imaginary Geometry field with marked points $x_0, ..., x_{N-1}$ with boundary values and angles determined by $\underline{W}$.

Random modulus = partition function

- The value $\prod_{1 \le i < j \le n} (x_j - x_i)^{\frac{\rho_i \rho_j}{2\kappa}}$ can be viewed as the partition function of the Imaginary Geometry field (Dubédat).

SLE boundary Green's function

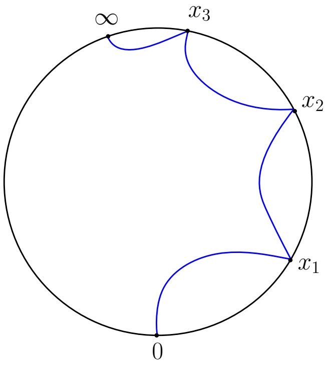
- Let $b_2 = \frac{8}{\kappa} - 1$, $x_j \in \mathbb{R}\backslash\{0\}$, and η be an SLE_κ curve. The n-point SLE boundary Green's function is defined by the limit

$$G(x_1, ..., x_n) = \lim_{r_1,...,r_n \to 0^+} r_1^{-b_2} ... r_n^{-b_2} \, \mathbb{P}\big(\mathrm{dist}(\eta, x_j) < r_j, \ 1 \leq j \leq n\big) \tag{10}$$

- The existence of the limit is proved by [Lawler'15] for $n = 1$ or $n = 2$ with $x_2 > x_1 > 0$ and [Fakhry-Zhan'22] for general case.

SLE boundary Green's function

- For $0 < x_1 < \ldots < x_n$, one can recursively define a measure $M(x_1, \ldots, x_n)$ on $n+1$ curves whose size is $G(x_1, \ldots, x_n)$, and can be interpreted as SLE_κ conditioned on hitting $x_1, \ldots, x_n$.



Relation with SLE boundary Green's function

Theorem (Ang-Sun-Y.'23+)

Let $n \geq 2$, $x_1 = 1$, $\beta = \gamma - \frac{2}{\gamma}$ and $\beta_2 = \gamma - \frac{4}{\gamma}$. Consider the conformal welding of QD induced by the previous picture. Then the output curve-decorated surface we get can be embedded as ($x_1 = 1$)

$$c \cdot \int_{0<x_1<\ldots<x_n} \left[\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,\infty),(\beta_2,x_1),\ldots,(\beta_2,x_n)} \times M(x_1, \ldots, x_n) \right] dx_2 \ldots dx_n.$$

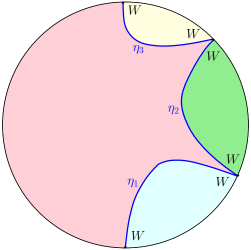
Following [Zhan'21] on 2-point boundary Green's function for $\mathrm{SLE}_\kappa(\underline{\rho})$, a similar result also holds for $n = 2$ and $\mathrm{SLE}_\kappa(\rho)$ process.

Relation with SLE boundary Green's function

Theorem (Ang-Sun-Y.'23+)

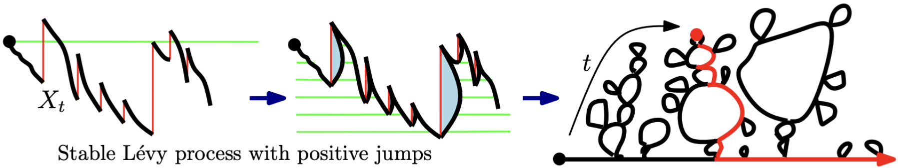
Let $\rho > -2$ *and* $W = \rho + 2$. *Let* $\beta_\rho = \gamma - \frac{2+\rho}{\gamma}$ *and* $\beta_{2,\rho} = \gamma - \frac{2+2\rho}{\gamma}$. *Consider the conformal welding below. Then for some constant* $c \in (0, \infty)$, *the output curve-decorated quantum surface can be embedded as* $(\mathbb{H}, \phi, 0, 1, x, \infty, \eta_1, \eta_2, \eta_3)$ *where* $(\phi, x, \eta_1, \eta_2, \eta_3)$ *has law*

$$c \int_1^\infty \mathrm{LF}_{\mathbb{H}}^{(\beta_\rho,0),(\beta_{2,\rho},1),(\beta_{2,\rho},x),(\beta_\rho,\infty)}(d\phi) \times M(\rho; 1, x)(d\eta_1 d\eta_2 d\eta_3) dx.$$



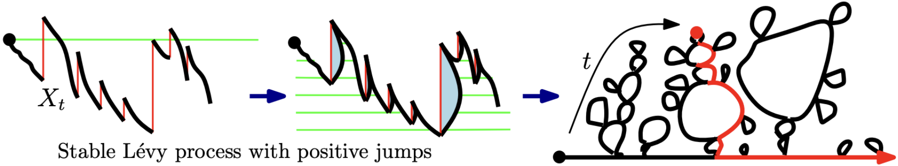
Forested line

- Sample a stable Lévy process $(X_t)_{t>0}$ of index $\frac{\kappa}{4} = \frac{4}{\gamma^2}$ with upward jumps.
- Add a curve for each jump and identify the points on the same green horizontal line.
- For each blue disk, assign a sample from QD with boundary length according to the jump.



Forested line

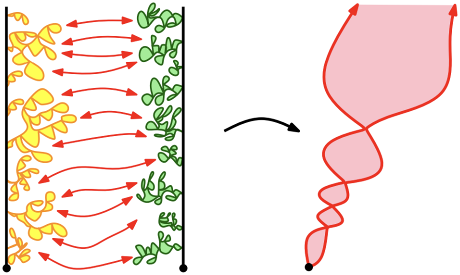
- Points on the horizontal line: record minima of $(X_t)_{t>0}$. Parameterized by quantum length.
- Lévy tree of disks: quantum natural parametrization, i.e., $Y_t = \inf\{s > 0 : X_s \le -t\}$.



Conformal welding of forested lines

Theorem (Duplantier-Miller-Sheffield'14)

Let $\gamma = 4/\sqrt{\kappa}$. If we draw an independent $\mathrm{SLE}_\kappa(\kappa/2-4;\kappa/2-4)$ process on a weight $2-\gamma^2/2$ quantum wedge, then we obtain the conformal welding of two independent forested lines.

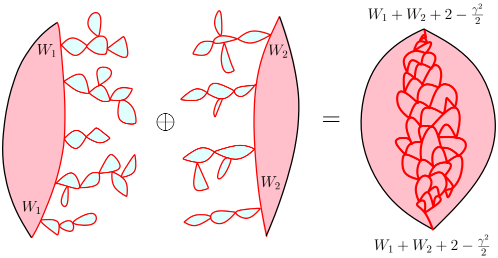


Conformal welding of forested quantum disks

Theorem (Ang-Holden-Sun-Y.'23+)

Let $W_1, W_2 > 0$ and $\rho_j = \frac{4}{\gamma^2}(2+\gamma^2 - W_j)$ for $j = 1, 2$.

$$\mathcal{M}_2^{\mathrm{disk}}(W_1 + W_2 + 2 - \frac{\gamma^2}{2}) \otimes \mathrm{SLE}_\kappa(\rho_1; \rho_2)$$
$$= c\int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{f.d.}}(W_1;\ell), \mathcal{M}_2^{\mathrm{f.d.}}(W_2;\ell))d\ell.$$



Conformal welding of forested quantum disks

- The weight $\gamma^2 - 2$ forested quantum disk $\mathcal{M}_2^{\mathrm{f.d.}}(\gamma^2 - 2)$ shares similar property as $\mathcal{M}_2^{\mathrm{disk}}(2)$ in $\kappa < 4$ regime. This allows us to define $\mathrm{GQD}_{0,n}$ analogously.
- We can consider the similar conformal welding problem of GQD according to a given link pattern.

Conformal welding of forested disks by link pattern

Theorem (Ang-Holden-Sun-Y. '23+)

Let $\gamma \in (\sqrt{2}, 2)$, $\kappa = 16/\gamma^2$ and $\beta = \frac{4}{\gamma} - \frac{\gamma}{2}$. Let $N \geq 2$ and $\alpha \in \mathrm{LP}_N$ be a link pattern. Then there exists a constant $c \in (0, \infty)$ such that

$$\int_{0<y_1<\ldots<y_{2N-3}<1} \Big[\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,1),(\beta,\infty),(\beta,y_1),\ldots,(\beta,y_{2N-3})} \times$$

$$\mathrm{mSLE}_{\kappa,\alpha}(\mathbb{H}, 0, y_1, \ldots, y_{2N-3}, 1, \infty) \Big] dy_1 \ldots dy_{2N-3} = c\,\mathrm{Weld}_\alpha(\mathrm{GQD})|_E$$

where E is the event that the welding output is simply connected, and the left hand side is understood as the law of a curve-decorated quantum surface.

Conformal welding of forested disks by link pattern

- The measure $\mathrm{mSLE}_{\kappa,\alpha}$ is constructed in an iterative way as [Wu'20], and for $\kappa \in (6, 8)$ we are able to show that when weighting by the partition function, the measure we get is still finite.
- Following the arguments from [Peltola'19], one can show that the partition function for $\mathrm{mSLE}_{\kappa,\alpha}$ when $\kappa \in (6, 8)$ is conformally covariant and solves the PDE.
- The resampling properties also uniquely characterizes the measure $\mathrm{mSLE}_{\kappa,\alpha}$ for $\kappa \in (4, 8)$.

Thanks for listening!