The Baxter permutation and Liouville quantum gravity

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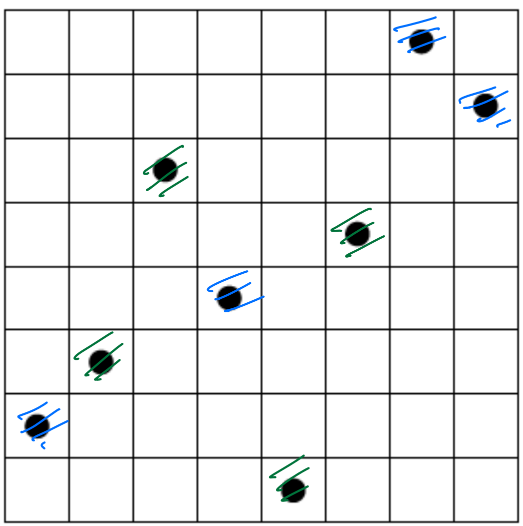
Discrete level objects

Baxter permutation ($\mathcal{P}$)

A permutation $\sigma \in S_n$ is *Baxter* if it is not possible to find $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$.

Bipolar orientations ($\mathcal{O}$)

A directed planar graph is called a *bipolar orientation* if it is acyclic and has a single source and a single sink.



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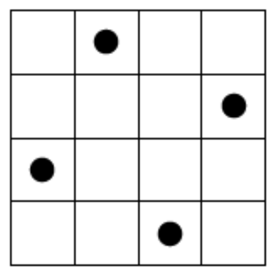

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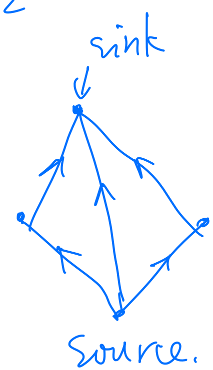


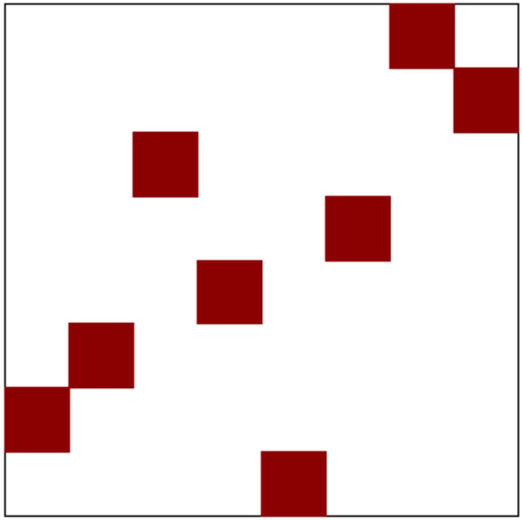
Permutons

(x, y)

Definition

A permuton is a measure on $[0,1]^2$, whose marginals are the uniform distributions.

A permutation σ naturally induces a measure and hence a permuton μ_σ.



total mass $= 1$

Discrete level objects

Tandem walks ($\mathcal{W}$)

A tandem walk is the process $W = (X_k, Y_k)_{1 \le k \le n}$ on $\mathbb{Z}^2_{\ge 0}$ from the y-axis to the x-axis with increments in $\{(1,-1)\} \cup \{(-i,j) : i,j \ge 0\}$.

Coalescent-walk process ($\mathcal{C}$)

Given a walk $W = (X_k, Y_k)_{1 \le k \le n}$ on $\mathbb{Z}^2$, the coalescent-walk process associated with W is a family of walks $\{Z^{(k)}\}_{1 \le k \le n}$ defined recursively by setting $Z^{(k)}_k = 0$ and for $\ell \ge k$:

- $Z^{(k)}_{\ell+1} - Z^{(k)}_{\ell} = Y_{\ell+1} - Y_\ell$ if $Z^{(k)}_\ell \ge 0$; ← Discrete ODE
- $Z^{(k)}_{\ell+1} - Z^{(k)}_{\ell} = X_\ell - X_{\ell+1}$ if $Z^{(k)}_\ell < \min\{0, X_{\ell+1} - X_\ell\}$;
- $Z^{(k)}_{\ell+1} = Y_{\ell+1} - Y_\ell$ if $X_{\ell+1} - X_\ell \le Z^{(k)}_\ell < 0$.

An example of coalescent-walk process

The definition above induces a mapping WC.

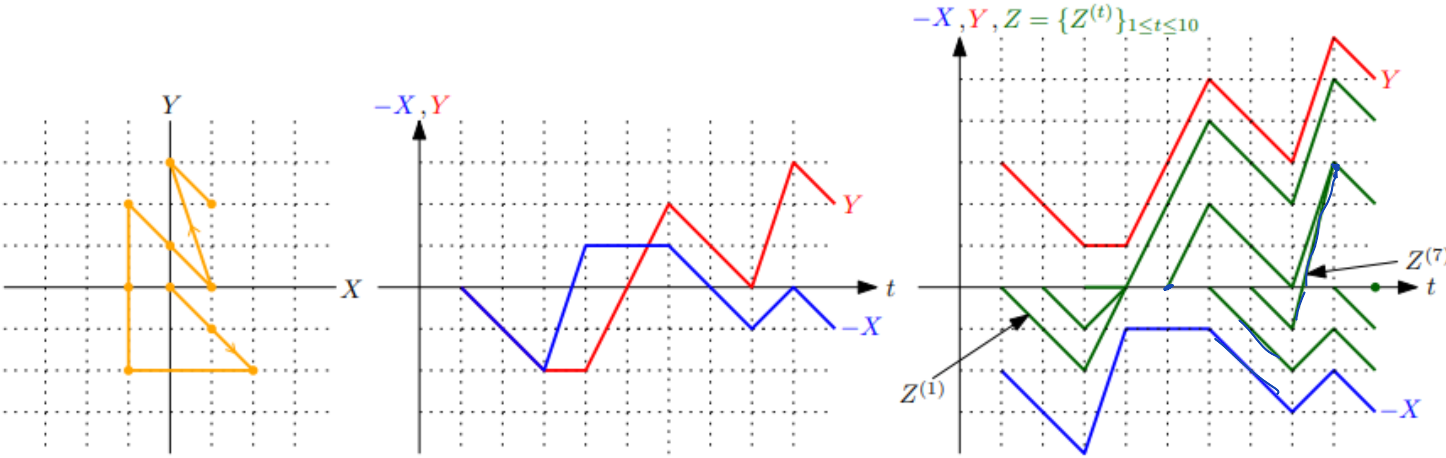


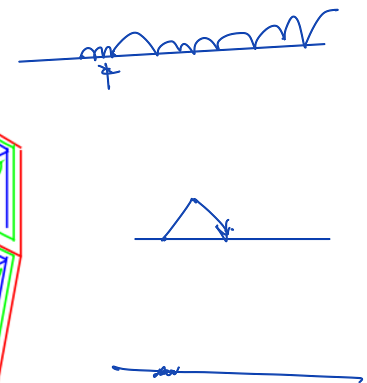
Figure: A coalescent-walk process induced by the walk $W = (X, Y)$. Figure from [Borga-Maazoun'20].

Bipolar orientation and space-filling curve

- For a bipolar orientation m, one can cut every incoming edge but the rightmost one at each vertex, yielding a tree $T(m)$ with root at the source of m.
- One can do the same for the map with *reverse orientation*, which gives a tree $T(m^{**})$ with root at the sink of m.

$SLE_\kappa(\rho)$.

- The two trees gives a space-filling curve η_m in the graph.



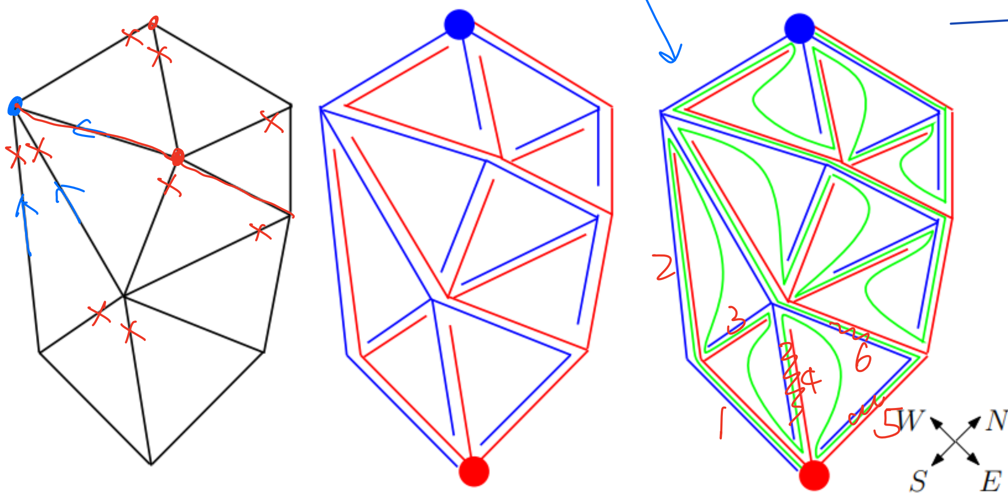


Figure: [Gwynne-Holden-Sun'16].

Bijections between discrete objects

Let e_k be the k-th edge visited by η_m, X_k be the distance between e_k and the source in $T(m)$, and Y_k be the distance e_k and the sink in $T(m^{**})$.

Mapping OW from [Kenyon-Miller-Sheffield-Wilson'19]

The walk $W = (X_k, Y_k)_{1 \le k \le n}$ constructed from m induces a bijection between bipolar orientations and tandem walks.

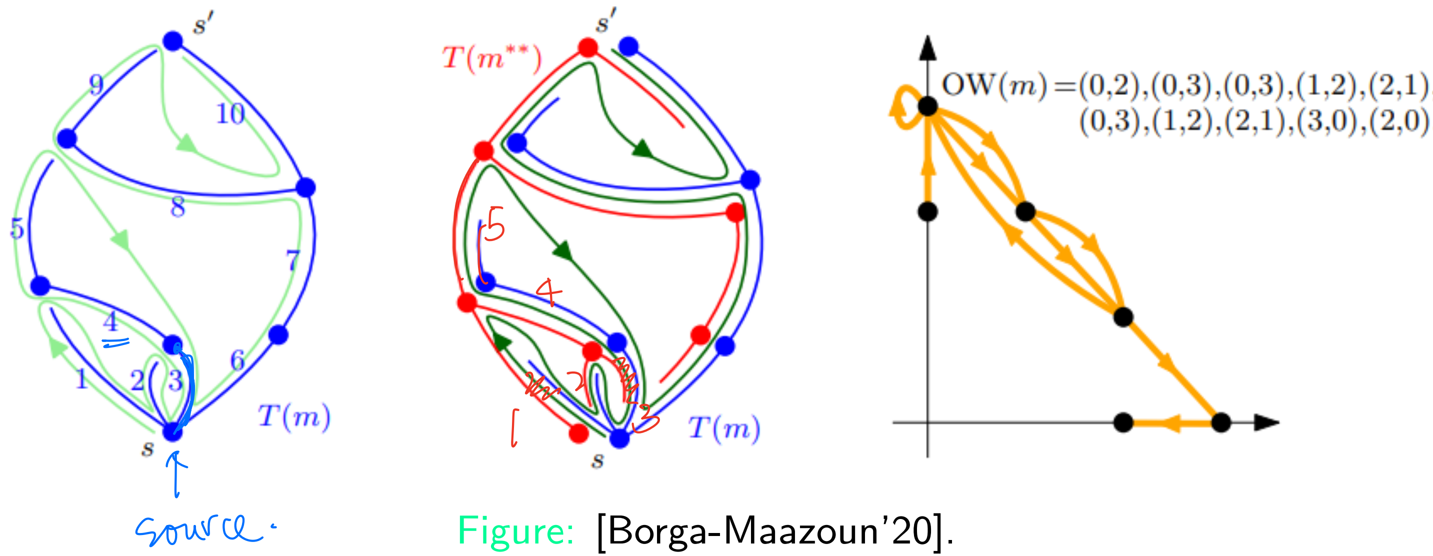


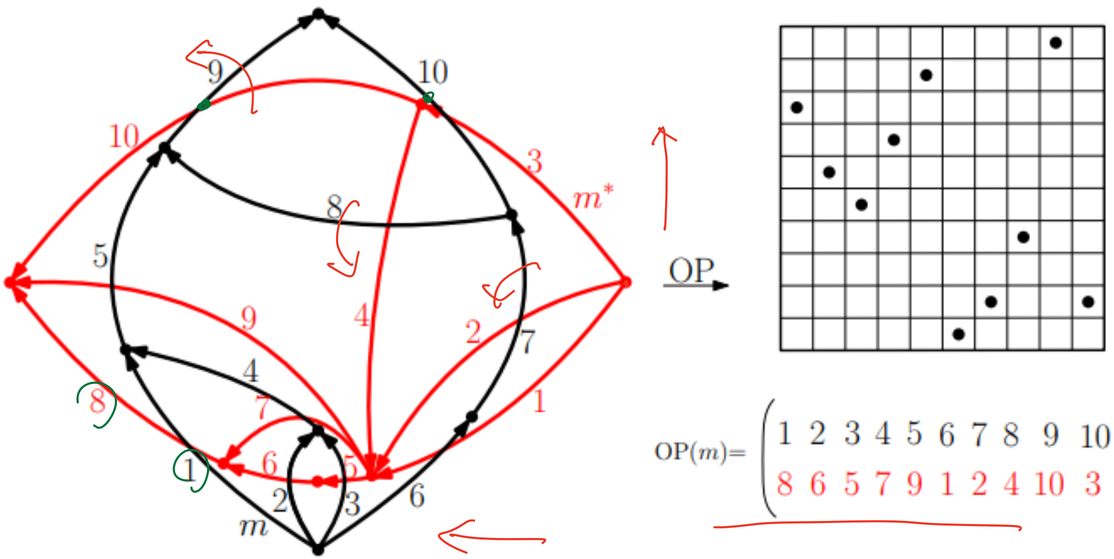
Figure: [Borga-Maazoun'20].

One can also consider the dual map m^* and its corresponding space-filling curve η_{m^*}. Let $\pi(k)$ be the time when η_{m^*} crosses e_k.

Mapping OP from [Bonichon-Bousquet-Mélou-Fusy'11]

The permutation $\pi \in S_n$ constructed from m induces a bijection between bipolar orientations and Baxter permutations.

η_m. η_{m^*}



$$\mathrm{OP}(m) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 6 & 5 & 7 & 9 & 1 & 2 & 4 & 10 & 3 \end{pmatrix}$$

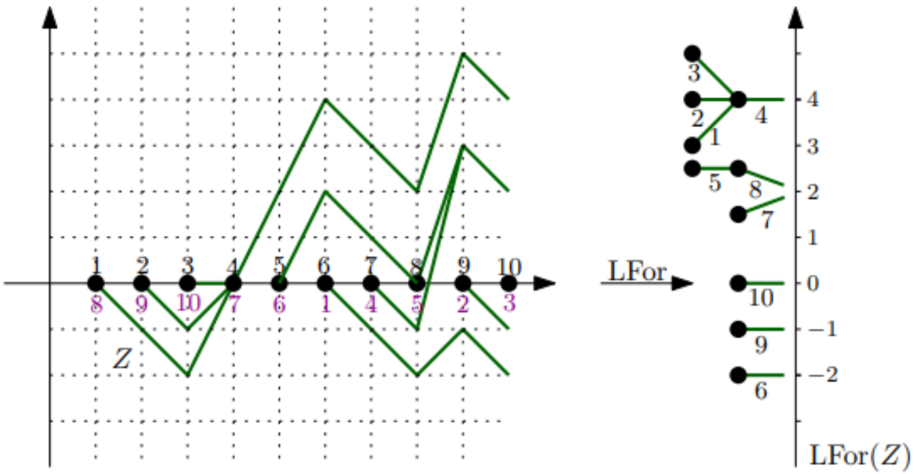
Figure: [Borga-Maazoun'20].

Bijections between discrete objects

Given a coalescent-walk process $Z = \{Z^{(k)}\}_{1 \le k \le n}$, for $1 \le i \le j \le n$, if $Z_j^{(i)} < 0$ then let $i \le_Z j$, otherwise let $j \le_Z i$.

Mapping CP from [Borga-Maazoun'20]

The ordering $\le_Z$ gives a permutation π, which further induces a mapping from coalescent-walk processes to Baxter permutations. Furthermore one has the relation $\mathrm{OP} = \mathrm{CP} \circ \mathrm{WC} \circ \mathrm{OW}$.



$6 \le_Z 9 \le_Z 10 \le_Z 7 \le_Z 8$
$\le_Z 5 \le_Z 4 \le_Z 1 \le_Z 2 \le_Z 3$

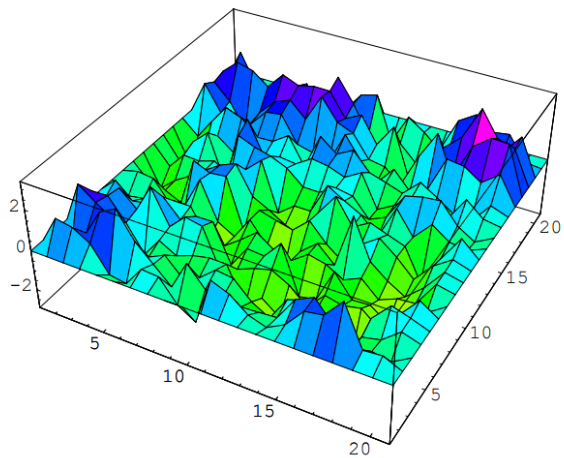
$\mathrm{CP}(Z)$=8 9 10 7 6 1 4 5 2 3

Figure: [Borga-Maazoun'20].

The Gaussian Free Field (GFF)

- Let $D \subset \mathbb{C}$ be a domain, and $H(D)$ be the (closure of the) space of smooth functions on D with finite Dirichlet energy modulo a global constant.
- Let $(f, g)_\nabla = \int_D \nabla f \cdot \nabla g dx$ be the inner product, and $\{\alpha_n\}_{n\geq 1}$ be i.i.d normal.
- The *free boundary* Gaussian Free Field is defined by the random generalized function $h = \sum_{n=1}^{\infty} \alpha_n f_n$ where $\{f_n\}$ is an orthonormal basis of $H(D)$.

$\underline{\underline{H^{-1}(D)}}$.



D

$H^1(D)$

Figure: A discrete GFF from [Sheffield'03].

Liouville quantum gravity (LQG)

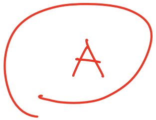
- Variants of the GFF: quantum spheres, disks, wedges and cones parameterized by a weight parameter.

W. Bessel.

- Let $\gamma \in (0,2)$ and fix some variant of the free boundary GFF h on D.
- The Liouville quantum gravity: "Riemannian metric induced by $e^{\gamma h}(dx^2 + dy^2)$".

$\frac{e^{\gamma h}}{\Delta}$

- LQG area: $\mu_h(d^2z) =$ "$e^{\gamma h} d^2 z$".
- LQG boundary length: $\nu_h(dx) =$ "$e^{\gamma h/2} dx$".



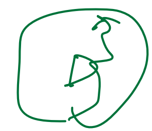
$\int_A \frac{e^{\gamma h}}{\Delta} d^2z.$

The Schramm-Loewner Evolution (SLE)

- Fix $\kappa > 0$, and let $\{B_t\}_{t\geq 0}$ be the standard Brownian motion.
- The (chordal) SLE_κ curve η on the upper half plane $\mathbb{H}$ can be characterized by

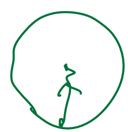
$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}; \; g_0(z) = z \tag{1}$$

where g_t is the conformal map from $\mathbb{H}\backslash\eta([0,t])$ to $\mathbb{H}$ with $\lim_{|z|\to\infty} |g_t(z) - z| = 0$.



- For $\kappa \in (0, 4]$, η is a.s. a simple curve, and for $\kappa \geq 8$, η is a.s. space-filling.
- The definition is extended to other domains via *conformal invariance*.
- Variants of SLE: $\text{SLE}_\kappa(\rho)$ processes, radial and whole plane $\text{SLE}_\kappa(\rho)$ processes.

weight



The Imaginary Geometry

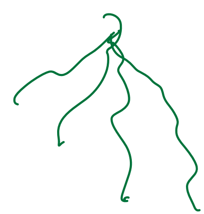
$\kappa = \gamma^2 = \frac{4}{3}$ $\kappa' = \frac{16}{\gamma^2}$

- Given a GFF h with certain boundary conditions, one can make sense of the equation

$$\frac{d}{dt}\eta(t) = e^{i(h(\eta(t))/\chi+\theta)}; \ \eta(0) = z \tag{2}$$

where $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ with $\kappa = \gamma^2 \in (0,4)$. The curve η is called the *angle θ flow line of h.*

- Flow lines are typically $\text{SLE}_\kappa(\rho)$ processes.
- All the θ angle flow lines merges into a flow line tree, which is *angle* $\theta - \frac{\pi}{2}$ *space-filling counter-flow lines* and $\text{SLE}_{\kappa'}$ type curves. ($\kappa' = \frac{16}{\kappa} > 4$.)



$\kappa' = 12$.

The Imaginary Geometry

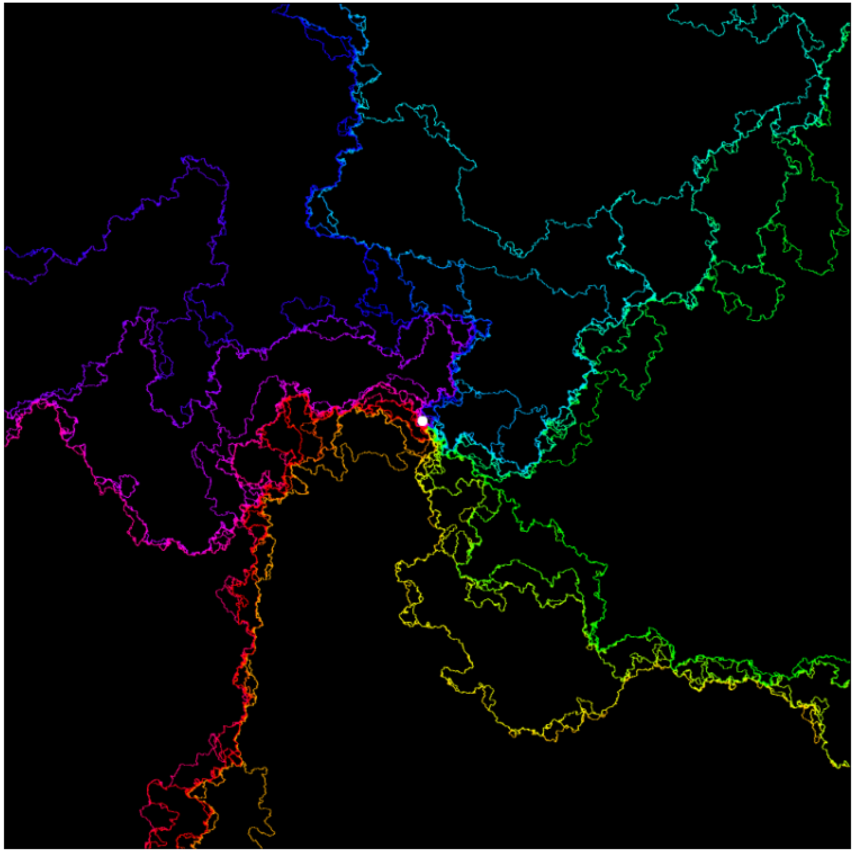


Figure: An example of numerically generated flow lines of whole plane GFF with different angles from [Miller-Sheffield'17].

The scaling limit of the discrete objects

$4-\gamma^2.$

Bipolar orientation ($\mathcal{O}$) [Kenyon-Miller-Sheffield-Wilson'19]

The scaling limit of the uniform bipolar orientation, along with its space-filling interface, is the unit area quantum sphere with $\gamma = \sqrt{4/3}$ decorated by an independent SLE_{12} curve, in the sense of peanosphere topology. $(\mathbb{C}, h, 0, \infty).$

It has also been argued in [Gwynne-Holden-Sun'16] that the scaling limit is *joint* for the space-filling curve of the bipolar orientation and its dual map.

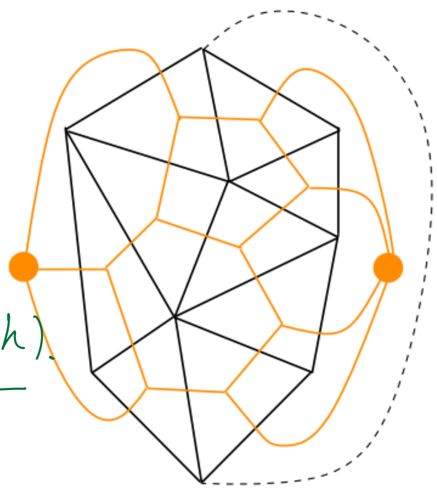
random planar map $\to$ LQG surface.

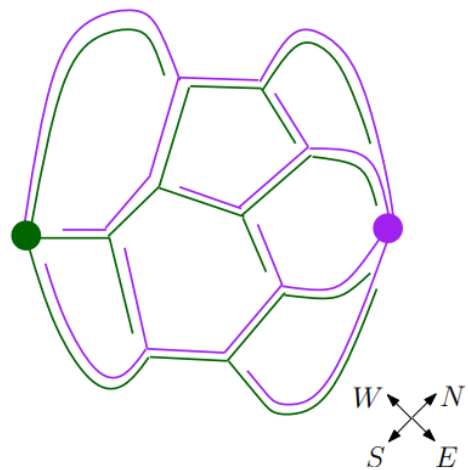
η'_m. η'_{m^*}

$\downarrow$

$\hat{h}$ (index of h).

η'_0. $\eta'_{-\frac{\pi}{2}}$





The scaling limit of the discrete objects

Recall that a permuton is a measure on $[0,1]^2$, whose marginals are the uniform distributions, while a permutation naturally induces a permuton.

Baxter permutation ($\mathcal{P}$) [Borga-Maazoun'20]

The random measure induced by uniform Baxter permutation converges weakly to a random measure on $[0,1]^2$, which is the *Baxter permuton*.

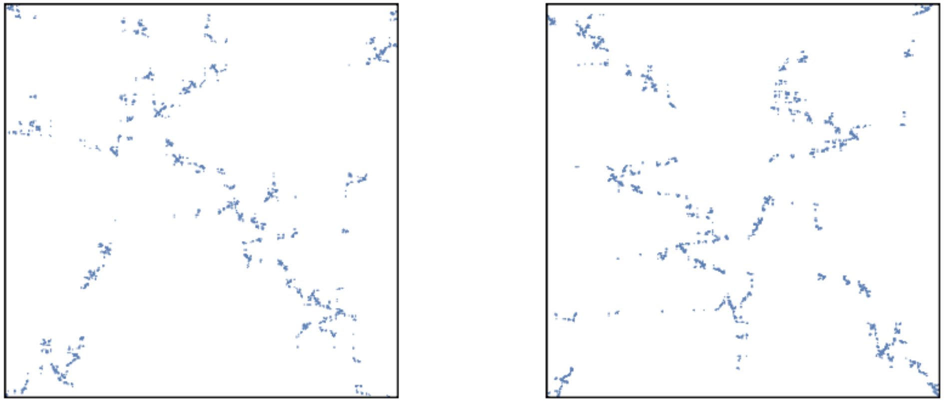
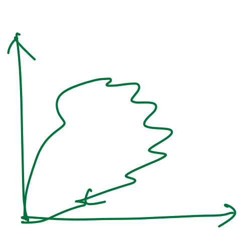
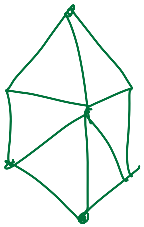


Figure: A simulation of uniform Baxter permutation for large n from [Borga-Maazoun'20].

The scaling limit of the discrete objects



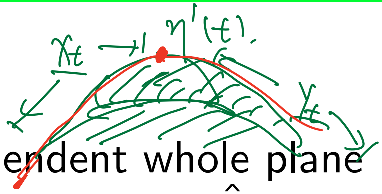


- The tandem walks $(\mathcal{T})$ scales to the Brownian loop $(X_t, Y_t)_{t\in[0,1]}$ with $(X_0, Y_0) = (X_1, Y_1) = 0$ conditioned on staying in the first quadrant (and correlation $-1/2$) .
- The scaling limit of coalescent-walk process $(\mathcal{C})$ is the Tanaka's SDE

$$dZ_t^{(u)} = 1_{Z_t^{(u)}>0} dY_t - 1_{Z_t^{(u)}<0} dX_t;\ \ t \in [u,1];\ \ Z_u^{(u)} = 0. \tag{3}$$

The Scaling limit of the discrete objects

$\eta_0'([x_1, x_2)) = x_2 - x_1$



Let $(\mathbb{C}, h, 0, \infty)$ be a unit area quantum sphere, $\hat{h}$ be an independent whole plane GFF, η_0' and $\eta_{-\frac{\pi}{2}}'$ be the angle 0 and $-\frac{\pi}{2}$ space-filling counterflow line of $\hat{h}$, both parameterized by γ-LQG area.

OW: The Mating of trees [Miller-Sheffield'19]

Let $(X_t, Y_t)_{0<t<1}$ be the left and right boundary length of $\eta_0'([0, t])$. Then $(X_t, Y_t)_{0<t<1}$ has the law of Brownian loop rooted at 0 in the first quadrant with correlation $-\cos(\pi\gamma^2/4)$.

OP: [Borga'21]

Let $\psi(t)$ be the time when $\eta_{-\frac{\pi}{2}}'$ hits $\eta_0'(t)$. For $\gamma = \sqrt{4/3}$, the random measure $(\mathrm{Id}, \psi)_* \mathrm{Leb}$ has the law of Baxter permuton.

The Scaling limit of discrete objects

WC: [Prokaj'13] and [Borga-Maazoun'20]

Almost surely, for a set A with full Lebesgue measure, the Tanaka's equation (3) has a unique solution $\{Z_t^{(u)}\}_{u<t<1}$ for *every* $u \in A$.

$[0,1]$

CP: [Borga-Maazoun'20]

Given the solution $\{Z_t^{(u)}\}_{0<u<t<1}$ to the SDE (3), consider the ordering $\leq_Z$: for $0 < s < t < 1$, $s \leq_Z t$ if $Z_t^{(s)} < 0$; otherwise $t \leq_Z s$. Then this ordering further induces the Baxter permuton.

The intensity of the Baxter permuton

Based on the LQG description, we compute the following formula for the intensity of the Baxter permuton $\mu_B := \mu_{\sqrt{4/3},1/2}$.

$\mathbb{E}\mu_B$

Theorem (Borga-Holden-Sun-Y.22')

Define the function

$$\rho(t,x,r) := \frac{1}{t^2}\left(\left(\frac{3rx}{2t}-1\right)e^{-\frac{r^2+x^2-rx}{2t}} + e^{-\frac{(x+r)^2}{2t}}\right). \tag{4}$$

Then the intensity measure $\mathbb{E}[\mu_B]$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]^2$. Moreover, it has the following density function

$$\begin{aligned} p_B(x,y) = c \int_{\max\{0,x+y-1\}}^{\min\{x,y\}} \int_{\mathbb{R}_+^4} & \rho(y-z,\ell_1,\ell_2)\rho(z,\ell_2,\ell_3) \\ & \cdot \rho(x-z,\ell_3,\ell_4)\rho(1+z-x-y,\ell_4,\ell_1)\, d\ell_1 d\ell_2 d\ell_3 d\ell_4 \, dz \end{aligned} \tag{5}$$

where c is a normalizing constant.

Ingredients for the proof

- SLE duality: For $0 < \kappa < 4$, the boundaries of $\mathrm{SLE}_{\frac{16}{\kappa}}$-type curves are SLE_κ-type curves.
- *Conformal welding* of quantum disks [Ang-Holden-Sun'20]: If one draw an $\mathrm{SLE}_\kappa(W_1 - 2; W_2 - 2)$ curve on an independent weight $W_1 + W_2$ disk, then the resulting surfaces are independent weight W_1 and W_2 disk conditioned on having the same interface length. Similarly if one draw an independent whole plane $\mathrm{SLE}_\kappa(W - 2)$ on a weight W quantum sphere, one get a weight W quantum disk.

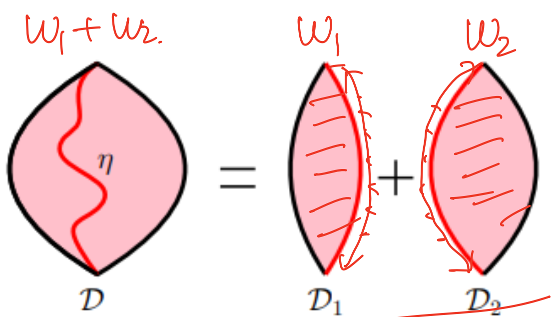
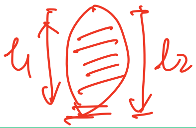


Figure: Welding of quantum disk.

Ingredients for the proof



$\frac{4-\gamma^2}{4} = \frac{\gamma^2}{2}$

$\gamma^2 = \frac{4}{3}$

Proposition (Ang-Holden-Sun'20)

The quantum area of a weight $\frac{\gamma^2}{2}$ disk with boundary length ℓ_1 and ℓ_2 has the same law as the duration of a Brownian excursion from $\ell_1\sqrt{2\sin\phi}$ to $\ell_2\sqrt{2\sin\phi}e^{i\phi}$ in the cone $\{z : \arg z \in (0,\phi)\}$.

Proposition

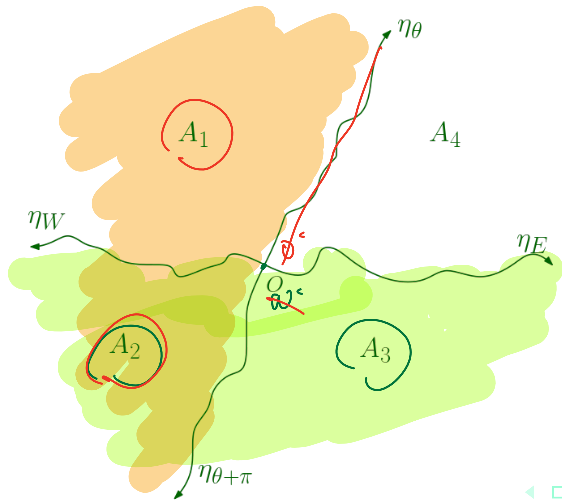
For $\phi = \frac{\pi}{3}$, $x, r > 0$ the duration of a Brownian motion from x to $re^{\pi i/3}$ in the cone $\{z : \arg z \in (0, \frac{\pi}{3})\}$ has density

$$\tilde{p}(t,x,r) := \left(\left(\frac{3xr}{2t} - 1 \right) e^{-\frac{x^2+r^2-\ell r}{2t}} + e^{-\frac{(\ell+r)^2}{2t}} \right) \frac{(x^3+r^3)^2}{18\ell^2 r^2} \cdot \frac{1}{t^2} \cdot 1_{t>0}. \tag{6}$$

Proof outline

- Consider a quantum sphere $(\mathbb{C}, h, 0, \infty)$ and the corresponding curves η' and $\eta'_{-\frac{\pi}{2}}$. Then $\mu_B([x_1, x_2] \times [y_1, y_2])$ is the quantum area of $\eta'([x_1, x_2]) \cap \eta'_{-\frac{\pi}{2}}([y_1, y_2])$.
- Uniformly sample a point $\mathbf{w}$ according to the quantum area. Expected quantum area is the same as the prob $\mathbf{w}$ falls into $\eta'([x_1, x_2]) \cap \eta'_{-\frac{\pi}{2}}([y_1, y_2])$. Consider the curves η' and $\eta'_{-\frac{\pi}{2}}$ stopped when hitting $\mathbf{w}$.
- By our parameterization, this is the same as the probability that $A_1 + A_2 \in [y_1, y_2]$ and $A_2 + A_3 \in [x_1, x_2]$.
- Then one can apply the *rerooting* invariance and the conformal welding.

$x_2 - x_1$ $\quad y_2 - y_1$ $\quad 4 - \gamma^2$ $\quad \mathbb{E}$



$A_2 + A_3 \in [x_1, x_2]$.

Positive occurrence

(21)

Occurence of permutation patterns

- Fix $n \geq k \geq 1$. Given a permutation $\sigma \in S_n$ and $I \subset \{1, ..., n\}$, the restriction of σ to I naturally induces a permutation in $S_{|I|}$, denoted by $\text{pat}_I(\sigma)$.
- For $\pi \in S_k$, the occurrence of π in σ is defined by

$$\widetilde{\text{occ}}(\pi, \sigma) := \frac{\#\{I \subset [n] : \text{pat}_I(\sigma) = \pi\}}{\binom{n}{k}} \tag{7}$$

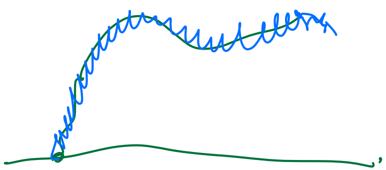
Theorem (Borga-Holden-Sun-Y.'22)

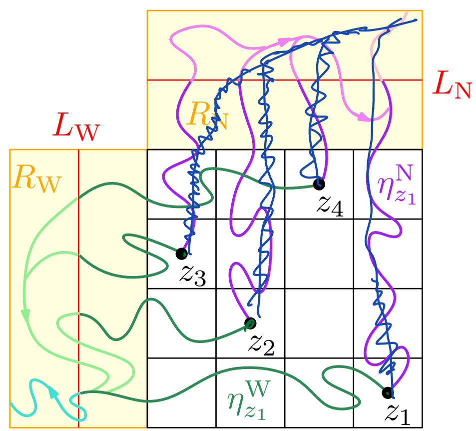
For any $k \geq 0$ and $\pi \in S_k$, the uniform Baxter permutation μ_n satisfies

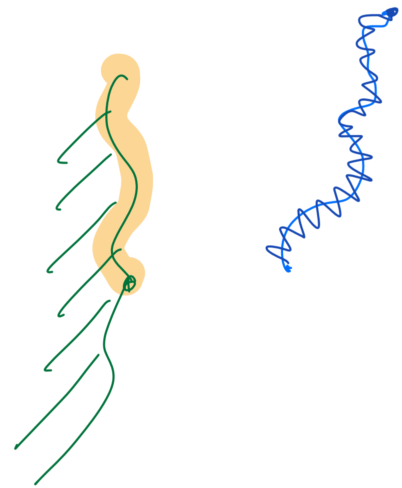
$$\lim_{n\to\infty} \widetilde{\text{occ}}(\pi, \mu_n) > 0, \ a.s.$$

Ingredients for the proof

- The notion of occurrence can be extended to permutons, and the above theorem is proved for general skew Brownian permutons.
- The main input is that, SLE$_\kappa$ curves can approximate any continuous simple curves with positive probability ([Miller-Sheffield'17]).
- The permutation pattern problem can be rephrased into the crossing and merging pattern of the flow lines.







The skew Brownian permuton

- From the coalescent-walk perspective, the construction can be generalized to the following SDE

$$dZ_t^{(u)} = 1_{Z_t^{(u)}>0} dY_t - 1_{Z_t^{(u)}<0} dX_t + (2q-1) d\mathcal{L}_t; \; t \in (u,1); \; Z_u^{(u)} = 0 \quad (8)$$

where $\mathcal{L}_t$ is the local time of $Z^{(u)}$ at 0, $q \in (0,1)$, and (X_t, Y_t) is the Brownian loop with correlation $-\cos(\pi\gamma^2/4)$.

- It has been shown in [Borga'21] that this SDE has unique solution and leads to the *skew Brownian permuton*.

The skew Brownian permuton and LQG

The skew Brownian permuton, as a random measure $\mu_{\gamma,q}$, can also be constructed on the LQG and OP side with general $\gamma \in (0,2)$ and the $\theta - \frac{\pi}{2}$-angle space-filling counterflow line $\eta'_{\theta-\frac{\pi}{2}}$ of the GFF *h*.

$q=0.$ $\gamma=\sqrt{\frac{4}{3}}$ $\theta=0.$

Proposition (Borga-Holden-Sun-Y.22')

There exists some constant $\theta := \theta_\gamma(q)$, such that a.s. for any $0 \le x_1 \le x_2 \le 1, 0 \le y_1 \le y_2 \le 1$, $\mu_{\gamma,q}([x_1,x_2]\times[y_1,y_2])$ is the γ-LQG area of $\eta'([x_1,x_2]) \cap \eta'_{\theta-\frac{\pi}{2}}([y_1,y_2])$.

Thank you for listening!