OBSTRUCTIONS TO APPLYING THE BAKER–BILU METHOD FOR DETERMINING INTEGRAL POINTS ON CURVES

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ABSTRACT. We prove that for every smooth projective integral curve X of genus at least 2 over $\mathbb{C}$, there exists $x \in X(\mathbb{C})$ such that no connected finite étale cover of $X - \{x\}$ admits a nonconstant morphism to $\mathbb{G}_m$. This has implications for the applicability of Baker's method to determining integral points on curves.

1. INTRODUCTION

It is not known if there is an algorithm to find all the integer solutions to an arbitrary polynomial equation in two variables. More generally, one can ask about solutions in a ring of S-integers of a number field. Equivalently, one can ask about effectively bounding the height of S-integral points on an affine curve U. One can reduce to the case where U is smooth and geometrically integral; then $U = X - R$ for some nice curve X and some finite set R of closed points of X. (Here, "nice" means smooth, projective, and geometrically integral.) Let g be the genus of X. We assume that the Euler characteristic $\chi(U) = 2 - 2g - \#R$ is negative; in this case, the set of S-integral points is finite by Siegel's theorem [Sie29], but the question is whether the result can be made effective.

Baker's method together with Dirichlet's S-unit theorem handles all cases with $g \leq 1$ [Bak66, Bak68a, Bak68b, Bak68c, BC70]. It also handles some (X, R) with $g \geq 2$, such as those in which X is a cyclic cover of $\mathbb{P}^1$ and R is an orbit of $\operatorname{Aut}(X/\mathbb{P}^1)$ [Bak69]. Bilu [Bil95, Theorem E] generalized Baker's approach to handle all cases in which U after base field extension has a connected finite étale cover with a nondegenerate morphism to $\mathbb{G}_m \times \mathbb{G}_m$; here nondegenerate means that the image is not contained in a coset of a proper algebraic subgroup. Bilu's theorem begs the question:

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Question 1.1. Does *every* smooth integral affine curve of genus at least 2 over $\overline{\mathbb{Q}}$ admit a connected finite étale cover with a nondegenerate morphism to $\mathbb{G}_m \times \mathbb{G}_m$?

We conjecture that the answer is *no*. In fact, we conjecture the following stronger statement:

Conjecture 1.2. For *any* nice curve X of genus at least 2, there exists $x \in X(\overline{\mathbb{Q}})$ such that the affine curve $X - \{x\}$ has no connected finite étale cover with even a *single* nonconstant morphism to $\mathbb{G}_m$.

As evidence, we prove this statement with $\mathbb{C}$ in place of $\overline{\mathbb{Q}}$:

Theorem 1.3. *Let X be a nice curve of genus at least* 2 *over* $\mathbb{C}$. *Then there exists $x \in X(\mathbb{C})$ such that $X - \{x\}$ has no connected finite étale cover with a nonconstant morphism to* $\mathbb{G}_m$.

Thus if, for some $g \geq 2$, Question 1.1 has a positive answer for all curves of genus g over $\overline{\mathbb{Q}}$, it cannot be because of a universal algebraic construction of a cover of degree depending only on g, as was the case for $g = 1$ — instead the degree would have to depend on the height of the curve as well, as happens in Belyi's theorem [Bel79].

Remark 1.4. For a smooth integral affine curve $U = X - R$ as above, having a nonconstant morphism $U \to \mathbb{G}_m$ is equivalent to having a nontrivial integer relation between the classes of the points of R in $\operatorname{Pic}(X)$.

For any nice curve X of genus at least 2 over an algebraically closed field K, let $X(K)_{\text{special}}$ be the set of $x \in X(K)$ such that $X - \{x\}$ has a connected finite étale cover with a nonconstant morphism to $\mathbb{G}_m$. Theorem 1.3 is an immediate consequence of the following, which we prove in section 2.

Theorem 1.5. *Let K be an algebraically closed field of characteristic* 0. *Let X be a nice curve of genus at least* 2 *over K. Then $X(K)_{\text{special}}$ is countable.*

One approach to proving a negative answer to Question 1.1 (over $\overline{\mathbb{Q}}$) might be to prove that for a nice curve X of genus at least 2 over $\overline{\mathbb{Q}}$, the set $X(\overline{\mathbb{Q}})_{\text{special}}$ is of bounded height. Our Theorem 3.2 implies the slightly weaker result that the set of $x \in X(\overline{\mathbb{Q}})$ such that $X - \{x\}$ admits such a cover of degree $\leq d$ is a set of bounded height. Another approach to Question 1.1 might be to prove that $X(\overline{\mathbb{Q}})_{\text{special}}$ is not p-adically dense in $X(\overline{\mathbb{Q}})$. Perhaps $X(\overline{\mathbb{Q}})_{\text{special}}$ is even *finite* for most X.

Example 1.6 (Ihara, Tamagawa)**.** If X is of genus 2, then $X(\overline{\mathbb{Q}})_{\text{special}}$ is infinite, because it contains the set called M just before Proposition 6.1 in [Bil95].

(Bilu received this example in a letter from Yasutaka Ihara, who wrote that it emerged in a discussion with Akio Tamagawa.)

Remark 1.7. Corvaja and Zannier recently gave many new examples of genus 2 curves punctured at one point that are amenable to the Baker–Bilu method [CZ23].

One might also ask whether any curve with *sufficiently many* punctures is amenable to the Baker–Bilu method. We conjecture that the answer is again no, in a strong sense:

Conjecture 1.8. For any nice curve X of genus at least 2 over $\overline{\mathbb{Q}}$, there exist distinct $x_1, \ldots, x_r \in X(\overline{\mathbb{Q}})$ such that $X - \{x_1, \ldots, x_r\}$ has no connected finite étale cover with a nonconstant morphism to $\mathbb{G}_m$.

As evidence, we prove the following:

Theorem 1.9. *If $g > r \geq 1$, there exist a genus g curve X over $\mathbb{C}$ and distinct $x_1, \ldots, x_r \in X(\mathbb{C})$ such that $X - \{x_1, \ldots, x_r\}$ has no connected finite étale cover with a nonconstant morphism to $\mathbb{G}_m$.*

Proof. We may enlarge r to assume $r = g - 1$. Let Z be any genus 2 curve over $\mathbb{C}$. Let $z \in Z(\mathbb{C})$ be as in Theorem 1.3. Let X be any connected finite étale cover of Z of degree $g - 1$; such a cover exists since the fundamental group of Z has $\mathbb{Z}/(g-1)\mathbb{Z}$ as a quotient. By the Riemann–Hurwitz formula, X has genus g. Let $x_1, \ldots, x_r \in X(\mathbb{C})$ be the preimages of z. Any finite étale cover of $X - \{x_1, \ldots, x_r\}$ is also a finite étale cover of $Z - \{z\}$, so the result follows from the conclusion of Theorem 1.3 for $Z - \{z\}$. $\square$

Remark 1.10. Theorem 1.3 is related to a number of other conjectures in algebraic geometry and geometric topology.

Prill's problem asked whether there is a nice curve X of genus at least 2 and a finite cover $f \colon Y \to X$ such that $H^0(Y, \mathscr{O}_Y(f^{-1}(x))) \geq 2$ for every $x \in X$ [ACGH85, p. 268, Chapter VI, Exercise D]. Call such a cover *Prill exceptional*. Any Prill exceptional cover has the property that for each $x \in X$, the curve $Y - f^{-1}(x)$ has a nonconstant map to $\mathbb{A}^1$. It is known that for every genus 2 curve X over $\mathbb{C}$, there is a finite étale Prill exceptional cover of X [LL22c], but it remains wide open whether there exists any X of genus at least 3 with a Prill exceptional cover.

Moreover, by [LL22a, Lemma 5.5], any Prill exceptional cover $f : Y \to X$ of a general curve X of genus g gives a counterexample to another conjecture in geometric topology, the Putman–Wieland conjecture, as stated in [PW13, Conjecture 1.2]. In fact, [LL22b, Lemma 6.10] implies that the Putman–Wieland conjecture is equivalent to a statement about maps from covers of curves to abelian varieties (in place of $\mathbb{G}_m$ as in Theorem 1.3): the statement is that for each $g \geq 2$, $n \geq 0$, and abelian variety A over $\mathbb{C}$, a general n-pointed genus g curve X has no finite cover $Y \to X$ branched only over the n points with a nonconstant map $Y \to A$. This statement is false for $g = 2$ [Mar22, Theorem 1.3] but open for $g \geq 3$. The Putman–Wieland conjecture, in turn, is closely related to another longstanding conjecture in geometric topology, Ivanov's conjecture [PW13, Theorem 1.3].

2. Proof of Theorem 1.5

Idea of proof 2.1. Fix K and X as in Theorem 1.5. First, we reduce to proving countability of the set $X(K)_{\text{special},G}$ of $x \in X(K)$ such that $X - \{x\}$ has a G-*Galois* finite étale cover with a nonconstant morphism to $\mathbb{G}_m$, for each finite group G. Next, we construct a moduli space M parametrizing G-covers $f\colon Y \to X$ branched at a varying point $x \in X$, together with a universal curve $S \to M$. We show that if there is a relation between the points in $f^{-1}(x)$ for a very general x, then there is a relation between certain divisors in S. Finally, we rule out such a relation by showing that the intersection matrix of these divisors is invertible.

Remark 2.2. Since every finite étale cover is dominated by a Galois finite étale cover, $\bigcup_G X(K)_{\text{special},G} = X(K)_{\text{special}}$. From now on, we fix G. It remains to prove that $X(K)_{\text{special},G}$ is countable.

We next construct a moduli space of G-covers of X branched over at most one point.

Lemma 2.3. *There is a finite-type K-scheme $\mathscr{M}$ parametrizing $(x, Y, f, \mathcal{B}, y_1, \ldots, y_n)$, where x is a point of X, Y is a nice curve, $f\colon Y \to X$ is a Galois cover with Galois group G that is étale above $X - \{x\}$ (at least), and $\mathcal{B}$ is a basis for $J[3]$, where J is the Jacobian of Y, and $y_1, \ldots, y_n$ are the distinct points of $f^{-1}(x)$ (here n is constant on each irreducible component of $\mathscr{M}$). Each irreducible component of $\mathscr{M}$ is a nice curve.*

Proof. This essentially follows from [Wew98, Theorem 4]. Let the $D \subset X \to S$ of Wewers be the diagonal $\Delta \subset X \times X \to X$. By [Wew98, Theorem 4], there is a finite-type algebraic stack $\mathcal{H} = \mathcal{H}^G_{X\times X}(G)$ over X such that for any $S' \to X$, the groupoid $\mathcal{H}(S')$ parameterizes G-Galois finite locally free covers $Y' \to S' \times_X (X \times X)$ that are tamely ramified above $S' \times_X \Delta$ and étale elsewhere. By [Wew98, Theorem 4], $\mathcal{H} \to X$ is étale. It follows from [Wew98, Theorem 3.2.4] that $\mathcal{H} \to X$ is proper.

By adding the data of $\mathcal{B}$ and the sections $y_1, \ldots, y_n$ to our moduli stack, we obtain a finite étale cover $\mathscr{M}$ of $\mathcal{H}$. Each groupoid $\mathscr{M}(S')$ is a setoid because $\operatorname{Aut} Y \to \operatorname{Aut} J[3]$ is injective [KS99, 10.5.6], and hence $\mathscr{M}$ is represented by an algebraic space. Since $\mathscr{M} \to \mathcal{H}$ is finite étale and $\mathcal{H} \to X$ is proper étale,

$\mathscr{M}$ is proper étale over X. By the previous two sentences, $\mathscr{M}$ is finite étale over X and is therefore a scheme. Since $\mathscr{M}$ is finite étale over the nice curve X, each irreducible component of $\mathscr{M}$ is a nice curve. $\square$

Notation 2.4. Let M be an irreducible component of $\mathscr{M}$. Then M is a nice curve. Let η be the generic point of M. Let $\pi\colon S \to M$ be the universal morphism whose fiber above $m = (x, Y, f, \mathcal{B}, y_1, \dots, y_n) \in M$ is Y. Thus S is a nice surface, and $\pi\colon S \to M$ is a relative curve. Let $h\colon S \to X$ be the morphism whose restriction to each fiber Y of π is the map $f\colon Y \to X$. Let $\mu\colon M \to X$ be $(x, Y, f, \mathcal{B}, y_1, \dots, y_n) \mapsto x$. Let e be the positive integer such that for any $m \in M$, the ramification index of the corresponding map $f\colon Y \to X$ at any point above x is e. Let $s_1, \dots, s_n\colon M \to S$ be the sections such that $s_i(m) = y_i$ for each i. Let $D_i = s_i(M) \in \operatorname{Div}(S)$. Let $F \in \operatorname{Div}(S)$ be a closed fiber of $S \to M$. Let $\operatorname{NS}(S)$ be the Néron–Severi group of S.

Proposition 2.5. *The classes of $D_1, \dots, D_n, F$ in $\operatorname{NS}(S)$ are $\mathbb{Z}$-independent.*

Proof. It suffices to prove that the intersection matrix is nonsingular. We know $D_i \cdot D_j = 0$ for $i \neq j$, $D_i \cdot F = 1$, $F \cdot F = 0$, and $D_i^2 = d$ for some d independent of i, because G acts transitively on $D_1, \dots, D_n$. Then the intersection matrix is

$$\begin{pmatrix} d & 0 & 0 & \cdots & 0 & 1 \\ 0 & d & 0 & \cdots & 0 & 1 \\ 0 & 0 & d & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix},$$

which has determinant $-nd^{n-1}$. In Lemma 2.6, we verify that $d \neq 0$. $\square$

Lemma 2.6. *We have $D_i^2 = (\deg \mu)(2 - 2g)/e < 0$.*

Proof. Observe that $\mu = h \circ s_i$ because $\mu(x, Y, f, \mathcal{B}, y_1, \dots, y_n) = x = h(y_i)$ and $y_i = s_i(x, Y, f, \mathcal{B}, y_1, \dots, y_n)$. Let $\delta\colon X \to X \times X$ be the diagonal. The diagram

$$\begin{tikzcd} S \arrow[r, "{(\pi,h)}"] & M \times X \arrow[r, "\mu \times \mathrm{id}"] & X \times X \\ & M \arrow[ul, "s_i"] \arrow[u, "{(\mathrm{id},\mu)}"'] \arrow[r, "\mu"] & X \arrow[u, "\delta"'] \end{tikzcd} \tag{2.1}$$

commutes, since $\pi \circ s_i = \mathrm{id}$ and $h \circ s_i = \mu$.

Let $\Delta = \delta(X) \subset X \times X$ be the diagonal. We will pull back the class of Δ to a line bundle on M along the two outer paths in (2.1). First, we determine the pullback of Δ to S by calculating its restriction to each fiber of $\pi\colon S \to M$.

If $m = (x, Y, f, \mathcal{B}, y_1, \dots, y_n) \in M$, so Y is the fiber of $S \to M$ over m, then the composition

$$Y \hookrightarrow S \xrightarrow{(\pi,h)} M \times X \xrightarrow{\mu \times \mathrm{id}} X \times X$$

is $Y \xrightarrow{(x,f)} X \times X$ (constant first coordinate), so $(\pi, h)^*(\mu \times \mathrm{id})^*\Delta|_Y = f^{-1}(x) = e\sum_{j=1}^n y_j$ in $\operatorname{Div} Y$. Thus $(\pi, h)^*(\mu \times \mathrm{id})^*\Delta = e\sum_{j=1}^n D_j$ in $\operatorname{Div} S$, and pulling this back to M along s_i yields $D_i \cdot (e\sum_{j=1}^n D_j) \in \operatorname{Div} M$, which has degree eD_i^2. On the other hand, the class of $\Delta \subset X \times X$ pulls back to a divisor of degree $\Delta \cdot \Delta = 2 - 2g$ on X, which pulls back to a divisor of degree $(\deg \mu)(2 - 2g)$ on M. Thus

$$eD_i^2 = (\deg \mu)(2 - 2g), \qquad \square$$

which is negative, since $g \geq 2$.

Corollary 2.7. *The images of $D_1, \dots, D_n$ in $\operatorname{Pic}(S_\eta)$ are $\mathbb{Z}$-independent.*

Proof. If there were a relation, the exact sequence

$$\bigoplus_{t \in M(K)} \mathbb{Z} \cdot \pi^{-1}(t) \longrightarrow \operatorname{Pic}(S) \longrightarrow \operatorname{Pic}(S_\eta) \longrightarrow 0 \qquad (2.2)$$

would give a relation between $D_1, \dots, D_n, F$ in $\operatorname{NS}(S)$, since the class of each fiber $\pi^{-1}(t)$ in $\operatorname{NS}(S)$ is F. $\square$

End of proof of Theorem 1.5. As mentioned in Remark 2.2, it suffices to show $X(K)_{\mathrm{special},G}$ is countable. It is the image under $\mathscr{M} \to X$ of

$$\mathscr{M}(K)' := \{(x, Y, \dots) \in \mathscr{M}(K) : Y \text{ has a nonconstant morphism to } \mathbb{G}_m\}, \qquad (2.3)$$

which is the finite union of sets $M(K)' := \mathscr{M}(K)' \cap M$. Each $M(K)'$ is the union over nonzero $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ of

$$V_a := \{(x, Y, f, \mathcal{B}, y_1, \dots, y_n) \in M(K) : \sum_{i=1}^n a_i y_i = 0 \text{ in } \operatorname{Pic}(Y)\},$$

so it suffices to prove that V_a is finite. The set V_a is the zero locus of a section of the relative Picard scheme $\mathbf{Pic}_{S/M} \to M$, so $V_a \subset M(K)$ is closed. If $V_a = M(K)$, then $\sum_{i=1}^n a_i D_i = 0$ in $\operatorname{Pic}(S_\eta)$, contradicting Corollary 2.7. Hence V_a is finite. $\square$

3. Bounded height

In this section, we set $K = \overline{\mathbb{Q}}$ and prove in Theorem 3.2 that each set $X(\overline{\mathbb{Q}})_{\mathrm{special},G}$ is of *bounded height*.

First, we introduce notation for Theorem 3.1, which records known results about heights and specialization. Let $\pi \colon S \to M$ be a morphism from a nice surface to a nice curve over $\overline{\mathbb{Q}}$. For each field extension $L \supset \overline{\mathbb{Q}}$ and $t \in M(L)$, let $S_t = \pi^{-1}(t)$; if S_t is a *nice* curve, let J_t be its Jacobian, a principally polarized abelian variety over L. By a fibral component we mean an irreducible component of S_t for some $t \in M(\overline{\mathbb{Q}})$. Let η be the generic point of M. We assume that S_η is a *nice* curve (over the function field $k(M)$).

Let k be a field with a product formula as in [Ser97, §2.1]. Given a polarized abelian variety A over k, one can enlarge k so that the polarization arises from a symmetric divisor on A and then define a canonical height pairing on $A(\bar{k})$, or its subgroup $A(k)$, as in [Sil83, pp. 200–201]. Applying this to J_η over $k(M)$ yields a (geometric) canonical height pairing $\langle\ ,\ \rangle$ on $J_\eta(k(M)) = \operatorname{Pic}^0(S_\eta)$ or on $\operatorname{Div}^0(S_\eta)$. For any $t \in M(\overline{\mathbb{Q}})$ such that S_t is a nice curve, applying this to J_t (over a number field to which it descends) yields a canonical height pairing $\langle\ ,\ \rangle_t$ on $J_t(\overline{\mathbb{Q}}) = \operatorname{Pic}^0(S_t)$ or on $\operatorname{Div}^0(S_t)$.

For each closed point $P \in S_\eta$, let $\overline{P}$ be its Zariski closure in S. Extend $\mathbb{Z}$-linearly to define $\overline{P} \in \operatorname{Div} S$ for any $P \in \operatorname{Div}(S_\eta)$; then for any $t \in M(\overline{\mathbb{Q}})$, define the specialization $P_t := \overline{P}|_{S_t} \in \operatorname{Div}(S_t)$. Let $h\colon M(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a Weil height associated to a divisor of nonzero degree on M, as in [Sil83, p. 205].

Theorem 3.1. *Let S, M, π, S_η, S_t, $\langle\ ,\ \rangle$, $\langle\ ,\ \rangle_t$, and h be as above.*

(a) *Let $P \in \operatorname{Div}^0(S_\eta)$. Then there exists a $\mathbb{Q}$-linear combination Φ of fibral components such that $D_P := \overline{P} + \Phi \in \operatorname{Div}(S) \otimes \mathbb{Q}$ is orthogonal to each fibral component. Moreover, Φ is unique modulo $\pi^* \operatorname{Div}(M) \otimes \mathbb{Q}$.*
(b) *For $P, P' \in \operatorname{Div}^0(S_\eta)$, we have $\langle P, P' \rangle = -D_P \cdot D_{P'}$.*
(c) *Fix $P, Q \in \operatorname{Div}^0(S_\eta)$. Then $\langle P_t, Q_t \rangle_t / h(t) \to \langle P, Q \rangle$ as $h(t) \to \infty$.*
(d) *Let $P_1, \ldots, P_n \in \operatorname{Div}^0(S_\eta)$. If the matrix $(\langle P_i, P_j \rangle)$ is positive definite, then the set of $t \in M(\overline{\mathbb{Q}})$ such that $P_{1,t}, \ldots, P_{n,t}$ are $\mathbb{Z}$-dependent in $\operatorname{Pic}^0(S_t)$ is of bounded height.*

Proof.

(a) Let $t \in M(\overline{\mathbb{Q}})$. Let $F_1, \ldots, F_n$ be the irreducible components of S_t. Let F be the class of S_t in $\operatorname{Div}(S)$. The existence and uniqueness of the part of Φ supported above t follows from the nondegeneracy of the intersection pairing on $(\bigoplus \mathbb{Q}F_i)/\mathbb{Q}F$, as in [Gro86, end of §3] (the finiteness of the residue field assumed there is not needed for this).
(b) This too follows from the same argument as in [Gro86, end of §3]. (The relation between canonical heights and intersection numbers was discovered earlier in the context of an abelian variety and its dual, by Manin [Man64] and Néron [Nér65].)

(c) This is a version of Silverman's specialization theorem, which extended ideas of Dem′janenko [Dem68] and Manin [Man69]. Specifically, the case $P = Q$ is contained in [Sil83, Theorem B]. The general case follows from this case applied to P, Q, and $P + Q$.

(d) If $h(t)$ is sufficiently large, then ((c)) implies that $(\langle P_{i,t}, P_{j,t}\rangle)$ is positive definite too, so $P_{1,t}, \ldots, P_{n,t}$ are $\mathbb{Z}$-dependent in $\operatorname{Pic}^0(S_t)$. □

Theorem 3.2. *Let X be a nice curve of genus at least 2 over $\overline{\mathbb{Q}}$. Then for each finite group G, the set $X(\overline{\mathbb{Q}})_{\mathrm{special},G}$ is of bounded height.*

Proof. Let $S \to M$ and $D_1, \ldots, D_n$ be as in Notation 2.4. For $i = 2, \ldots, n$, let $P_i = (D_i - D_1)|_{S_\eta}$. Since $S \to M$ has irreducible fibers, we can always take $\Phi = 0$ in Theorem 3.1((a)). By Lemma 2.6, the matrix $(D_i \cdot D_j)_{1 \le i,j \le n}$ is dI for some $d < 0$, so $((D_i - D_1) \cdot (D_j - D_1))_{2 \le i,j \le n}$ is negative definite too. By Theorem 3.1((b)), $(\langle P_i, P_j\rangle)$ is *positive* definite. For $t \in M(\overline{\mathbb{Q}})$, the following are equivalent:

- t belongs to the subset $M(\overline{\mathbb{Q}})'$ defined in (2.3);
- $D_1|_{S_t}, \ldots, D_n|_{S_t}$ are $\mathbb{Z}$-dependent in $\operatorname{Pic}(S_t)$;
- $P_2, \ldots, P_n$ are $\mathbb{Z}$-dependent in $\operatorname{Pic}^0(S_t)$.

By Theorem 3.1((d)), the set of such t is of bounded height. Thus $M(\overline{\mathbb{Q}})'$ is of bounded height, and so is its image in $X(\overline{\mathbb{Q}})$. Taking the union over the finitely many irreducible components M of $\mathscr{M}$ shows that $X(\overline{\mathbb{Q}})_{\mathrm{special},G}$ is of bounded height. □

Remark 3.3. To prove the weaker statement that $X(\overline{\mathbb{Q}}) - X(\overline{\mathbb{Q}})_{\mathrm{special},G}$ is infinite, one could use Néron's specialization theorem [Nér52, Théorème 6] in place of [Sil83, Theorem B].

Question 3.4. Can we refine the proof of Theorem 3.2 to obtain a height bound that is uniform in G? If so, then Question 1.1 has a negative answer.

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