DRP 2022: THE HEAT EQUATION

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SYLLABUS

Meetings take place virtually at 9am Boston time via Zoom (link).

Topics	**References**	**Date**
Introduction, heuristics, uniqueness	[C, §1]	Jan 3
Gradient flow formulation, parabolic maximum principle	[C, §2-3]	Jan 5
Heat kernel	[C, §5.1-5.2]	Jan 7
Green's function, parabolic mean value inequality	[C, §5.3-5.4]	Jan 10
Central limit theorem, Hölder inequality	[C, §6.1]	Jan 12
Shannon entropy, Fisher information, Perelman's $\mathcal{W}$-functional	[C, §9.1]	Jan 14
Logarithmic Sobolev inequality, Renyi entropy	[C, §9.2]	Jan 17
Differential Harnack inequality	[C, §10.1]	Jan 19
Matrix maximum principle, discuss presentation topics	[C, §10.3], [H, §4]	Jan 21
Vector maximum principle I, choose presentation topic	[H, §4]	Jan 24
Vector maximum principle II, presentation outline due	[H, §4]	Jan 26
Hamilton's matrix Harnack inequality, practice presentation	[C, §10.2]	Jan 28
DRP Symposium	–	Feb 1

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References

[C] Colding, T., & Lee, T-K. (2021). *Topics in the heat equation.* 18.966 Lecture notes. (Link)

[H] Hamilton, R. S. (1986). Four-manifolds with positive curvature operator. *Journal of Differential Geometry, 24*(2), 153-179. (Link)

[W] Weinberger, H. F. (1975). Invariant sets for weakly coupled parabolic and elliptic systems. *Rend. Mat, 8*(6), 295-310. (Link)

Exercises

Exercises labeled with a ★ are highly recommended.

Exercise 1 (★ Separable solutions)**.** Find all solutions $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ of the 1-dimensional heat equation $\partial_t u = \partial_x \partial_x u$ of the form $u(x, t) = f(t)g(x)$.

Exercise 2 (★ Solutions with zero boundary condition)**.** Find all solutions $u : [0, 1] \times [0, \infty) \to \mathbb{R}$ of the 1-dimensional heat equation $\partial_t u = \partial_x \partial_x u$ satisfying $0 = u(0, t) = u(1, t)$ for all $t \geq 0$.

(Hint: use Fourier series.)

Exercise 3 (★ Green's identity)**.** Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. Let $u, v \in C^2(\overline{\Omega})$, where $\overline{\Omega}$ denotes the closure of Ω. Show that

$$\int_\Omega (u\Delta v - (\Delta u)v)\, dx = \int_{\partial\Omega} (u\nabla_\nu v - (\nabla_\nu u)v)\, d\sigma,$$

where ν is the outward unit normal to $\partial\Omega$ and $d\sigma$ is the integration form of $\partial\Omega$.

(Hint: apply the divergence theorem to the vector field $X = u\nabla v - v\nabla u$, and use the fact that $\Delta = \mathrm{div} \circ \nabla$.)

Exercise 4 (★ Hölder's inequality)**.** Prove that if $u, v : \mathbb{R}^n \to \mathbb{R}$, then whenever the integrals make sense,

$$\int_{\mathbb{R}^n} |uv|\, dx \leq \left(\int_{\mathbb{R}^n} |u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |v|^2\, dx\right)^{\frac{1}{2}}.$$

Exercise 5 (★ 1D Poincaré inequality)**.** Let $u : [a, b] \to \mathbb{R}$ be C^1 and let $\bar{u} = \frac{1}{b-a}\int_a^b u\, dx$ be the average of u on $[a, b]$. Prove that

$$\int_a^b |u - \bar{u}|^2\, dx \leq |b - a|^2 \int_a^b |\nabla u|^2\, dx.$$

(Hint: use Hölder's inequality and the fundamental theorem of calculus.)

Exercise 6 (★ Convergence of solutions)**.**

(a) Let $u : \Omega \times [0, \infty) \to \mathbb{R}$ solve the heat equation with Dirichlet boundary condition, i.e. $u = 0$ on $\partial\Omega \times [0, \infty)$. Prove the following exponential decay estimate: there exists a constant $C > 0$ such that for all $t > 0$, there holds

$$\int_\Omega |u(x,t)|^2 \, dx \le e^{-Ct} \int_\Omega |u(x,0)|^2 \, dx.$$

(Hint: use the Poincaré inequality.)

(b) Let $u : \Omega \times [0, \infty) \to \mathbb{R}$ solve the heat equation with Neumann boundary condition, i.e. $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega \times [0, \infty)$. Let $\bar{u}_0$ denote the average of $u(\cdot, 0)$ over Ω. Prove the following exponential decay estimate: there exists a constant $C > 0$ such that for all $t > 0$, there holds

$$\int_\Omega |u(x,t) - \bar{u}_0|^2 \, dx \le e^{-Ct} \int_\Omega |u(x,0) - \bar{u}_0|^2 \, dx.$$

(Hint: show that the average of u over Ω is constant in time and use the Poincaré inequality.)

In what follows, let $H(x,y,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$ be the heat kernel on $\mathbb{R}^n$.

Exercise 7 (★ Heat kernel)**.** Show that H satisfies the heat equation in both the x and y variables:

$$\partial_t H(x,y,t) = \Delta_x H(x,y,t) = \Delta_y H(x,y,t).$$

Exercise 8 (★ Fundamental solution)**.** Let $u_0 \in C_b(\mathbb{R}^n)$ be a continuous and bounded function, and define for $x \in \mathbb{R}^n$ and $t > 0$,

$$u(x,t) = \int_{\mathbb{R}^n} H(x,y,t) u_0(y) \, dy.$$

Show that:

(a) $u(\cdot, t) \in C^\infty(\mathbb{R}^n)$ for all $t > 0$, i.e. that all partial derivatives of u exist as long as $t > 0$.

(Hint: differentiate under the integral sign and use properties of H.)

(b) $\partial_t u = \Delta u$.

(Hint: differentiate under the integral sign and use Exercise 7.)

(c) For each $x \in \mathbb{R}^n$, $\lim_{t \searrow 0} u(x,t) = u_0(x)$.

(Hint: Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|x - y| < \delta$ implies $|u_0(x) - u_0(y)| < \varepsilon$. Show that $|u_0(x) - u(x,t)| \to 0$ as ε and t go to zero, by splitting up the integral $|u_0(x) - u(x,t)|$ into two pieces, one over $B_\delta(x)$ and the other over $\mathbb{R}^n \setminus B_\delta(x)$.)

In what follows, define the Green's function for the Laplacian for $x \neq y \in \mathbb{R}^n$ by

$$G(x,y) = \int_0^\infty H(x,y,t) \, dt.$$

Exercise 9 (★ Green's function)**.**

(a) Show that for $n \geq 3$, there exists a constant $c_n > 0$ such that for all $x \neq y \in \mathbb{R}^n$, there holds

$$G(x,y) = \frac{c_n}{|x-y|^{n-2}}.$$

In particular, the integral defining $G(x,y)$ is well-defined when $n \geq 3$.

(Hint: rewrite the integral defining the Green's function in terms of the Gamma function.)

(c) Show by directly differentiating the equation from part (a) that for $x \neq y \in \mathbb{R}^n$, G solves the Laplace equations in the x and y variables:

$$\Delta_x G(x,y) = \Delta_y G(x,y) = 0.$$

For $t < 0$ and $x, y \in \mathbb{R}^n$, define the backwards heat kernel by $H_b(x,y,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x-y|^2}{4t}}$.

Exercise 10 (★ Parabolic mean value inequality)**.** Let $u : \mathbb{R}^n \times [-T, 0] \to \mathbb{R}$ be a subsolution of the heat equation, i.e. $\partial_t u \leq \Delta u$.

(a) Show that for each fixed $y \in \mathbb{R}^n$, the function

$$I_y(t) = \int_{\mathbb{R}^n} u(x,t) H_b(x,y,t)\, dx$$

is monotone decreasing in time.

(Hint: show that $I_y' \leq 0$ by integrating by parts. Apply the Green's identity from Exercise 3 on balls of larger and larger radii, and show that the boundary term vanishes in the limit.)

(b) Show that for each fixed $y \in \mathbb{R}^n$,

$$\lim_{t \nearrow 0} I_y(t) = u(y,0).$$

(Hint: use Exercise 8, part (c).)

(c) Deduce, for each fixed $y \in \mathbb{R}^n$, the parabolic mean value inequality

$$u(y,0) \leq \int_{\mathbb{R}^n} u(x,-T) H_b(x,y,-T)\, dx.$$

(Hint: combine parts (a) and (b).)

Exercise 11 (★ Hölder inequality)**.** Let $f, g : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}_{>0}$ be positive supersolutions of the heat equation: $\partial_t f \geq \Delta f$ and $\partial_t g \geq \Delta g$. Let $1 < p, q < \infty$ be Hölder conjugates, i.e. satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f^{\frac{1}{p}} g^{\frac{1}{q}}$ is a supersolution of the heat equation:

$$(\partial_t - \Delta) f^{\frac{1}{p}} g^{\frac{1}{q}} \geq 0.$$

Further, show that $f^{\frac{1}{p}} g^{\frac{1}{q}}$ *solves* the heat equation only if $f = cg$ for some constant $c \in \mathbb{R}$.

(Hint: Let $u = \log(f^{\frac{1}{p}} g^{\frac{1}{q}})$ and compute $e^{-u}(\partial_t - \Delta)e^u$.)

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the weighted Laplacian Δ_f is defined by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle \qquad \text{for all } u \in C^2(\mathbb{R}^n).$$

This operator also goes under the names of drift Laplacian, f-Laplacian, and Witten Laplacian.

Exercise 12 (★ Weighted Laplacian)**.**

(a) Prove that Δ_f is self-adjoint with respect to the weighted $L^2(e^{-f}dx)$ inner product. That is, prove that for all functions u, v on $\mathbb{R}^n$ decaying suitably rapidly at infinity,

$$\int_{\mathbb{R}^n} (\Delta_f u) v \, e^{-f} dx = \int_{\mathbb{R}^n} u(\Delta_f v) \, e^{-f} dx.$$

(b) Formulate and prove a "weighted divergence theorem" involving the weighted measure $e^{-f}dx$. What is your definition of the "weighted divergence" div_f?

(c) Prove the following weighted Bochner formula holds for all $u \in C^3(M)$:

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla \Delta_f u, \nabla u \rangle + \mathrm{Hess}_f(\nabla u, \nabla u).$$

A subset $C \subset \mathbb{R}^k$ is a *cone* with vertex $v \in \mathbb{R}^k$ if for every $w \in C$ and every $t \geq 0$, the vector $v + t(w - v)$ lies in C. The *tangent cone* $C_v X$ of a closed, convex set $X \subset \mathbb{R}^k$ at a boundary point $v \in \partial X$ is defined to be the intersection of all closed half-spaces containing X and whose boundary contains v.

Exercise 13 (★ Tangent cone)**.**

(a) Prove that the tangent cone $C_v X$ is the smallest closed, convex cone with vertex v containing X.

(b) Prove that if ∂X is C^1 at v, then $C_v X$ is a half-space.

(c) Prove that every closed, convex set is the intersection of its tangent cones: $X = \bigcap_{v \in \partial X} C_v X$.

(d) Prove that the sum of two vectors in the tangent cone of a closed convex set lies in the tangent cone.