DRP 2021

Spin geometry and the positive mass theorem

Mentor: Julius Baldauf

1 Syllabus

Spin geometry

Topic	**Reference** [LM]	**Reference** [B]	**Date**
Clifford algebras	I.1	1.1.1	Dec 23
Pin and Spin groups	I.2	1.2.1	Jan 3
The algebras $C\ell_n$ and $C\ell_{r,s}$	I.3	1.1.1	Jan 4
Classification of Clifford algebras	I.4	1.1.2	Jan 6
Representations, part 1	I.5	1.2.2	Jan 8
Representations, part 2	I.5	1.2.2	Jan 11
Lie algebra structures	I.6	1.2.1	Jan 13
Clifford and spin bundles	II.1-II.3	2.1.1	Jan 15
Connections on spin bundles	II.4	2.1.2	Jan 18
Dirac operators	II.5	2.3.4	Jan 20
Lichnerowicz formula	II.8	2.5	Jan 22

Positive mass theorem

Topic	**Reference** [LP]	**Reference** [PT]	**Date**
Dominant energy condition, asymptotically flat manifolds, ADM mass	§8, §9	§1, §4	Jan 25
Weighted function spaces and well-definedness of ADM mass	Def. 8.2, Thm. 9.6	§4	Jan 27
Green's function for the Dirac operator	Thm. 9.2(d)	§5	Jan 29
Witten's formula for the mass	Appendix	§3, §4	Feb 1

References

[B] Bourguignon, J. P., Hijazi, O., Milhorat, J. L., Moroianu, A., & Moroianu, S. (2015). *A spinorial approach to Riemannian and conformal geometry.* European Mathematical Society.

[LM] Lawson, H. B., & Michelsohn, M. L. (1989). *Spin geometry.* Princeton University Press.

[LP] Lee, J. M., & Parker, T. H. (1987). The Yamabe problem. *Bulletin (New Series) of the American Mathematical Society, 17*(1), 37-91.

[PT] Parker, T., & Taubes, C. H. (1982). On Witten's proof of the positive energy theorem. *Communications in Mathematical Physics, 84*(2), 223-238.

2 Exercises

Exercises labeled with a ★ are used in the proof of the positive mass theorem.

Algebraic aspects

1 (Clifford vs. exterior algebra)**.** Let V be a vector space over the field $K = \mathbb{R}$ or $K = \mathbb{C}$, and let q be a quadratic form on V. Show that if $e_1, \ldots, e_n$ is a q-orthogonal basis of V, then the following map is an isomorphism of vector spaces

$$\mathrm{C}\ell(V, q) \to \Lambda^\bullet V, \qquad e_{j_1} \cdots e_{j_p} \mapsto e_{j_1} \wedge \cdots \wedge e_{j_p}.$$

2 (Connected components of SO)**.** Show that for all $n \geq 1$, $\mathrm{SO}(n)$ is connected, and that $\mathrm{SO}(n-1,1)$ has exactly two connected components, where $\mathrm{SO}(r,s)$ is the Lie group

$$\mathrm{SO}(r,s) = \{\lambda \in \mathrm{GL}(\mathbb{R}^n) \mid \lambda^* q = q,\ \det(\lambda) = 1\},$$

and q is the quadratic form on $\mathbb{R}^n$ given by

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2.$$

Hint: Use the Cartan-Dieudonné theorem to find paths connecting elements in $\mathrm{SO}(r,s)$ to either plus or minus the identity.

3 (★ SU(2) is double cover of SO(3))**.** Prove that there exists a homomorphism $\xi : SU(2) \to SO(3)$ which is surjective and has kernel $\{1, -1\} \subset \mathrm{SU}(2)$.

Hint: Show first that the Lie algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$ is isomorphic to the 3-dimensional real vector space of traceless, skew-hermitian 2×2 complex matrices, which has a basis given by

$$\sigma_1 = \begin{bmatrix} & i \\ i & \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} i & \\ & -i \end{bmatrix}. \tag{2.1}$$

Then show that for all $U \in \mathrm{SU}(2)$, the adjoint action $\mathrm{Ad}_U : \mathfrak{su}(2) \to \mathfrak{su}(2)$ by U, defined for $X = X_i\sigma_i$ by

$$\mathrm{Ad}_U(X) = UXU^{-1}, \tag{2.2}$$

is an element of SO(3), is a homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, is surjective, and has kernel $\{1, -1\}$.

4 (Real vs. complex Clifford algebras)**.** Let q and $q^{\mathbb{C}}$ be the non-degenerate quadratic forms on $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, defined by

$$q(x) = \sum_{i=1}^{r} x_i^2 - \sum_{i=r+1}^{s} x_i^2, \qquad q^{\mathbb{C}}(z) = \sum_{i=1}^{n} z_i^2.$$

Consider the Clifford algebras

$$\mathrm{C}\ell_{r,s} = \mathrm{C}\ell(\mathbb{R}^n, q), \qquad \mathbb{C}\ell_n = \mathrm{C}\ell(\mathbb{C}^n, q^{\mathbb{C}}).$$

Show that there exists an isomorphism

$$\mathrm{C}\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_n.$$

In particular,

$$\mathbb{C}\ell_n \cong \mathrm{C}\ell_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{C}\ell_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \ldots \cong \mathrm{C}\ell_{0,n} \otimes_{\mathbb{R}} \mathbb{C}.$$

Hint: Use the universal property of Clifford algebras.

5 (Canonical representation of $\mathbb{C}\ell_n$)**.** Prove that the complex Clifford algebra $\mathbb{C}\ell_{2n}$ is isomorphic to the matrix algebra $\mathbb{C}(2^n)$, and that $\mathbb{C}\ell_{2n+1}$ is isomorphic to $\mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$.

6 (Canonical representation of $\mathrm{C}\ell_3$ and $\mathrm{C}\ell_{3,1}$)**.** Prove that $\mathrm{C}\ell_3$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ and that $\mathrm{C}\ell_{3,1}$ is isomorphic to the matrix algebra $\mathbb{H}(2)$.

7 (★ Exceptional isomorphisms)**.** Prove that there exists an isomorphism $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

8 (★ Exceptional isomorphisms)**.** Prove that there exists a diffeomorphism $\mathrm{SU}(2) \to S^3$, where $S^3 \subset \mathbb{R}^4$ is the set of unit vectors.

9 (★ Exceptional isomorphisms)**.** Prove that there exists an isomorphism $\mathrm{Spin}(3,1) \cong \mathrm{SL}(2, \mathbb{C})$.

10 (Complex volume element)**.** Let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of $\mathbb{R}^n$ (with respect to the standard inner product) and let

$$\omega^{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n \in \mathbb{C}\ell_n$$

be the complex volume element, where "$\cdot$" denotes the product in the Clifford algebra $\mathbb{C}\ell_n$. Show that

$$(\omega^{\mathbb{C}})^2 = 1,$$

and, for all $x \in \mathbb{R}^n$, there holds

$$x \cdot \omega^{\mathbb{C}} = (-1)^{n-1} \omega^{\mathbb{C}} \cdot x.$$

Hint: Use the Clifford algebra relation $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} 1$.

11 (★ 3D Spin representation)**.** Prove that the spin representation of $\mathrm{Spin}(3)$ is the standard representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$.

12 (Spinorial inner product)**.** For n even, let $\rho : \mathbb{C}\ell_n \to \mathrm{End}(\Sigma_n)$ be the unique irreducible representation for the complex Clifford algebra, and for n odd, let $\rho_\pm : \mathbb{C}\ell_n \to \mathrm{End}(\Sigma_n)$ be the two inequivalent irreducible representations, where Σ_n is a complex vector space with

$$\dim_{\mathbb{C}} \Sigma_n = 2^{\lfloor \frac{n}{2} \rfloor}. \tag{2.3}$$

Construct a Hermitian inner product $\langle \cdot, \cdot \rangle$ on Σ_n with respect to which Clifford multiplication is orthogonal, i.e. such that for all $x \in \mathbb{R}^n$ and all $\varphi, \psi \in \Sigma_n$, there holds

$$\langle \rho(x)\varphi, \rho(x)\psi \rangle = \|x\|^2 \langle \varphi, \psi \rangle. \tag{2.4}$$

Show that this inner product is unique, up to scaling by a constant factor.

Hint: See Proposition I.5.16 of [LM] or Proposition 1.35 of [B] for help.

13 (★ 3D Clifford multiplication)**.** Prove that the Clifford multiplication map $c : \mathbb{R}^3 \to \mathrm{End}(\mathbb{C}^2)$ is given by

$$c(x, y, z) = \begin{pmatrix} ix & iy + z \\ iy - z & -ix \end{pmatrix} \in \mathfrak{su}(2).$$

14 (★ Lie algebra representation of $\mathrm{Spin}(n)$)**.** Prove formula (A.1) of Lee-Parker [LP]; that is, prove that the Lie algebra representation

$$\mathfrak{spin}(n) \to \mathrm{End}(V)$$

can be written in terms of Clifford multplication as follows:

$$A \quad \mapsto \quad -\frac{1}{4} A_{ij} c(e^i) c(e^j) = -\frac{1}{8} A_{ij} [c(e^i), c(e^j)],$$

where $\{e^i\}$ is the standard basis of $\mathbb{R}^n$.

Hint: For help, see Proposition I.6.2 of Lawson-Michelsohn [LM] or Theorem 1.25 of Bourguignon et al. [B].

Geometric aspects

Let (M, g) be a Riemannian spin n-manifold and let $\{e_i\}$ be a local orthonormal frame of TM around $p \in M$, with dual coframe $\{e^i\}$.

15 (★ Spin connection is metric compatible)**.** Prove that the spin connection is metric compatible (in either the real or complex case), i.e. prove that for all vector fields X and all spinors φ, ψ on M,

$$X\langle\varphi,\psi\rangle = \langle\nabla_X\varphi,\psi\rangle + \langle\varphi,\nabla_X\psi\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the canonical inner product on the spin bundle.

Hint: Use the fact that Clifford multiplication is skew-Hermitian.

16 (★ Clifford multiplication is covariantly constant)**.** Prove that Clifford multiplication $\rho : \Gamma(\mathbb{C}\ell(M)) \to \mathrm{End}(\Sigma M)$ is covariantly constant with respect to the spin connection, i.e. prove that for all $\alpha \in \Gamma(\mathbb{C}\ell(M))$ and all spinors ψ on M,

$$\nabla_X(\rho(\alpha)\psi) = \rho(\nabla_X^{\mathrm{LC}}\alpha)\psi + \rho(\alpha)\nabla_X\psi.$$

Hint: For help, see Proposition II.4.11 of Lawson-Michelsohn [LM].

17 (★ Spin connection in local coordinates)**.** Prove that the spin connection ∇ can locally be written as

$$\nabla_i\psi = \partial_i\psi + \frac{1}{4}\sum_{j,k=1}^{n}\Gamma_{ij}^{k}c(e_j)c(e_k)\psi,$$

where $\Gamma_{ij}^k := g(\nabla_i e_j, e_k)$ are the Christoffel symbols and c denotes Clifford multiplication.

Hint: See Theorem 2.7 of Bourguignon et al. [B] for help.

18 (★ Spin curvature in terms of Riemannian curvature)**.** Prove that if

$$\mathcal{R}_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

is the curvature of the spin connection, then for any spinor ψ,

$$\mathcal{R}_{X,Y}\psi = \frac{1}{4}\sum_{i,j=1}^{n} g(R_{X,Y}e_i, e_j)c(e_i)c(e_j)\psi,$$

where R is the curvature of the Levi-Civita connection on TM.

Hint: Use Exercise 14. See Theorem 2.7 of Bourguignon et al. [B] for help.

19 (★ Lichnerowicz' vanishing theorem)**.** Prove that if M is closed (i.e. compact with empty boundary) and has positive scalar curvature, then the Dirac operator of (M, g) has trivial kernel.

Hint: Use the Schrödinger-Lichnerowicz formula for the Dirac operator.