Localization Genus of Classifying Spaces

by

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Abstract

We show that for a large class of torsionfree classifying spaces, K-theory filtered ring is an invariant of the genus. We apply this result in two ways. First, we use it to show that the powerseries ring on n indeterminates over the integers admits uncountably many mutually non-isomorphic λ-ring structures. Second, we use it to study the genus of infinite quaternionic projective space. In particular, we describe spaces in the genus of infinite quaternionic projective space which occur as targets of essential maps from infinite complex projective space, and we compute explicitly the homotopy classes of maps in these cases.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction and Statement of Results

In this thesis, we study the genus of the classifying space of a compact connected Lie group, with applications to the studies of λ-rings and classification of maps from infinite complex projective space to spaces in the genus of infinite quaternionic projective space. This introduction contains the statements of all of the main results in this thesis, which are taken from the author's papers [25, 26].

All spaces considered here are nilpotent of finite type, unless the contrary is explicitly stated. For a space X its integral K-theory is denoted, as usual, by $K(X)$. Localization and completion are always meant in the sense of Bousfield and Kan [5].

Let us first recall the relevant definitions. The *genus* of a nilpotent finite type space X, denoted $\text{Genus}(X)$, consists of the homotopy types of all nilpotent finite type spaces Y such that the p-completions of X and Y are homotopy equivalent for all primes p and also their rationalizations are homotopy equivalent. One often speaks of a space rather than its homotopy type when considering genus. McGibbon's survey article [12] is a good reference for results about genus.

1.1 Genus of classifying spaces

We are interested in the genus of the classifying space BG of a compact connected Lie group G. The genus of BG is a very rich set. Indeed, Møller [16] proved that whenever G is a compact connected non-abelian Lie group, the genus of its classifying space BG contains uncountably many distinct homotopy types. Then Notbohm [18] succeeded in finding an algebraic invariant which can tell different spaces in the genus of BG apart. More precisely, Notbohm showed that if X and Y are two spaces in the genus of BG, where G is a fixed simply-connected compact Lie group, then X is homotopy equivalent to Y if and only if $K(X)$ and $K(Y)$ are isomorphic as λ-rings. Here and throughout the rest of the paper, $K(-)$ denotes the complex K-theory with integral coefficients of a space.

With this result of Notbohm in mind, a natural question arises: How much of the difference between these uncountably many K-theory λ-rings is detected by the ring structure?

Our first main result below shows that for a large class of torsionfree classifying spaces, the K-theory ring structure cannot detect the difference between different spaces in the genus of the classifying space BG. Therefore, from the point of view of K-theory, the differences between these spaces lie entirely in the λ-operations, or equivalently, the Adams operations.

Theorem 1.1. *Let X be a simply-connected space of finite type whose integral homology is torsionfree and is concentrated in even dimensions, and whose K-theory filtered ring is a finitely generated powerseries ring over the integers. If Y belongs to the genus of X, then there exists a filtered ring isomorphism from $K(X)$ to $K(Y)$.*

Here, a *filtered ring* is a pair $(R, \{I_n\})$ consisting of:

1. A commutative ring R with unit;
2. A decreasing filtration $R = I_0 \supset I_1 \supset \cdots$ of ideals of R such that $I_i I_j \subset I_{i+j}$ for all $i, j \geq 0$.

A map between two filtered rings is a ring homomorphism which preserves the filtrations. With these maps as morphisms, the filtered rings form a category. Every space Z of the homotopy type of a CW complex gives rise naturally to an object $(K(Z), \{K_n(Z)\})$, which is usually abbreviated to $K(Z)$, in this category. Here $K_n(Z)$ denotes the kernel of the restriction map $K(Z) \to K(Z_{n-1})$, where Z_{n-1} denotes the $(n-1)$-skeleton of Z. Using a different CW structure of Z will not change the filtered ring isomorphism type of $K(Z)$, as can be easily seen by using the Cellular Approximation Theorem.

In Theorem 1.1 the space X could be, for example, $BSp(n)$ for $n \geq 1$, $BSU(n)$ for $n \geq 2$, or any finite product of copies of such spaces and $\mathbf{CP}^\infty$.

It should be noted that Theorem 1.1 has a variant in which complex K-theory (resp. $\mathbf{Z}$) is replaced with orthogonal K-theory, KO^* (resp. $KO^*(\mathrm{pt})$), provided the integral homology of the space X satisfies the more restrictive condition that it is torsionfree and is concentrated in dimensions divisible by 4. For example, the space $X = BSp(n)$ satisfies these conditions. This variant admits a proof which is essentially identical with that of Theorem 1.1 itself.

We will now discuss both algebraic and topological applications of Theorem 1.1.

1.2 Lots of λ-ring structures over powerseries rings

The first application of Theorem 1.1 is about λ-rings. The reader can consult Atiyah-Tall [3] and Knutson [11] for more information about λ-rings .

Let us first recall the relevant definition. A λ-ring is a commutative ring R with unit which is equipped with functions $\lambda^i \colon R \to R$ $(i \geq 0)$, called λ-operations. These operations are required to satisfy the following conditions: For any elements r and s in R one has

- $\lambda^0(r) = 1$
- $\lambda^1(r) = r$, $\lambda^n(1) = 0$ for all $n > 1$
- $\lambda^n(r + s) = \sum_{i=0}^n \lambda^i(r)\lambda^{n-i}(s)$
- $\lambda^n(rs) = P_n(\lambda^1(r), \ldots, \lambda^n(r); \lambda^1(s), \ldots, \lambda^n(s))$
- $\lambda^n(\lambda^m(r)) = P_{n,m}(\lambda^1(r), \ldots, \lambda^{nm}(r))$

Here the P_n and $P_{n,m}$ are certain universal polynomials with integer coefficients; see [3] for details. (Note that in some part of the literature, for example, in Atiyah-Tall, these rings are referred to as *special* λ-rings.) The K-theory $K(X)$ of a space and the complex representation ring $R(G)$ of a group are examples of λ-rings.

Now the classifying space $BSp(n)$ has an n-variable powerseries K-theory filtered ring:

$$K(BSp(n)) \;=\; \mathbf{Z}[[x_1, \ldots, x_n]]. \tag{1.2}$$

Therefore, by combining Theorem 1.1 with the results of Møller [16] and Notbohm [18] in this special case, we obtain the following purely algebraic consequence about λ-rings.

Corollary 1.3. *There exist uncountably many mutually non-λ-isomorphic λ-ring structures over the powerseries ring $\mathbf{Z}[[x_1, \ldots, x_n]]$ on n indeterminates for any integer $n \geq 1$.*

As far as the author is aware, this result is new for any positive integer n.

Remark 1.4. It should be noted that this result is obtained by combining several topological statements about classifying spaces. It would be nice to see a purely algebraic proof of it. The analogous question of how many λ-ring structures the *polynomial ring* $\mathbf{Z}[x]$ supports has been studied by Clauwens [6], who showed that there are essentially only two non-isomorphic λ-ring structures on the polynomial ring $\mathbf{Z}[x]$. The corresponding question for the n-variable ($n > 1$) polynomial ring is still open.

1.3 Genus of infinite quaternionic projective space revisited

We will now use Theorem 1.1 to study various aspects of the genus of infinite quaternionic projective space $\mathbf{HP}^\infty$, considered as a model for the classifying space BS^3.

The original motivation to study spaces in the genus of a classifying space came from Rector's idea to study Lie groups through their classifying spaces. To do that one has to understand how many different loop structures the homotopy type of a Lie group G can carry. Around 1970 Rector [20] considered the case when G is the three-sphere S^3, and classified (with the help of McGibbon [13] at the prime 2) the genus of $\mathbf{HP}^\infty$, all spaces in which are loop structures on S^3. Here we recall Rector's classification of the genus of $\mathbf{HP}^\infty$.

Theorem 1.5 (Rector [20]). *Let X be a space in the genus of $\mathbf{HP}^\infty$. Then for each prime p there exists a homotopy invariant $(X/p) \in \{\pm 1\}$ such that the following statements hold.*

1. *The (X/p) for p primes provide a complete list of homotopy classification invariants for the genus of $\mathbf{HP}^\infty$.*

2. *Any combination of values of the (X/p) can occur. In particular, the genus of $\mathbf{HP}^\infty$ is uncountable.*

3. *The invariant $(\mathbf{HP}^\infty/p)$ is 1 for all primes p.*

4. *The space X has a maximal torus if and only if X is homotopy equivalent to $\mathbf{HP}^\infty$.*

The invariant (X/p) is now known as the *Rector invariant at the prime p*. Actually, for the last statement about the maximal torus, Rector only proved it for the odd primes.

That is, if X has a maximal torus, then (X/p) is equal to 1 for all odd primes p. Then McGibbon [13] proved it for the prime 2 as well. Here X is said to have a maximal torus if there exists a map from $\mathbf{CP}^\infty$, the infinite complex projective space, to X whose homotopy theoretic fiber has the homotopy type of a finite complex.

The question then became whether or not every loop structure on S^3 belongs to the genus of $\mathbf{HP}^\infty$. A positive answer to this question was given by the work of Dwyer, Miller, and Wilkerson [8] in the mid-1980's. They showed that at each prime p, the p-local three-sphere, in strong contrast with the integral three-sphere, has only one loop structure. This homotopical uniqueness result prompted much development in the last fifteen years in the subject of classifying spaces at a prime, the so-called p-compact groups.

Since the work of Rector about three decades ago, our understanding of classifying spaces of compact connected Lie groups has expanded a great deal, thanks to the work of many authors (Adams, Dwyer, Mahmud, Miller, Notbohm, Smith, Wilkerson, Zabrodsky, etc.). Many interesting questions about classifying spaces, however, remain open, and in this section we will try to answer a couple of them in the case when G is S^3. We are particularly interested in questions about maps out of classifying spaces of tori.

Our first application of Theorem 1.1 to the genus of $BS^3 = \mathbf{HP}^\infty$ is to show that it is classified by KO-theory filtered λ-rings.

Theorem 1.6. *Let X and Y be spaces in the genus of $\mathbf{HP}^\infty$. Then X and Y are homotopy equivalent if, and only if, there exists a filtered λ-ring isomorphism from $KO^*(X)$ to $KO^*(Y)$.*

Here, by a *filtered λ-ring*, we mean a filtered ring $(R, \{I_n\})$ which is also a λ-ring such that the ideals I_n are closed under the λ-operations λ^i for $i > 0$.

Actually, a more general result was obtained by Notbohm [18] by using topological realization properties of self homomorphisms of K-theory λ-rings of classifying spaces. Our proof of Theorem 1.6 is independent of Notbohm's and uses directly Rector's classification (Theorem 1.5) and a special case of the KO-theory analogue of Theorem 1.1.

A glance at Rector's classification (Theorem 1.5) reveals the importance of the notion of a maximal torus. Now for a space X in the genus of $\mathbf{HP}^\infty$ which is not homotopy equivalent to $\mathbf{HP}^\infty$, the nonexistence of a maximal torus does not rule out the possibility that there could be some essential (that is, non-nullhomotopic) maps from $\mathbf{CP}^\infty$ to X. So for which spaces X can this happen?

Our next result gives an answer to this question by characterizing spaces in the genus of $\mathbf{HP}^\infty$ which admit essential maps from $\mathbf{CP}^\infty$.

Theorem 1.7. *Let X be a space in the genus of $\mathbf{HP}^\infty$. Then the following statements are equivalent.*

1. *There exists an essential map from $\mathbf{CP}^\infty$ to X.*

2. *There exists a nonzero integer k such that $(X/p) = (k/p)$ for all but finitely many primes p.*

3. *There exists a cofinite set of primes L such that $\mathbf{HP}^\infty$ and X become homotopy equivalent after localization at L.*

Here (k/p) is the number-theoretic Legendre symbol of k, which is defined whenever p does not divide k. If p is odd and if p does not divide k, then $(k/p) = 1$ (resp. -1) if k is a quadratic residue (resp. non-residue) mod p. If $p = 2$ and if k is odd, then $(k/2) = 1$ (resp. -1) if k is a quadratic residue (resp. non-residue) mod 8.

Before discussing related issues, let us first record the following immediate consequence of Theorem 1.7.

Corollary 1.8. *There exist only countably many homotopically distinct spaces in the genus of $\mathbf{HP}^\infty$ which admit essential maps from $\mathbf{CP}^\infty$.*

Indeed, each nonzero integer k can determine only countably many homotopically distinct spaces X in the genus of $\mathbf{HP}^\infty$ satisfying the second condition in Theorem 1.7.

The second condition of Theorem 1.7 gives an arithmetic description of spaces in the genus of $\mathbf{HP}^\infty$ which occur as the targets of essential maps from $\mathbf{CP}^\infty$. Since it involves Rector invariants, it is specific to the genus of $\mathbf{HP}^\infty$ and is not very convenient for generalizations. The last condition of Theorem 1.7, on the other hand, is geometric and is more suitable for possible generalizations of the theorem.

Having characterized spaces in the genus of $\mathbf{HP}^\infty$ which admit nontrivial maps from $\mathbf{CP}^\infty$, we proceed to compute the maps themselves. Now for any space X in the genus of $\mathbf{HP}^\infty$, the K-theory $K(X)$ of X, as a filtered ring, is a powerseries ring $\mathbf{Z}[[b^2 u_X]]$ (see Theorem 1.1), where u_X is some element in $K^4(X)$ and b is the Bott element in $K^{-2}(\mathrm{pt})$.

So if $f\colon \mathbf{CP}^\infty \to X$ is any map, then its induced map in K-theory defines an integer $\deg(f)$, called the *degree of* f, by the equation

$$(1.9) \qquad f^*(b^2 u_X) \;=\; \deg(f)(b\xi)^2 + \text{higher terms in } b\xi,$$

where $b\xi$ is the ring generator in the powerseries ring $K(\mathbf{CP}^\infty) = \mathbf{Z}[[b\xi]]$. Note that $\deg(f)$ is, up to a sign, simply the degree of the induced map of f in integral homology in dimension 4. According to a result of Dehon and Lannes [7], the homotopy class of such a map f is determined by its degree. We will therefore identify such a map with its degree in the sequel. The degrees of a self-map of X, a self-map of the p-localization $\mathbf{HP}^\infty_{(p)}$ of $\mathbf{HP}^\infty$, or a map from $\mathbf{CP}^\infty$ to $\mathbf{HP}^\infty_{(p)}$ can be defined similarly.

To describe the maps from $\mathbf{CP}^\infty$ to $X \in \mathrm{Genus}(\mathbf{HP}^\infty)$ up to homotopy, we need only describe the possible degrees of such maps. Let's first consider the classical case. There is a maximal torus inclusion $i\colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty$ of degree 1, and any other map $f\colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty$ factors through i up to homotopy. A special case of a theorem of Ishiguro, Møller, and Notbohm [9, Theorem 1] says that for any space X in the genus of $\mathbf{HP}^\infty$, the degrees of essential self-maps of X consist of precisely the squares of odd numbers. For the classical case, $X = \mathbf{HP}^\infty$, this result is due to Sullivan [22, p. 58-59]. Therefore, the degrees of essential maps from $\mathbf{CP}^\infty$ to $\mathbf{HP}^\infty$ also consist of precisely the odd squares.

The situation in general is quite similar. Recall that any space X in the genus of $\mathbf{HP}^\infty$ can be obtained as a homotopy inverse limit [5]

$$(1.10) \qquad X \;=\; \mathrm{holim}_q \left\{ \mathbf{HP}^\infty_{(q)} \xrightarrow{r_q} \mathbf{HP}^\infty_{(0)} \xrightarrow{n_q} \mathbf{HP}^\infty_{(0)} \right\}.$$

Here q runs through all primes, r_q is the natural map from the q-localization to the rationalization of $\mathbf{HP}^\infty$, and n_q is an integer relatively prime to q, satisfying $(n_q/q) = (X/q)$. The integer n_2 also satisfies $n_2 \equiv 1 \pmod 4$.

Now if X admits an essential map from $\mathbf{CP}^\infty$, and thus satisfies the second condition in Theorem 1.7 for some nonzero integer k, then the integers n_q can be chosen so that the set $\{n_q\colon q \text{ primes}\}$ contains only finitely many distinct integers. So it makes sense to talk about the least common multiple of the integers n_q, denoted $\mathrm{LCM}(n_q)$. Now we define an integer T_X as

$$(1.11) \qquad T_X \;=\; \min\{\mathrm{LCM}(n_q) :\; X = \mathrm{holim}_q\,(n_q \circ r_q)\}.$$

That is, choose the integers n_q as in (1.10) so as to minimize their least common multiple, and T_X is defined to be $\mathrm{LCM}(n_q)$.

We are now in a position to describe the maps from $\mathbf{CP}^\infty$ to $X \in \mathrm{Genus}(\mathbf{HP}^\infty)$.

Theorem 1.12. *Let X be a space in the genus of $\mathbf{HP}^\infty$ which admits an essential map from $\mathbf{CP}^\infty$ (see Theorem 1.7). Then the following statements hold.*

1. *There exists a map $i_X \colon \mathbf{CP}^\infty \to X$ of degree T_X.*
2. *The map $i_{\mathbf{HP}^\infty} \colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty$ is the maximal torus inclusion.*
3. *Given any map $f \colon \mathbf{CP}^\infty \to X$, there exists a self-map g of X such that f is homotopic to $g \circ i_X$.*
4. *The degrees of essential maps from $\mathbf{CP}^\infty$ to X are precisely the odd squares multiples of T_X.*

It should be noted that the integer T_X does **not** determine the homotopy type of X. For example, consider the spaces X and Y in the genus of $\mathbf{HP}^\infty$ with Rector invariants

$$(1.13) \qquad (X/p) = \begin{cases} 1 & \text{if } p \neq 3 \\ -1 & \text{if } p = 3 \end{cases}, \qquad (Y/p) = \begin{cases} 1 & \text{if } p \neq 5 \\ -1 & \text{if } p = 5. \end{cases}$$

Then, of course, X is not homotopy equivalent to Y because their Rector invariants at the prime 3 are distinct. But it is easy to see that $T_X = 2 = T_Y$.

Theorems 1.7 and 1.12 are closely related to the (non)existence of Adams-Wilkerson type embeddings of finite H-spaces in integral K-theory. As mentioned before, a map $f \colon \mathbf{CP}^\infty \to X \in \mathrm{Genus}(\mathbf{HP}^\infty)$ is essential if and only if $\deg(f)$ is nonzero. Thus, if there exists an essential map from $\mathbf{CP}^\infty$ to $X \in \mathrm{Genus}(\mathbf{HP}^\infty)$, then $K(X)$ can be embedded into $K(\mathbf{CP}^\infty)$ as a sub-λ-ring. The converse is also true. Indeed, a theorem of Notbohm and Smith [19, Theorem 5.2] says that the function

$$\alpha \colon [\mathbf{CP}^\infty, X] \to \mathrm{Hom}_\lambda(K(X), K(\mathbf{CP}^\infty))$$

which sends (the homotopy class of) a map to its induced map in K-theory, is a bijection. (Here $[-,-]$ and $\mathrm{Hom}_\lambda(-,-)$ denote, respectively, sets of homotopy classes of maps between spaces and of λ-ring homomorphisms.) So a λ-ring embedding $K(X) \to K(\mathbf{CP}^\infty)$ must be induced by an essential map from $\mathbf{CP}^\infty$ to X. Therefore, Theorem 1.7 and Corollary 1.8 can be restated in this context as follows.

Theorem 1.14. *Let X be a space in the genus of $\mathbf{HP}^\infty$. Then $K(X)$ can be embedded into $K(\mathbf{CP}^\infty)$ as a sub-λ-ring if, and only if, there exists a nonzero integer k such that $(X/p) = (k/p)$ for all but finitely many primes p. This is true if, and only if, there exists a cofinite set of primes L such that $\mathbf{HP}^\infty$ and X become homotopy equivalent after localization at L.*

In particular, there exist only countably many homotopically distinct spaces X in the genus of $\mathbf{HP}^\infty$ whose K-theory λ-rings can be embedded into that of $\mathbf{CP}^\infty$ as a sub-λ-ring.

Before Theorem 1.14, there is at least one space in the genus of $\mathbf{HP}^\infty$ whose K-theory λ-ring was known to be non-embedable into the K-theory λ-ring of $\mathbf{CP}^\infty$. This example was due to Adams [1, p. 79].

Remark 1.15. Theorems 1.7 and 1.12 can also be regarded as an attempt to understand the set of homotopy classes of maps from X to Y, where $X \in \text{Genus}(BG)$ and $Y \in \text{Genus}(BK)$ with G and K some connected compact Lie groups. This problem, especially the case $G = K = S^3 \times \cdots \times S^3$, was studied extensively by Ishiguro, Møller, and Notbohm [9].

This finishes the presentation of the main results in this thesis.

Organization

The rest of the thesis is organized as follows. Chapter 2 contains the proof of Theorem 1.1. All the preliminary results that are used in the proof of that theorem are stated in §2.1, and the proofs of these preliminary results are given in §2.2.

Chapter 3 contains the proofs of Theorems 1.6, 1.7, and 1.12, one in each section. In §3.1 all the preliminary results that are used in the proof of Theorem 1.6 are stated, and they are proved in §3.1.1 and §3.1.2. The proof of Theorem 1.7, which consists of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), is presented in §3.2. Finally, the proof of Theorem 1.12 is given in §3.3.

Chapter 2

K-theory Filtered Rings

In this chapter we prove Theorem 1.1, which consists of a few lemmas. In §2.1 the lemmas which are used in the proof of Theorem 1.1 are stated. One can see the structure of the proof of the theorem by reading that section. These lemmas are proved in §2.2.

Throughout this chapter we work in the category of filtered rings.

2.1 Proof of Theorem 1.1

We will make use of the following observation, whose proof is a straightforward adaptation of Wilkerson's proof of the classification theorem [23, Theorem I] of spaces of the same n-type for all n. Now let X be as in Theorem 1.1.

Lemma 2.1. *There is a bijection between the following two pointed sets:*

1. *The pointed set of isomorphism classes of filtered rings $(R, \{I_n\})$ with the properties:*
   - *(a) The natural map $R \to \varprojlim_n R/I_n$ is an isomorphism, and*
   - *(b) R/I_n and $K(X)/K_n(X)$ are isomorphic as filtered rings for all $n > 0$.*
2. *The pointed set $\varprojlim_n^1 \operatorname{Aut}(K(X)/K_n(X))$.*

Here $\operatorname{Aut}(-)$ denotes the group of filtered ring automorphisms, and the $\varprojlim^1$ of a tower of not-necessarily abelian groups is as defined in Bousfield-Kan [5].

The two lemmas below will show that, for every space Y in the genus of X, the object $K(Y)$ lies in the first pointed set in Lemma 2.1.

Lemma 2.2. *For every space Y in the genus of X (as in Theorem 1.1), the natural map $K(Y) \to \varprojlim_j K(Y)/K_j(Y)$ is an isomorphism.*

Lemma 2.3. *For every space Y in the genus of X (as in Theorem 1.1), the filtered rings $K(Y)/K_n(Y)$ and $K(X)/K_n(X)$ are isomorphic for all $n > 0$.*

In view of Lemma 2.1, to prove Theorem 1.1 we are only left to show that the classifying object $\varprojlim^1_n \operatorname{Aut}(K(X)/K_n(X))$ is the one point set. To do this, it suffices to show that almost all the structure maps in the tower are surjective. This is shown in the following lemma.

Lemma 2.4. *The maps* $\operatorname{Aut}(K(X)/K_{j+1}(X)) \to \operatorname{Aut}(K(X)/K_j(X))$ *are surjective for all j sufficiently large.*

Therefore, to complete the proof of Theorem 1.1, we only have to prove the three lemmas above, which we will do in the next section. The reader is now offered the opportunity to skip to Chapter 3 on page 25.

2.2 Proof of some lemmas

In this section the proofs of Lemmas 2.2, 2.3, and 2.4, which are used in the proof of Theorem 1.1 (see §2.1), are given.

2.2.1 Proof of Lemma 2.2

Proof. According to [4, 2.5 and 7.1] the natural map from $K(Y)$ to $\varprojlim_j K(Y)/K_j(Y)$ is an isomorphism if the following condition holds:

$$\varprojlim{}^1_r E_r^{p,q} = 0 \quad \text{for all pairs} \quad (p,q). \tag{2.5}$$

Here $E_r^{*,*}$ is the E_r-term in the K^*-Atiyah-Hirzebruch spectral sequence (AHSS) for Y. This condition is satisfied, in particular, when the AHSS degenerates at the E_2-term. Thus, to prove (2.5) it suffices to show that $H^*(Y;\mathbf{Z})$ is concentrated in even dimensions, since in that case there is no room for differentials in the AHSS. So pick an odd integer N. We must show that

$$H^N(Y;\mathbf{Z}) \cong 0. \tag{2.6}$$

By the Universal Coefficient Theorem it suffices to show that the integral homology of Y is torsionfree and is concentrated in even dimensions, which hold because Y lies in the genus of X. □

This finishes the proof of Lemma 2.2.

2.2.2 Proof of Lemma 2.3

Proof. We have to show that for each $j > 0$ there is an isomorphism of filtered rings

$$K(Y)/K_j(Y) \cong K(X)/K_j(X). \tag{2.7}$$

It follows from the hypothesis that for each $j > 0$, the filtered ring $K(Y)/K_j(Y)$ belongs to $\mathrm{Genus}(K(X)/K_j(X))$, where $\mathrm{Genus}(R)$ for a filtered ring R is defined in terms of $R \otimes \mathbf{Q}$ and $R \otimes \widehat{\mathbf{Z}}$ in exactly the same way the genus of a space is defined. To finish the proof we will adapt two results of Wilkerson [24, 3.7 and 3.8], which we now recall.

- For a nilpotent finite type space X, Wilkerson showed that there is a surjection

  $$\sigma\colon \mathrm{Caut}(\hat{X}_0) \to \mathrm{Genus}(X),$$

  where $\hat{X}_0$ is the rationalization of the formal completion of X. Notice that each homotopy group $\pi_*(\hat{X}_0)$ is a $\mathbf{Q} \otimes \widehat{\mathbf{Z}}$-module, and $\mathrm{Caut}(\hat{X}_0)$ is by definition the group of homotopy classes of self-homotopy equivalences of $\hat{X}_0$ whose induced maps on homotopy groups are $\mathbf{Q} \otimes \widehat{\mathbf{Z}}$-module maps.

- Note that the definitions $\mathrm{Genus}(-)$ and $\mathrm{Caut}(-)$ also make sense in both the categories of nilpotent groups and of filtered rings. For instance, if $R = (R, \{I^n\})$ is a filtered ring, then $\mathrm{Caut}(R \otimes \mathbf{Q} \otimes \widehat{\mathbf{Z}})$ is the group of filtered ring automorphisms of $R \otimes \mathbf{Q} \otimes \widehat{\mathbf{Z}}$ which are also $\mathbf{Q} \otimes \widehat{\mathbf{Z}}$-module maps. Now if A is a finitely generated abelian group, then Wilkerson showed that for any class $[\varphi] \in \mathrm{Caut}(A \otimes \mathbf{Q} \otimes \widehat{\mathbf{Z}})$, the image $\sigma([\varphi])$ is isomorphic to A as groups; that is, the image of σ is constant at A.

It is straightforward to adapt Wilkerson's proofs of these results to show that for each j the map

$$\sigma\colon\ \mathrm{Caut}((K(X)/K_j(X)) \otimes \mathbf{Q} \otimes \widehat{\mathbf{Z}}) \to \mathrm{Genus}(K(X)/K_j(X))$$

is surjective and that the image of σ is constant at $K(X)/K_j(X)$. In other words, the genus of $K(X)/K_j(X)$ is the one-point set. Therefore, $K(Y)/K_j(Y)$ is isomorphic to $K(X)/K_j(X)$.

This finishes the proof of Lemma 2.3. □

2.2.3 Proof of Lemma 2.4

Proof. First note that by hypothesis the K-theory filtered ring of X has the form

$$K(X) = \mathbf{Z}[[c_1, \ldots, c_n]] \tag{2.8}$$

in which the generators c_i are algebraically independent over $\mathbf{Z}$ and $K_j(X)$ is the ideal generated by the monomials of filtrations at least j. Suppose that d_i is the largest integer k for which c_i lies in filtration k. Let N be the integer $\max\{d_i \colon 1 \leq i \leq n\} + 1$ (or any integer that is strictly greater than d_i for $i = 1, \ldots, n$). We will show that the structure maps $\mathrm{Aut}(K(X)/K_{j+1}(X)) \to \mathrm{Aut}(K(X)/K_j(X))$ are surjective for all $j > N$.

So fix an integer $j > N$ and pick a filtered ring automorphism σ of $K(X)/K_j(X)$. We must show that σ can be lifted to a filtered ring automorphism of $K(X)/K_{j+1}(X)$. For $1 \leq i \leq n$ pick any lift of the element $\sigma(c_i)$ to $K(X)/K_{j+1}(X)$ and call it $\hat{\sigma}(c_i)$. Since there are no relations among the c_i in $K(X)$, it is easy to see that $\hat{\sigma}$ extends to a well-defined filtered ring endomorphism of $K(X)/K_{j+1}(X)$, and it will be a desired lift of σ once it is shown to be bijective.

To show that $\hat{\sigma} \colon K(X)/K_{j+1}(X) \to K(X)/K_{j+1}(X)$ is surjective, it suffices to show that the image of each c_i in $K(X)/K_{j+1}(X)$ lies in the image of $\hat{\sigma}$, since $K(X)/K_{j+1}(X)$ is generated as a filtered ring by the images of the c_i. So fix an integer i with $1 \leq i \leq n$. We know that there exists an element $g_i \in K(X)/K_j(X)$ such that

$$\sigma(g_i) \;=\; c_i. \tag{2.9}$$

Pick any lift of g_i to $K(X)/K_{j+1}(X)$, call it g_i again, and observe that (2.9) implies that

$$\hat{\sigma}(g_i) \;=\; c_i + \alpha_i \tag{2.10}$$

in $K(X)/K_{j+1}(X)$ for some element $\alpha_i \in K_j(X)/K_{j+1}(X)$. We will alter g_i to obtain a $\hat{\sigma}$-pre-image of c_i as follows.

Observe that the ideal $K_j(X)/K_{j+1}(X)$ is generated by certain monomials in $c_1, \ldots, c_n$. Namely, the monomials

$$\mathbf{c}^{\mathbf{i}} \;=\; c_1^{i_1} \cdots c_n^{i_n}, \quad \mathbf{i} \;=\; (i_1, \ldots, i_n) \in J_j \tag{2.11}$$

where J_j is the set of ordered n-tuples $\mathbf{i} = (i_1, \ldots, i_n)$ of nonnegative integers satisfying $\mathbf{d} \cdot \mathbf{i} = \sum_{l=1}^{n} d_l i_l = j$. Thus, for every element $\mathbf{i}$ in the set J_j, there exists a corresponding integer $a_{\mathbf{i}}$ such that we can write the element α_i as the sum

$$\alpha_i \;=\; \sum_{\mathbf{i} \in J_j} a_{\mathbf{i}}\, \mathbf{c}^{\mathbf{i}}. \tag{2.12}$$

Now define the element $\bar{g}_i$ in $K(X)/K_{j+1}(X)$ by the formula

$$\bar{g}_i \;\equiv\; g_i \;-\; \sum_{\mathbf{i} \in J_j} a_{\mathbf{i}}\, \mathbf{g}^{\mathbf{i}}, \quad \text{where } \mathbf{g}^{\mathbf{i}} = g_1^{i_1} \cdots g_n^{i_n}. \tag{2.13}$$

We claim that $\bar{g}_i$ is a $\hat{\sigma}$-pre-image of c_i. That is, we claim that

$$\hat{\sigma}(\bar{g}_i) \;=\; c_i \quad \text{in} \quad K(X)/K_{j+1}(X). \tag{2.14}$$

In view of (2.10), (2.12), and (2.13), it clearly suffices to prove the following equality for each element $\mathbf{i}$ in J_j:

$$\hat{\sigma}\left(\mathbf{g}^{\mathbf{i}}\right) \;=\; \mathbf{c}^{\mathbf{i}} \quad \text{in } K(X)/K_{j+1}(X). \tag{2.15}$$

Now in the quotient $K(X)/K_{j+1}(X)$, one computes

$$\begin{aligned}
\hat{\sigma}\left(\mathbf{g}^{\mathbf{i}}\right) \;&=\; \prod_{j=1}^{n} \hat{\sigma}(g_j)^{i_j} \\
&=\; \prod_{j=1}^{n} (c_j + \alpha_j)^{i_j} \quad \text{by (2.10)} \\
&=\; \mathbf{c}^{\mathbf{i}} \;+\; (\text{terms of filtrations } > j) \\
&=\; \mathbf{c}^{\mathbf{i}}.
\end{aligned}$$

This proves (2.15), and hence (2.14), and therefore $\hat{\sigma}$ is surjective.

It remains to show that $\hat{\sigma}$ is injective. Since any surjective endomorphism of a finitely generated abelian group is also injective and since $K(X)/K_{j+1}(X)$ is a finitely generated abelian group, it follows that $\hat{\sigma}$ is injective as well. Thus, $\hat{\sigma}$ is a filtered-ring automorphism of $K(X)/K_{j+1}(X)$ and is a lift of σ.

This finishes the proof of Lemma 2.4. □

Chapter 3

Genus of infinite quaternionic projective space

In this chapter, we prove Theorems 1.6, 1.7, and 1.12, in this order. The proof of each theorem occupies one section. In §3.1 all the preliminary results that are used in the proof of Theorem 1.6 are stated, and they are proved in §3.1.1 and §3.1.2. The proof of Theorem 1.7, which consists of the implications (1) ⇒ (2) ⇒ (3) ⇒ (1), is presented in §3.2. Finally, the proof of Theorem 1.12 is given in §3.3.

3.1 Proof of Theorem 1.6

In this section we prove Theorem 1.6. The arguments in this section, especially Lemma 3.4 below, are inspired by Rector's [21, §4].

Let KO^* denote orthogonal K-theory. We begin by noting that an argument entirely similar to the proof of Theorem 1.1 implies that whenever X belongs to the genus of $\mathbf{HP}^\infty$, one has that

$$KO^*(X) \;\cong\; KO^*[[x]] \tag{3.1}$$

as filtered rings, where x is an element in $KO^4_4(X)$ and is a representative of an integral generator x_4 in $H^4(X;\mathbf{Z}) = E_2^{4,0}$ in the KO^*-Atiyah-Hirzebruch spectral sequence for X. Here $KO^a_b(X)$ denotes the subgroup of $KO^a(X)$ consisting of elements u which restrict to 0 under the natural map $KO^a(X) \to KO^a(X_{b-1})$.

In preparation for the proof of Theorem 1.6, we first recall the relevant notations and definitions regarding Rector's classification (Theorem 1.5) of the genus of $\mathbf{HP}^\infty$.

Let X be a space in the genus of $\mathbf{HP}^\infty$. Let $\xi \in KO^{-4}(\text{pt})$ and $b_R \in KO^{-8}(\text{pt})$ be the generators so that $\xi^2 = 4b_R$. As usual, denote by ψ^k $(k = 1, 2, \ldots)$ the Adams operations. Since ΩX is homotopy equivalent to S^3, it follows as in [21, §4] that there exists an integer a, depending on the choice of the representative x, such that the following statements hold.

1. $\psi^2(\xi x) \;=\; 4\xi x + 2ab_R x^2 \pmod{KO_9^0(X)}$.

2. The integer a is well-defined (mod 24). This means that if x' is another representative of x_4 with corresponding integer a', then $a \equiv a' \pmod{24}$, and if x_4 is replaced with $-x_4$, then a will be replaced with $-a$. We can, and we will, therefore write $a(X)$ for a.

3. $a(X) \;\equiv\; \pm 1, \pm 5, \pm 7$, or $\pm 11 \pmod{24}$.

The last condition above follows from the examples constructed by Rector in [21, §5] and James' result [10] which says that there are precisely eight homotopy classes of homotopy-associative multiplications on S^3. These eight classes can be divided into four pairs with each pair consisting of a homotopy class of multiplication and its inverse.

The Rector invariants (X/p) for odd primes p are defined as follows [20]. The Adem relation $P^1P^1 = 2P^2$ implies that

$$P^1\,\overline{x}_4 \;=\; \pm\, 2\overline{x}_4^{(p+1)/2} \tag{3.2}$$

in $H^*(X; \mathbf{Z}/p)$, where $\overline{x}_4$ is the mod p reduction of the integral generator x_4. Then $(X/p) \in \{\pm 1\}$ is defined as the sign on the right-hand side of (3.2).

The Rector invariant $(X/2)$ and a canonical choice of orientation of the integral generator x_4 are given as follows. Using the (mod 24) integer $a(X)$, define

$$((X/2), (X/3)) = \begin{cases} (1,1) & \text{if } a(X) \equiv \pm 1 \bmod 24; \\ (1,-1) & \text{if } a(X) \equiv \pm 5 \bmod 24; \\ (-1,1) & \text{if } a(X) \equiv \pm 7 \bmod 24; \\ (-1,-1) & \text{if } a(X) \equiv \pm 11 \bmod 24. \end{cases} \tag{3.3}$$

The orientation of x_4 is then chosen so that $(X/3)$ is as given in (3.6). This definition of the Rector invariants coincides with the original one (cf. [12, §9]).

From now on in this section, let X and Y be two fixed spaces in the genus of $\mathbf{HP}^\infty$. As explained above, $KO^*(X) = KO^*[[x]]$ and $KO^*(Y) = KO^*[[y]]$ with $x \in KO_4^4(X)$ and $y \in KO_4^4(Y)$ representing, respectively, the integral generators $x_4 \in H^4(X;\mathbf{Z})$ and $y_4 \in H^4(Y;\mathbf{Z})$.

Proof of Theorem 1.6. In view of Rector's Theorem 1.5, it suffices to show that if there exists a filtered λ-ring isomorphism σ from $KO^*(X)$ to $KO^*(Y)$, then $a(X) \equiv \pm a(Y)$ (mod 24) and $(X/p) = (Y/p)$ for all odd primes p, which hold by Lemmas 3.4 and 3.5 below. □

Lemma 3.4. *If there exists a filtered λ-ring isomorphism σ from $KO^*(X)$ to $KO^*(Y)$, then $a(X) \equiv \pm a(Y)$ (mod 24).*

Lemma 3.5. *If there exists a filtered λ-ring isomorphism σ from $KO^*(X)$ to $KO^*(Y)$, then $(X/p) = (Y/p)$ for each odd prime p.*

The proofs of these two lemmas are given below. The reader is now offered the opportunity to skip to §3.2 on page 30.

3.1.1 Proof of Lemma 3.4

Proof. Since σ is a filtered ring isomorphism, we have

$$\sigma(\xi x) = \varepsilon \xi y + \sigma_2 b_R y^2 \pmod{KO_9^0(Y)} \tag{3.6}$$

for some integer σ_2 and $\varepsilon \in \{\pm 1\}$. Computing modulo $KO_9^0(Y)$ we have

$$4\sigma(b_R x^2) = \sigma(\xi x)^2 = \xi^2 y^2 = 4 b_R y^2, \tag{3.7}$$

and therefore

$$\sigma(b_R x^2) = b_R y^2 \pmod{KO_9^0(Y)}. \tag{3.8}$$

First we claim that there is an equality

$$a(X) = 6\sigma_2 + \varepsilon a(Y). \tag{3.9}$$

To prove (3.9) we will compute both sides of

$$\sigma\psi^2(\xi x) = \psi^2\sigma(\xi x) \pmod{KO_9^0(Y)}. \tag{3.10}$$

Computing modulo $KO_9^0(Y)$ we have, on the one hand,

$$\begin{aligned}\sigma\psi^2(\xi x) &= \sigma(4\xi x + 2a(X)b_R x^2)\\ &= 4(\varepsilon\xi y + \sigma_2 b_R y^2) + 2a(X)b_R y^2 \quad \text{(by (3.6) and (3.8))}\\ &= 4\varepsilon\xi y + (4\sigma_2 + 2a(X))b_R y^2.\end{aligned}$$

On the other hand, still working modulo $KO_9^0(Y)$, we have

$$\begin{aligned}\psi^2\sigma(\xi x) &= \varepsilon\psi^2(\xi y) + \sigma_2\psi^2(b_R y^2) \quad \text{(by (3.6))}\\ &= \varepsilon(4\xi y + 2a(Y)b_R y^2) + \sigma_2(2^4 b_R y^2)\\ &= 4\varepsilon\xi y + (16\sigma_2 + 2\varepsilon a(Y))b_R y^2.\end{aligned}$$

Equation (3.9) now follows by equating the coefficients of $b_R y^2$.

In view of (3.9), to finish the proof of Lemma 3.4 it is enough to establish the equality

(3.11) $$\sigma_2 \equiv 0 \pmod 4.$$

To prove (3.11), note that since σ is a KO^*-module map, we have $\xi\sigma(x) = \sigma(\xi x)$. Since σ is a filtered ring isomorphism, we also have

(3.12) $$\sigma(x) = \varepsilon' y + \sigma_2'\xi y^2 \pmod{KO_9^4(Y)}$$

for some integer σ_2' and $\varepsilon' \in \{\pm 1\}$. Therefore, computing modulo $KO_9^0(Y)$ we have

$$\begin{aligned}\xi\sigma(x) &= \varepsilon'\xi y + \sigma_2'\xi^2 y^2 \quad \text{(by (3.12))}\\ &= \varepsilon'\xi y + 4\sigma_2' b_R y^2\\ &= \varepsilon\xi y + \sigma_2 b_R y^2.\end{aligned}$$

In particular, by equating the coefficients of $b_R y^2$, we obtain $\sigma_2 = 4\sigma_2'$, thereby proving (3.11).

This completes the proof of Lemma 3.4. □

3.1.2 Proof of Lemma 3.5

Proof. It follows from Theorem 1.1 that $K^*(X) \cong K^*[[u_x]]$ with $u_x \in K_4^4(X)$ a representative of the integral generator $x_4 \in H^4(X;\mathbf{Z}) = E_2^{4,0}$ in the K^*-Atiyah-Hirzebruch spectral sequence. Moreover, we may choose u_x so that $c(x) = u_x$, where $c\colon KO^*(X) \to K^*(X)$ is the complexification map. Similar remarks apply to Y so that $K^*(Y) \cong K^*[[u_y]]$.

Now denote by $b \in K^{-2}(\mathrm{pt})$ the Bott element and let p be a fixed odd prime. We first claim that

$$\psi^p(b^2u_x) = (b^2u_x)^p + 2(X/p)\,p\,(b^2u_x)^{(p+1)/2} + p\,w_x + p^2x_0 \tag{3.13}$$

for some $w_x \in K^0_{2p+3}(X)$ and some $x_0 \in K^0_4(X)$. To see this, note that since $b^2u_x \in K^0_4(X)$, it follows from Atiyah's theorem [2, 5.6] that

$$\psi^p(b^2u_x) = (b^2u_x)^p + p\,x_1 + p^2x_0 \tag{3.14}$$

for some $x_i \in K^0_{4+2i(p-1)}(X)$ $(i = 0, 1)$. Moreover, $\overline{x}_1 = P^1\,\overline{b^2u_x}$, where $\overline{z}$ is the mod p reduction of z. Thus, to prove (3.13) it is enough to show that

$$x_1 = 2(X/p)(b^2u_x)^{(p+1)/2} + w_x + p\,z_x \tag{3.15}$$

for some $w_x \in K^0_{2p+3}(X)$ and some $z_x \in K^0_{2p+2}(X)$. Now in $H^*(X;\mathbf{Z}) \otimes \mathbf{Z}/p$ we have

$$\overline{x}_1 = P^1\,\overline{b^2u_x} = P^1\,\overline{x}_4 = 2(X/p)\,\overline{x}_4^{(p+1)/2} = 2(X/p)\,\overline{b^2u_x}^{(p+1)/2}, \tag{3.16}$$

from which (3.15) follows immediately. As remarked above, this also establishes (3.13).

Now the λ-ring isomorphism σ induces, via the complexification map c, a λ-ring isomorphism $\sigma_c\colon K^*(X) \cong K^*(Y)$. By composing σ_c with a suitable λ-ring automorphism of $K^*(Y)$ if necessary, we obtain a λ-ring isomorphism $\alpha\colon K^*(X) \cong K^*(Y)$ with the property that

$$\alpha(b^2u_x) = b^2u_y + \text{ higher terms in } b^2u_y. \tag{3.17}$$

Using (3.13) and (3.17) it is then easy to check that

$$\alpha\psi^p(b^2u_x) = 2(X/p)\,p\,(b^2u_y)^{(p+1)/2} \pmod{\langle K^0_{2p+3}(Y),\, p^2\rangle} \tag{3.18}$$

and

$$\psi^p\alpha(b^2u_x) = 2(Y/p)\,p\,(b^2u_y)^{(p+1)/2} \pmod{\langle K^0_{2p+3}(Y),\, p^2\rangle}. \tag{3.19}$$

Since $\alpha\psi^p = \psi^p\alpha$ it follows from (3.18) and (3.19) that

$$2(X/p)\,p \equiv 2(Y/p)\,p \pmod{p^2}, \tag{3.20}$$

or, equivalently,

$$2(X/p) \equiv 2(Y/p) \pmod{p}. \tag{3.21}$$

But p is assumed odd, and so $(X/p) \equiv (Y/p) \pmod{p}$. Hence $(X/p) = (Y/p)$, as desired.

This finishes the proof of Lemma 3.5. □

3.2 Proof of Theorem 1.7

In this section the proof of Theorem 1.7 is given.

Recall that the complex K-theory of $\mathbf{CP}^\infty$ as a filtered λ-ring is given by $K(\mathbf{CP}^\infty) = \mathbf{Z}[[b\xi]]$ for some $\xi \in K_2^2(\mathbf{CP}^\infty)$, where $b \in K^{-2}(\mathrm{pt})$ is the Bott element. The Adams operations on the generator are given by

$$\psi^r(b\xi) \;=\; (1+b\xi)^r - 1 \quad (r = 1, 2, \ldots). \tag{3.22}$$

Fix a space X in the genus of $\mathbf{HP}^\infty$ and write $\mathbf{Z}[[b^2u_X]]$ for its K-theory filtered ring (see Theorem 1.1).

We will prove Theorem 1.7 by proving the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Each implication is contained in one subsection below.

3.2.1 Proof of (1) implies (2)

This part of Theorem 1.7 is contained in the next Lemma.

Lemma 3.23. *Let p be an odd prime and k be a nonzero integer relatively prime to p. If there exists an essential map $f\colon \mathbf{CP}^\infty \to X$ of degree k, then $(X/p) = (k/p)$.*

Proof. We will compare the coefficients of $(b\xi)^{p+1}$ in the equation

$$f^*\psi^p\left(b^2u_X\right) \;=\; \psi^p f^*\left(b^2u_X\right) \quad \left(\mathrm{mod}\ \langle K^0_{2p+3}(\mathbf{CP}^\infty), p^2\rangle\right). \tag{3.24}$$

Working modulo $K^0_{2p+3}(\mathbf{CP}^\infty)$ and p^2, it follows from (1.9) and (3.13) that

$$\begin{aligned} f^*\psi^p\left(b^2u_X\right) &= 2\,(X/p)\,p\left(kb^2\xi^2\right)^{(p+1)/2} \\ &= 2\,(X/p)\,p\,k^{(p+1)/2}(b\xi)^{p+1}. \end{aligned}$$

Similarly, still working modulo $K^0_{2p+3}(\mathbf{CP}^\infty)$ and p^2, it follows from (1.9) and (3.22) that

$$\begin{aligned} \psi^p f^*\left(b^2u_X\right) &= k\psi^p(b^2\xi^2) \\ &= k\psi^p(b\xi)^2 \\ &= 2pk(b\xi)^{p+1}. \end{aligned}$$

Thus, we obtain the congruence relation

$$2\,(X/p)\,p\,k^{(p+1)/2} \;\equiv\; 2pk \quad (\mathrm{mod}\ p^2). \tag{3.25}$$

Since (k/p) is congruent to $k^{(p-1)/2} \pmod p$ (see, for example, [17, Theorem 3.12]) and since p is odd and relatively prime to k, (3.25) is equivalent to the congruence relation

$$(X/p)\,(k/p) \equiv 1 \pmod p. \tag{3.26}$$

Hence $(X/p) = (k/p)$, as desired.

This finishes the proof of Lemma 3.23. □

This shows (1) implies (2) in Theorem 1.7.

3.2.2 Proof of (2) implies (3)

Suppose that there exists a nonzero integer k such that $(X/p) = (k/p)$ for all primes p, except possibly $p_1, \ldots, p_s$. The prime factors of k are among the p_i. Let L be the cofinite set consisting of all primes except the p_i, $1 \le i \le s$. We will show that $\mathbf{HP}^\infty$ and X become homotopy equivalent after localization at L.

First note that for any space Y in the genus of $\mathbf{HP}^\infty$ and for any subset I of primes, the I-localization of Y can be obtained as

$$Y_{(I)} = \operatorname{holim}_{q\in I} \left\{ \mathbf{HP}^\infty_{(q)} \xrightarrow{n_q \circ r_q} \mathbf{HP}^\infty_{(0)} \right\}. \tag{3.27}$$

In particular, we have

$$X_{(L)} = \operatorname{holim}_{q\in L} \left\{ \mathbf{HP}^\infty_{(q)} \xrightarrow{k \circ r_q} \mathbf{HP}^\infty_{(0)} \right\} \tag{3.28}$$

and

$$\mathbf{HP}^\infty_{(L)} = \operatorname{holim}_{q\in L} \left\{ \mathbf{HP}^\infty_{(q)} \xrightarrow{r_q} \mathbf{HP}^\infty_{(0)} \right\}. \tag{3.29}$$

Now for each prime $q \in L$, let f_q be a self-map of $\mathbf{HP}^\infty_{(q)}$ of degree k^{-1}. Since k is a q-local unit (because q does not divide k), it is easy to see that each f_q is a homotopy equivalence. Moreover, the two maps

$$r_q,\ k \circ r_q \circ f_q \colon \mathbf{HP}^\infty_{(q)} \to \mathbf{HP}^\infty_{(0)} \tag{3.30}$$

coincide. Therefore, the maps f_q $(q \in L)$ glue together to yield a map

$$f \colon \mathbf{HP}^\infty_{(L)} \to X_{(L)} \tag{3.31}$$

which is a homotopy equivalence, since each f_q is. This shows (2) implies (3) in Theorem 1.7.

3.2.3 Proof of (3) implies (1)

Suppose that there exists a cofinite set of primes L such that $\mathbf{HP}^{\infty}_{(L)}$ and $X_{(L)}$ are homotopy equivalent. Write $p_1, \ldots, p_s$ for the primes not in L, and write r_L for the natural map from $X_{(L)}$ to $\mathbf{HP}^{\infty}_{(0)}$.

To construct an essential map from $\mathbf{CP}^{\infty}$ to X, first note that X can be constructed as the homotopy inverse limit of the diagram

$$\mathbf{HP}^{\infty}_{(p_i)} \xrightarrow{r_{p_i}} \mathbf{HP}^{\infty}_{(0)} \xrightarrow{n_{p_i}} \mathbf{HP}^{\infty}_{(0)} \xleftarrow{r_L} X_{(L)} \tag{3.32}$$

in which i runs from 1 to s.

For each i, $1 \leq i \leq s$, let f_{p_i} be a map from $\mathbf{CP}^{\infty}$ to $\mathbf{HP}^{\infty}_{(p_i)}$ of degree M/n_{p_i}, where $M = \prod_{j=1}^{s} n_{p_j}$. Also, let f_L denote a map from $\mathbf{CP}^{\infty}$ to $X_{(L)}$ of degree M, which exists because $X_{(L)}$ has the same homotopy type as $\mathbf{HP}^{\infty}_{(L)}$. It is then easy to see that the two maps

$$r_L \circ f_L,\ n_{p_i} \circ r_{p_i} \circ f_{p_i} \colon \mathbf{CP}^{\infty} \to \mathbf{HP}^{\infty}_{(0)} \tag{3.33}$$

coincide for any $1 \leq i \leq s$. Therefore, the maps f_{p_i} $(1 \leq i \leq s)$ and f_L glue together to yield an essential map

$$f \colon \mathbf{CP}^{\infty} \to X \tag{3.34}$$

through which all the maps f_{p_i} and f_L factor.

This shows (3) implies (1) in Theorem 1.7.

3.3 Proof of Theorem 1.12

In this section we prove Theorem 1.12.

Fix a space X in the genus of $\mathbf{HP}^{\infty}$ which admits an essential map from $\mathbf{CP}^{\infty}$.

First we note that part (2) follows from the discussion preceding Theorem 1.12, since it is obvious that the integer $T_{\mathbf{HP}^{\infty}}$ is 1.

Since any essential self-map of X is a rational equivalence, part (4) is an immediate consequence of parts (1) and (3) and a result of Ishiguro, Møller, and Notbohm [9, Theorem 1] which says that the degrees of essential self-maps of X are precisely the odd squares.

Now we consider part (1). Suppose that the integers n_q as in (1.10) are chosen so that there are only finitely many distinct integers in the set $\{n_q \colon q \text{ primes}\}$ and that T_X

is their least common multiple (see (1.11) on page 15 for the definition of T_X). Denote by $l_q\colon \mathbf{HP}^\infty \to \mathbf{HP}^\infty_{(q)}$ the q-localization map and by $i\colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty$ the maximal torus inclusion. A self-map of $\mathbf{CP}^\infty$ of degree m on $H^2(\mathbf{CP}^\infty; \mathbf{Z})$ is simply denoted by m. Now for each prime q define a map $f_q\colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty_{(q)}$ to be the composition

$$\mathbf{CP}^\infty \xrightarrow{M/n_q} \mathbf{CP}^\infty \xrightarrow{i} \mathbf{HP}^\infty \xrightarrow{l_q} \mathbf{HP}^\infty_{(q)}. \tag{3.35}$$

It is then easy to see that the two maps

$$n_q \circ r_q \circ f_q,\ n_{q'} \circ r_{q'} \circ f_{q'}\colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty_{(0)} \tag{3.36}$$

coincide for any two primes q and q'. Therefore, the maps f_q glue together to yield an essential map

$$f\colon \mathbf{CP}^\infty \to X \tag{3.37}$$

through which every map f_q factors. The map f has degree T_X because its induced map in rational cohomology in dimension 4 does.

Finally, for part (3), suppose that $f\colon \mathbf{CP}^\infty \to X$ is a map. Write $f_p\colon \mathbf{CP}^\infty \to \mathbf{HP}^\infty_{(p)}$ for the component map of f corresponding to the prime p. That is, f_p is the composition

$$\mathbf{CP}^\infty \xrightarrow{f} X \to \mathbf{HP}^\infty_{(p)} \tag{3.38}$$

where the second map is the natural map arising from the construction of X. Then for any prime p we have the equality

$$\deg(f) \;=\; n_p \deg(f_p). \tag{3.39}$$

Since each n_p divides $\deg(f)$, so does their least common multiple T_X. Moreover, by writing $(i_X)_p$ for the component map of i_X corresponding to the prime p, (3.39) implies that for any prime p we have the equalities

$$\frac{\deg(f_p)}{\deg(i_X)_p} \;=\; \frac{\deg(f)/n_p}{T_X/n_p} \;=\; \frac{\deg(f)}{T_X}. \tag{3.40}$$

Since there are self-maps of $\mathbf{HP}^\infty_{(q)}$ (q any prime) and $\mathbf{HP}^\infty_{(0)}$ of degree $\deg(f)/T_X$, one can construct a self-map g of X such that $\deg(g)$ is equal to $\deg(f)/T_X$ and that f is homotopic to $g \circ i_X$. This proves part (3).

The proof of Theorem 1.12 is complete.

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