A chromatic spectral sequence to study Ext over the Steenrod algebra

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Contents

List of Figures

Abstract

We develop a viewpoint for studying Ext_A (where A is the mod 2 Steenrod algebra), analogous to the chromatic viewpoint for studying spectra. The philosophy throughout is to replace v_n information in the chromatic picture with P_t^s information in the Steenrod algebra picture. The starting point is the *Margolis chromatic spectral sequence*, developed by Margolis; this decomposes $\mathrm{Ext}_A(M, N)$ into P_t^s information, for all $s < t$. Input for this is provided by the *Margolis Adams spectral sequence* and sequences of Bockstein spectral sequences. The former was constructed by Margolis in some unpublished notes, and he proved convergence in some cases. We give an account of this work, and a discussion of approaches to proving convergence in all cases.

The latter are new; one application of them computes, for example, $\mathrm{Ext}_A(M, \mathbf{F}_2)$ from $\mathrm{Ext}_A(M, A(n))$ under a vanishing line condition on M (which follows from M having no P_t^s-homology when $|P_t^s| \leq |P_{n+1}^0|$). Constructing the Bockstein spectral sequences involves a careful analysis of Mitchell's/Smith's A-module structure on $A(n)$.

We also work out one example in great detail, the calculation of $\{\mathbf{F}_2, \mathbf{F}_2\}_{A(1)}$ (and hence $\mathrm{Ext}_{A(1)}(\mathbf{F}_2, \mathbf{F}_2)$).

Acknowledgments

So look—it's pretty ridiculous for anyone to try to thank all of the people who have helped, directly or indirectly, in producing a thesis. Either the list will be much too long, or lots of people will be left out. On the other hand, MIT's "Specifications for Thesis Preparation" suggest a short biographical note and make no mention of an "Acknowledgments" page, so maybe I'll just do that instead. So here goes.

Born on the Russo-Japanese border at the start of the rainy season in 1927, the author has spent his life in pursuit of the elusive—what's that? You really *do* want acknowledgments? Well, don't come running to me when you find that I've left you out.

Mathematically, I should certainly acknowledge the math department at Swarthmore College; if nothing else, they guided me from the false road of physics onto the path of true enlightenment. Secondly, the topological community at MIT needs lots of thanks; if it hadn't been for them, I might have gone into combinatorics and be at Bell Labs now, making $50,000 a year. I've had many good mathematical (and non-mathematical) conversations with Tom Hunter and Hal Sadofsky, and I should also acknowledge Kathryn Lesh and Matthew Ando (and David Blanc and Jim Turner and . . .).

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Non-mathematically, I could get really carried away thanking people, so I won't; I'll just mention David Van Stone, who's been a good friend and housemate, all of my other housemates through the years, and the math grad students at MIT for providing frisbee games and political discussions, giving me lots of distractions from my work.

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Chapter 1

Introduction

Our goal is to describe a method for studying Ext over the mod 2 Steenrod algebra which is analogous to the chromatic picture (see [MRW]); this is done using tools developed by Margolis (see [Ma1]). Our method will describe Ext_A as being put together (in a rather complicated fashion) from information having to do with a single P_t^s at a time; this should include v_n-periodic information, as well as some other periodic information.

Here is a preliminary description; the terminology and facts cited here are discussed (with references) in the following two sections, and in Section 1.3 we describe this program more carefully. Let M and N be bounded below finite type modules over the mod 2 Steenrod algebra A, and assume that N is finite. Let $\{M, N\}^{u,v}$ be the collection of stable maps from M to N; note that this is isomorphic to $\mathrm{Ext}_A^{u,v}(M, N)$ when $u > 0$. Define $P_t^s \in A$ to be the Milnor basis element dual to $\xi_t^{2^s}$. When $s < t$, then $(P_t^s)^2 = 0$, so for any A-module M one may define the P_t^s-homology of M, $H(M, P_t^s)$. The P_t^s's with $s < t$ are linearly ordered by degree; let $\mathbf{P}$ be an interval in this ordering. Margolis has shown that given any module M there exists a module $M\langle \mathbf{P} \rangle$, so that if $P \in \mathbf{P}$, then $H(M, P) \cong H(M\langle \mathbf{P} \rangle, P)$, and if $P \notin \mathbf{P}$, then $H(M\langle \mathbf{P} \rangle, P) = 0$. He essentially constructs a spectral sequence, which we call the *Margolis chromatic spectral sequence*, with

$$E_1 \cong \{M\langle P_t^s \rangle, N\}_{P_t^s \in A, \quad s<t}$$

converging to $\{M, N\}$. We want to compute each column $\{M\langle P_t^s \rangle, N\}$ using homology data: there is a spectral sequence (the *Margolis Adams spectral sequence*) with

$$E_2 \cong \mathrm{Ext}_{A_t^s}(H(M, P_t^s), H(N, P_t^s)) \otimes \mathbf{F}_2[v_{s,t}^{\pm 1}]$$

abutting to $\{M\langle P_t^s\rangle, N\langle P_t^s, \infty\rangle\}$ (here A_t^s is the algebra of operations for P_t^s-homology, and $v_{s,t}$ is a periodicity operator associated to P_t^s—it is the polynomial generator of $\mathrm{Ext}_{E[P_t^s]}(\mathbf{F}_2, \mathbf{F}_2)$), and one can think of $v_{0,t}$ as v_{t-1} from BP-theory. Note that the abutment isn't quite the right thing; to fix that we construct a finite module $L_{s,t} = L_{s,t}\langle P_t^s, \infty\rangle$ and a sequence of Bockstein spectral sequences, starting with $\{M\langle P_t^s\rangle, L_{s,t} \otimes N\}$ and ending with $\{M\langle P_t^s\rangle, N\}$. Then the Margolis Adams spectral sequence can provide the link between P_t^s-homology and the beginning of this sequence.

The structure of the thesis is as follows: in the remainder of this chapter, we will discuss relevant background material; in particular, we will discuss the stable category of modules over a connected algebra, and some properties of the Steenrod algebra. We finish this chapter with a more detailed description of our program. In Chapter 2 we discuss the sequence of Bockstein spectral sequences; in Chapter 3 we discuss the Margolis Adams spectral sequence; and in Chapter 4 we use our machinery over the sub-Hopf algebra $A(1) \subseteq A$ to work out $\{\mathbf{F}_2, \mathbf{F}_2\}_{A(1)}$.

1.1 The stable category of modules

In this section we discuss the stable category of modules over an algebra, as presented in Chapter 14 of [Ma1]. First, we discuss a little homological algebra: let A be a graded connected algebra, and let $\mathcal{M}$ be the category of bounded below modules over A. We assume that $\mathcal{M}$ satisfies

Property 1.1.1. *Any projective in $\mathcal{M}$ is injective.*

(Margolis proves that the category of bounded below modules over the Steenrod algebra, or over any sub-Hopf algebra of it, satisfies this property—see [Ma1], 13.12 and 15.7.) We say that a map $f : M \to N$ is *stably trivial*, written $f \simeq 0$, if f factors through a projective module; similarly, one can define two maps $f, g : M \to N$ to be *stably equivalent* ($f \simeq g$) if $f - g$ is stably trivial. Two modules M and N are *stably equivalent* ($M \simeq N$) if we have $f : M \to N$ and $g : N \to M$ with $fg \simeq \mathrm{id}_N$ and $gf \simeq \mathrm{id}_M$. Given $M, N \in \mathcal{M}$, we define

$$\{M, N\}^{0,t} = \mathrm{Hom}_A^t(M, N)/\simeq .$$

We make $\{M, N\}$ bigraded as follows: define ΩM by the short exact sequence

$$0 \to \Omega M \to PM \to M \to 0,$$

where PM is a bounded below projective module; note that ΩM is well-defined up to stable equivalence. Of course, we can iterate this to define $\Omega^k M$ for any $k \geq 0$. Then we define

$$\{M,N\}^{s,t} = \begin{cases} \{\Omega^s M, N\}^{0,t} & \text{if } s \geq 0 \\ \{M, \Omega^{-s} N\}^{0,t} & \text{if } s \leq 0 \end{cases}$$

Now, define a new category $\overline{\mathcal{M}}$ to have the same objects as $\mathcal{M}$, but with maps given by $\mathrm{Hom}_{\overline{\mathcal{M}}}(M,N) = \{M,N\}^{*,*}$. Define the composition of $f : \Omega^i L \to M$ and $g : \Omega^j M \to N$ by $g \circ \Omega^j f : \Omega^{i+j} L \to \Omega^j M \to N$ (and similarly if $i < 0$ or $j < 0$); this is well-defined, essentially by Proposition 1.1.2. This new category is called the *stable* category of (bounded below) A-modules. We say that a module M is *deloopable* if $M \simeq \Omega N$ for some N. If we ever need to specify the ground ring A, we will write $\{M,N\}_A$ for the collection of stable maps.

Here are some simple facts about $\{-,-\}$; the first one is where we need Property 1.1.1:

Proposition 1.1.2 ([Ma1], 14.3 and 14.8). *(a) For all $k \geq 0$, then we have $\Omega^k : \{M,N\}^{s,t} \xrightarrow{\cong} \{\Omega^k M, \Omega^k N\}^{s,t}$.*

(b) If $s > 0$, then $\mathrm{Ext}_A^{s,t}(M,N) \cong \{M,N\}^{s,t}$. There is a surjection $\mathrm{Ext}_A^{0,t}(M,N) \to \{M,N\}^{0,t}$.

Part (b) illustrates the usefulness of this definition—our spectral sequences are going to compute $\{-,-\}$; this means that we are also computing Ext. There are also a number of results like the following:

Proposition 1.1.3 ([Ma1],14.6). *If $0 \to A \to B \to C \to 0$ is a short exact sequence of A-modules, then for any M there are long exact sequences*

$$\cdots \to \{M,A\}^s \to \{M,B\}^s \to \{M,C\}^s \to \{M,A\}^{s+1} \to \cdots$$

and

$$\cdots \to \{C,M\}^s \to \{B,M\}^s \to \{A,M\}^s \to \{C,M\}^{s+1} \to \cdots.$$

One can make an analogy between the homotopy category of spectra and the stable category of modules; for example, one can turn any map into a "fibration" and certain maps into "cofibrations": given a map $M \to N$, let PN be a projective cover for N. Then

$$\begin{array}{ccc} M & \simeq & M \oplus PN \\ \downarrow & & \downarrow \\ N & = & N \end{array}$$

commutes up to stable equivalence. Thus we can talk about the "fiber" of any map in the stable category of modules, namely the kernel of a stably equivalent surjection. Similarly, if M is deloopable, then there is an inclusion $M \hookrightarrow Q$ with Q bounded below projective; then we can replace $M \to N$ by the inclusion $M \to N \oplus Q$. So in this case the "cofiber" of $M \to N$ is a well-defined notion.

1.2 The Steenrod algebra

A good reference for the material in this section is [Ma1], Chapters 15, 19, 21, and 22; of course, for a basic reference, there is also [Mil].

Let $A^* = A$ denote the mod 2 Steenrod algebra. In [Mil], Milnor proved that the dual of A^* is $A_* \cong \mathbf{F}_2[\xi_1, \xi_2, \ldots]$ (isomorphic as algebras), where $|\xi_i| = 2^i - 1$. Let P^s_t be the element in the dual basis to the monomial basis which is dual to $\xi_t^{2^s}$ (this dual basis is called the *Milnor basis*; for a description of how to multiply using this basis, see [Mil] or Chapter 15 of [Ma1]). An easy calculation in the Milnor basis shows that $(P^s_t)^2 = 0$ if and only if $s < t$. Therefore, for any A-module M, if $s < t$ we can define

$$H_k(M, P^s_t) = \frac{\ker\ P^s_t : M_k \to M_{k+|P^s_t|}}{\mathrm{im}\ P^s_t : M_{k-|P^s_t|} \to M_k}.$$

(Thus we will sometimes refer to the P^s_t's with $s < t$ as *differentials*.) These homology groups have many applications; one of the main examples is this theorem, a "Whitehead" theorem for P^s_t-homology groups:

Theorem 1.2.1 ([AM1]). *Let B be a sub-Hopf algebra of A. Then a bounded below B-module M is free if and only if $H(M, P^s_t) = 0$ for every $P^s_t \in B$ with $s < t$.*

Given A-modules M and N one can put an A-module structure on $M \otimes N$, namely, the diagonal action. Unfortunately, the homology groups in general don't satisfy a Kunneth formula (they do only when $s = 0$, i.e., when P^s_t is primitive); instead there is a spectral sequence. We will need the following:

Proposition 1.2.2 ([Ma1], 19.18). *If $H(M, P^s_t) = 0$, then for any N we have $H(M \otimes N, P^s_t) = 0$.*

Here is another easy fact:

Proposition 1.2.3 ([Ma1], 19.3). *For any M, we have*

$$H_i(M, P^s_t) \cong \{A/AP^s_t, M\}^{0,-i}.$$

This co-representability leads to

Definition 1.2.4. *The algebra of operations for P^s_t-homology, A^s_t, is the opposite algebra to $\{A/AP^s_t, A/AP^s_t\}^{0,*}$.*

Since $\{A/AP^s_t, M\}^{0,*}$ is a right $\{A/AP^s_t, A/AP^s_t\}^{0,*}$-module, then $H(M, P^s_t)$ is a left A^s_t-module. This algebra of operations plays the role of the Steenrod algebra in the Margolis Adams spectral sequence.

Co-representability and Proposition 1.1.3 gives

Proposition 1.2.5 ([Ma1], 19.14). *If $0 \to K \to L \to M \to 0$ is a short exact sequence of A-modules, then for any differential P there is a long exact sequence of A^s_t-modules*

$$\cdots \to H_i(K, P) \to H_i(L, P) \to H_i(M, P) \to H_{i+|P|}(K, P) \to \cdots.$$

Margolis describes a method for "killing" homology groups; first, note that the P^s_t's are linearly ordered by dimension. Let $P^s_t + 1$ (respectively, $P^s_t - 1$) denote the successor (respectively, predecessor) of P^s_t in this ordering. Now, here are the appropriate definitions:

Definitions 1.2.6. *(a) Let $\mathbf{P} = [P^s_t, P^u_v]$ be an interval of differentials. An A-module M is said to be of* type $\langle \mathbf{P} \rangle$ *or* type $\langle P^s_t, P^u_v \rangle$ *if and only if $H(M, P) = 0$ when $P \notin \mathbf{P}$, i.e., $|P| < |P^s_t|$ or $|P^u_v| < |P|$. Modules of type $\langle P^s_t, \infty \rangle$ are defined in a similar way.*

(b) A module M is said to be monochromatic *if and only if $M \simeq M\langle P^s_t \rangle$ for some P^s_t.*

(c) Choose $P^s_t \in A$ with $s < t$. A module K is of type $M\langle P^0_1, P^s_t \rangle$ *if and only if there exists a map $f : K \to M$ with K of type $\langle P^0_1, P^s_t \rangle$ such that $H(f, P)$ is an isomorphism for $|P| \leq |P^s_t|$; we write $K \simeq M\langle P^0_1, P^s_t \rangle$.*

(d) Choose $P^s_t \in A$ with $s < t$. A module L is of type $M\langle P^s_t, \infty \rangle$ *if and only if there exists a map $g : M \to L$ with L of type $\langle P^s_t, \infty \rangle$ such that $H(g, P)$ is an isomorphism for $|P^s_t| \leq |P|$; we write $L \simeq M\langle P^s_t, \infty \rangle$.*

(e) Let $\mathbf{P} = [P^s_t, P^u_v]$ be an interval of differentials; given a module M, a module N is said to be of type $M\langle \mathbf{P} \rangle$, written $N \simeq M\langle \mathbf{P} \rangle$, if and only if N is of type $\langle \mathbf{P} \rangle$ and there is either a diagram

$$M \to L \leftarrow N$$

or a diagram

$$M \leftarrow K \rightarrow N$$

such that the maps are isomorphisms in P-homology for $P \in \mathbf{P}$.

Margolis proves the following:

Theorem 1.2.7 ([Ma1], 21.1)**.** *For any bounded below A-module M, and for any interval of differentials $\mathbf{P}$, there exists a module of type $M\langle\mathbf{P}\rangle$. If M is of finite type, then $M\langle\mathbf{P}\rangle$ may be chosen to be of finite type, also. This construction is functorial in $\overline{\mathcal{M}}$.*

Two other useful facts are:

Proposition 1.2.8. *Given a bounded below A-module M and a differential P, let Q be a projective cover for $M\langle P, \infty\rangle$. Then we have a short exact sequence*

$$0 \rightarrow M\langle P_1^0, P-1\rangle \rightarrow M \oplus Q \rightarrow M\langle P, \infty\rangle \rightarrow 0.$$

Proposition 1.2.9 ([Ma1], 22.1)**.** *For any M and N, we have*

$$\{M\langle P_1^0, P_t^s\rangle, N\langle P_t^s + 1, \infty\rangle\} = 0.$$

We can phrase these definitions and results in the language of Bousfield localization:

Definitions 1.2.10. *Let $\mathbf{P}$ be an interval of differentials.*

(a) A map $f : K \rightarrow L$ is a $\mathbf{P}$-equivalence *if and only if $H(f, P)$ is an isomorphism for all $P \in \mathbf{P}$.*

(b) A module N is $\mathbf{P}$-local *if and only if for any $\mathbf{P}$-equivalence $f : K \rightarrow L$, every map $K \rightarrow N$ factors uniquely through f.*

(c) $M \rightarrow N$ is a $\mathbf{P}$-localization *if and only if it is a $\mathbf{P}$-equivalence and N is $\mathbf{P}$-local.*

(d) A module J is $\mathbf{P}$-colocal *if and only if for any $\mathbf{P}$-equivalence $f : K \rightarrow L$, every map $J \rightarrow L$ factors uniquely through f.*

(e) $J \rightarrow M$ is a $\mathbf{P}$-colocalization *if and only if it is a $\mathbf{P}$-equivalence and J is $\mathbf{P}$-colocal.*

Then we have

Proposition 1.2.11. *(a) N is $\mathbf{P}$-local if and only if $\{K, N\}^* = 0$ whenever K is $\mathbf{P}$-trivial (i.e., $K\langle \mathbf{P} \rangle \simeq 0$).*

(b) N is $\mathbf{P}$-colocal if and only if $\{N, L\}^* = 0$ whenever L is $\mathbf{P}$-trivial.

Proof: This is standard, using the semi-triangulated structure of $\overline{\mathcal{M}}$. □

Theorem 1.2.12. *(a) N is $\langle P, \infty \rangle$-local if and only if N is of type $\langle P, \infty \rangle$.*

(b) J is $\langle P_1^0, P \rangle$-colocal if and only if J is of type $\langle P_1^0, P \rangle$.

Proof: In each case, if the module is of the given type, then it is local or colocal because of Propositions 1.2.11 and 1.2.9. Conversely, if N is $\langle P, \infty \rangle$-local, then $\{N\langle P_1^0, P-1 \rangle, N\} = 0$; therefore the canonical map $N\langle P_1^0, P-1 \rangle \to N$ factors through a projective module, and hence is zero on homology. On the other hand, this map is a $\langle P_1^0, P-1 \rangle$-equivalence; therefore, N must be $\langle P_1^0, P-1 \rangle$-trivial.

A similar argument proves that if J is $\langle P_1^0, P \rangle$-colocal, then J is of type $\langle P_1^0, P \rangle$. □

Corollary 1.2.13. *Given a bounded below module M, then for any P its $\langle P, \infty \rangle$-localization and $\langle P_1^0, P \rangle$-colocalization exist; they may be chosen to be of finite type if M is of finite type.*

Corollary 1.2.14. *(a) $\langle P, \infty \rangle$-localizations and $\langle P_1^0, P \rangle$-colocalizations are functorial in $\overline{\mathcal{M}}$.*

(b) $M \to M\langle P, \infty \rangle$ is initial among maps to $\langle P, \infty \rangle$-local modules, and terminal among $\langle P, \infty \rangle$-equivalences from M.

(c) $M\langle P_1^0, P \rangle \to M$ is initial among $\langle P_1^0, P \rangle$-equivalences to M, and terminal among maps from $\langle P_1^0, P \rangle$-colocal modules.

Proof: Part (a) is immediate. To prove (b), we use the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M\langle P, \infty \rangle \arrow[d] \\ N\langle P, \infty \rangle \arrow[r, equal] & N\langle P, \infty \rangle \end{tikzcd}$$

(where the horizontal maps are $\langle P, \infty \rangle$-localization) to show that $M \to M\langle P, \infty \rangle$ is initial among maps to $\langle P, \infty \rangle$-local objects, and the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"] & M\langle P, \infty \rangle \arrow[d, "\simeq"] \\ N \arrow[r] & N\langle P, \infty \rangle \end{tikzcd}$$

where $f : M \to N$ is a $\langle P, \infty\rangle$-equivalence, to show that $M \to M\langle P, \infty\rangle$ is terminal among $\langle P, \infty\rangle$-equivalences from M. Similar diagrams work for part (c). $\square$

Here is one useful property of local modules, originally due to Anderson and Davis:

Theorem 1.2.15 ([AD]). *Let M be a bounded below $\langle P^s_t, \infty\rangle$-local A-module; then there exists a number c so that $\mathrm{Ext}_A^{u,v}(M, \mathbf{F}_2) = 0$ when $v < |P^s_t|u + c$.*

Margolis rephrases this vanishing line result in terms of connectivity: let $\|M\|$ denote the *stable connectivity* of M, i.e., the connectivity of the module stably equivalent to M with no free summands. (This is well-defined—see [Ma1], 13.13.)

Proposition 1.2.16 ([Ma1], 22.6). *For each P^s_t, M is $\langle P^s_t, \infty\rangle$-local if and only if there exists $k = k(P^s_t) > 0$ such that for all $r \geq 0$, $\|\Omega^r M\| > |M| - k + r|P^s_t|$.*

Proof: Part of Margolis' proof, that if the appropriate homology groups vanish, then the stable connectivity is bounded as indicated, is not quite right; here is a correction.

Suppose that $H(M, P) = 0$ for P with $|P| < |P^s_t|$. Then $|H(\Omega^r M, P)| = |H(M, P)| + r|P| \geq |M| + r|P^s_t|$ for all P and $r \geq 0$. If P^s_t is in $A(n)$, then let $k = \max \deg(A(n)) + 2^{n+1}$, and assume inductively that $\|\Omega^r M\| > |M| - k + r|P^s_t|$. Then there is a module $N \simeq \Omega^r M$ with $|N| > |M| - k + r|P^s_t|$ and $|H(N, P)| \geq |M| + r|P^s_t|$. [Now comes the correction.] Applying 19.7 from [Ma1] with $B = A(n)$ shows that N is free through degree $|M| - k + (r + 1)|P^s_t|$. Therefore, $\|\Omega^{r+1} M\| = \|\Omega N\| > |M| - k + (r + 1)|P^s_t|$. $\square$

1.3 The Margolis chromatic spectral sequence

Using the machinery above, we now give a more detailed explanation of our program.

Given a bounded below A-module M, in Section 22.2 of [Ma1], Margolis constructs a tower

$$\begin{array}{ccccccccccc} 0 & \to & M\langle P^0_1\rangle & \to & M\langle P^0_1, P^0_2\rangle & \to & \cdots & \to & M\langle P^0_1, P^s_t\rangle & \to & \cdots \to M \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & M\langle P^0_1\rangle & & M\langle P^0_2\rangle & & & & M\langle P^s_t\rangle & & \end{array}$$

Applying $\{-, N\}$ gives a spectral sequence with

$$E_1 \cong \{M\langle P^s_t\rangle, N\}_{P^s_t \in A,\ s<t}.$$

The spectral sequence converges to $\{M, N\}$ when M is finite type, because of

Theorem 1.3.1 ([Ma1], 22.4). *If M is finite type, then*

$$\lim \{M\langle P_1^0, P_t^s\rangle, N\} \cong \{M, N\},$$

and

$$\lim{}^1\{M\langle P_1^0, P_t^s\rangle, N\} = 0.$$

We call this spectral sequence the *Margolis chromatic spectral sequence*.

Now we want to compute each column. Let M and N be bounded below finite type A-modules, and let N be finite. In some unpublished notes ([Ma2]), for A-modules K and L Margolis constructs and proves convergence for a spectral sequence with

$$E_2 \cong \mathrm{Ext}_{A_t^s}(H(K, P_t^s), H(L, P_t^s)) \otimes \mathbf{F}_2[v_{s,t}^{\pm 1}] \Rightarrow \{K\langle P_t^s\rangle, L\langle P_t^s, \infty\rangle\},$$

converging only when $s = 0$. We call this the *Margolis Adams spectral sequence*; it is described more fully in Chapter 3. Note that we cannot prove (yet) that it converges when $s \neq 0$. Assuming that it converges for all values of s, then this spectral sequence is a step closer to our goal—now given M we have a spectral sequence which gives $\{M\langle P_t^s\rangle, L\langle P_t^s, \infty\rangle\}$ for any L; thus we need to be able to choose L correctly and from that compute $\{M\langle P_t^s\rangle, N\}$. In analogy with the chromatic picture, this is accomplished via a sequence of Bockstein spectral sequences, as described in

Theorem 2.1.1 *Choose a differential $P_t^s \in A$; let M be a bounded below module which is $\langle P_t^s, \infty\rangle$-local. For any finite module N, there is a finite $\langle P_t^s, \infty\rangle$-local module L and a sequence of Bockstein spectral sequences, in which the first has E_1-term $\{M, L\}$, each converges to the E_1-term of the next, and the last converges to $\{M, N\}$.*

This theorem is proved in Chapter 2.

Chapter 2

Bockstein spectral sequences

2.1 Introduction and results

Our goal in this chapter is to prove the following theorem:

Theorem 2.1.1. *Choose a differential $P_t^s \in A$; let M be a bounded below module which is $\langle P_t^s, \infty\rangle$-local. For any finite module N, there is a finite $\langle P_t^s, \infty\rangle$-local module L and a sequence of Bockstein spectral sequences, in which the first has E_1-term $\{M, L\}$, each converges to the E_1-term of the next, and the last converges to $\{M, N\}$.*

In the chromatic picture, the purpose of the Bockstein spectral sequences is to add in, one at a time, torsion for v_{n-1}, v_{n-2}, …, v_1, p. In analogy, we need to add in torsion for the periodicity operators associated to the differentials P with $|P| < |P_t^s|$. Because of the structure of the Steenrod algebra, we also need to add in torsion for operators associated to the non-differential P_t^s's (e.g., the element $h_{11} \in \mathrm{Ext}_A^{1,2}(\mathbf{F}_2, \mathbf{F}_2)$ is associated to $\mathrm{Sq}^2 = P_1^1$). Our method for doing all of this may be somewhat inefficient (depending on P_t^s, we may have more spectral sequences than would seem to be required by this analogy); this inefficiency is discussed at the end of this section (see 2.1.10).

To discuss the theorem more precisely, we need some background on the A_n's. Recall that $A_n = A(n)$ is the sub-Hopf algebra of the Steenrod algebra A generated by Sq^1, Sq^2, …, Sq^{2^n}; it is well-known (see [Mil], for example) that $P_t^s \in A_n$ if and only if $s + t \leq n + 1$. We are interested in A_n as an A-module, as given by the following result:

Theorem 2.1.2 ([Mit1]). *A_n admits a self-dual A-module structure extending the left A_n-module structure (i.e., extending the multiplication).*

Unless otherwise stated, A_n will mean this A-module. Jeff Smith has given another description of the same structure; we will need to use his version, so we describe it in Section 2.3 (see also [Mit2]).

We can use the $A(n)$'s to build finite A-modules which are $\langle P^s_t, \infty\rangle$-local: let Φ denote the doubling functor (see [Ma1], chapter 15, for example); it interacts nicely with homology groups: $H(\Phi M, P^{s+1}_t) = \Phi H(M, P^s_t)$, if $s+1 < t$ ([Ma1], 19.19). We have

Proposition 2.1.3. *Fix P^s_t with $s < t$. The module $M = A_{s+t-2} \otimes \Phi^{s+1} A_{t-2}$ is a (finite) $\langle P^s_t, \infty\rangle$-local module, and $H(M, P^s_t) \neq 0$.*

Proof: M is $\langle P^s_t, \infty\rangle$-local by the "Whitehead" Theorem (1.2.1) and Proposition 1.2.2; a Poincaré series argument shows that $H(M, P^s_t) \neq 0$. $\square$

We now outline the proof of the main theorem (2.1.1); in Section 2.2 we do all of the easier parts of the proof, and in Section 2.3 we do the hard part.

First of all, we discuss the case where $N = \mathbf{F}_2$ and $P^s_t = P^0_{n+1} + 1$ (the successor differential of P^0_{n+1}); then we let $L = A_n = A_n\langle P^0_{n+1} + 1, \infty\rangle$. For $k \geq 1$, let $J_k = A(P^0_k, P^0_{k+1}, \ldots)$. Then we have the following results:

Proposition 2.1.4. *For each k, $1 \leq k \leq n+1$, there is a short exact sequence of A-modules*

$$0 \to \Sigma^{|P^0_k|} A_n/J_k A_n \to A_n/J_{k+1}A_n \to A_n/J_k A_n \to 0.$$

Proposition 2.1.5. $J_{n+2} A_n = 0$.

Proposition 2.1.6. *$A_n/J_1 A_n \cong \Phi A_{n-1}$ (as A-modules).*

The next corollary is immediate.

Corollary 2.1.7. *We have short exact sequences of A-modules*

$$\begin{array}{ccccccccc}
0 & \to & \Sigma^{|P^0_{n+1}|} A_n/J_{n+1}A_n & \to & A_n & \to & A_n/J_{n+1}A_n & \to & 0 \\
0 & \to & \Sigma^{|P^0_n|} A_n/J_n A_n & \to & A_n/J_{n+1}A_n & \to & A_n/J_n A_n & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \Sigma^{|P^0_1|} A_n/J_1 A_n & \to & A_n/J_2 A_n & \to & A_n/J_1 A_n & \to & 0 \\
 & & \| & & & & \| & & \\
 & & \Sigma^{|P^0_1|}\Phi A_{n-1} & & & & \Phi A_{n-1} & & \\
0 & \to & \Sigma^{|P^1_n|}\Phi A_{n-1}/\Phi J_n A_{n-1} & \to & \Phi A_{n-1} & \to & \Phi A_{n-1}/\Phi J_n A_{n-1} & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \Sigma^{|P^n_1|}\Phi^n A_0/\Phi^n J_1 A_0 & \to & \Phi^n A_0 & \to & \Phi^n A_0/\Phi^n J_1 A_0 & \to & 0 \\
 & & \| & & & & \| & & \\
 & & \Sigma^{|P^n_1|}\mathbf{F}_2 & & & & \mathbf{F}_2 & &
\end{array}$$

Applying $\{M, -\}^{*,*}$ to these short exact sequences gives the desired Bockstein spectral sequences (which converge—see Proposition 2.1.8).

Thus Propositions 2.1.4, 2.1.5 and 2.1.6 are enough (along with Proposition 2.1.8) to prove the main theorem, in the case $N = \mathbf{F}_2$ and $P_t^s = P_{n+1}^0 + 1$. If N is an arbitrary finite module and P_t^s is an arbitrary differential, then we let $L = N \otimes A_{s+t-2} \otimes \Phi^{s+1} A_{t-2}$; by Proposition 2.1.3, this is of type $\langle P_t^s, \infty\rangle$. This gives us short exact sequences as above, with $n = s + t - 2$, but tensored with $N \otimes \Phi^{s+1} A_{t-2}$. This takes us from L to $N \otimes \Phi^{s+1} A_{t-2}$; then we have short exact sequences as above, but doubled $s + 1$ times and with $n = t - 2$, taking us to N.

To finish the proof of Theorem 2.1.1, we use the following:

Proposition 2.1.8. *Given a bounded below A-module M and a short exact sequence of finite A-modules*

$$0 \to \Sigma^{|P|} K \to J \to K \to 0,$$

then

(a) *Applying $\{M, -\}^{*,*}$ gives a Bockstein spectral sequence with $E_1 \cong \{M, J\}^{*,*} \otimes \mathbf{F}_2[v^{-1}]$, where $|v| = (1, |P|)$, abutting to $\{M, K\}^{*,*}$.*

(b) *If $M \simeq M\langle P_t^s, \infty\rangle$ and $|P| < |P_t^s|$, then the spectral sequence in part (a) converges.*

The rest of this chapter is structured as follows: in Section 2.2 we prove Propositions 2.1.4, 2.1.5 and 2.1.8; in Section 2.3 we prove the remaining piece, Proposition 2.1.6.

We end this section with a few remarks.

Remark 2.1.9. *We should point out that one "obvious" choice for the order in which to add in "P-torsion" for all P with $|P| < |P_t^s|$, namely in decreasing order of degree, does not work. By construction, P_1^n acts trivially on $A_n/A(P_{n+1}^0, P_n^1, \ldots, P_1^n) \cdot A_n$, while it does not on Mitchell's A_{n-1} for $n \geq 2$: since the top class in A_1 is $\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^1$, then with any A-module structure on A_1 there must be a $\mathrm{Sq}^4 = P_1^2$ connecting Sq^1 to the class in dimension 5, $\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^4$. With Mitchell's A-module structure on A_{n-1}, there are A-linear surjections $A_{n-1} \to \Phi A_{n-2} \to \cdots \to \Phi^{n-2} A_1$ (see Corollary 2.3.6); since P_1^2 acts non-trivially on A_1, then P_1^n acts non-trivially on $\Phi^{n-2} A_1$, and hence on A_{n-1}. Even if one wants to work over A_n, then $A_n/A_n(P_{n+1}^0, P_n^1, \ldots, P_1^n) \not\cong A_{n-1}$: when $n = 3$ we have*

$$\begin{aligned} \operatorname{rank}_{\mathbf{F}_2} (A_2)_{(16)} &= 4, \\ \operatorname{rank}_{\mathbf{F}_2} A_3(P_4^0, P_3^1, P_2^2, P_1^3)_{(16)} &= 8, \text{ and} \\ \operatorname{rank}_{\mathbf{F}_2} (A_3)_{(16)} &= 11 \end{aligned}$$

(subscript denotes dimension). So there is a problem even on the vector space level. Presumably, there are similar problems for A_4, A_5, ...

Remark 2.1.10. *As mentioned earlier, our method of adding in P-torsion when $|P| < |P_t^s|$ is rather inefficient; the reader will notice that, depending on P_t^s, we may need to add in information for certain P's twice. It may be possible to do better than this; the choice of L is what needs to be improved. For example, let B be the sub-Hopf algebra of A generated by $\{P_v^u | u < v, |P_v^u| < |P_t^s|\}$. If B had an A-module structure compatible with its multiplication, then we could use B for L, and try to construct short exact sequences as in Proposition 2.1.4 to get exactly one Bockstein spectral sequence for each* differential *preceding P_t^s. Of course, this won't work—a theorem of Lin ([L]) says that B cannot be an A-module (unless $P_t^s = Q_1$, in which case the method we've just described coincides with the method above for A_0). In any case, it is quite possible that there are modules of type $\langle P_t^s, \infty\rangle$ which have smaller dimension over $\mathbf{F}_2$ than $A_{s+t-2} \otimes \Phi^{s+1} A_{t-2}$ (and, of course, smaller dimension is necessary to reduce the number of Bockstein spectral sequences).*

2.2 Short exact sequences and convergence

In this section we prove Propositions 2.1.4, 2.1.5 and 2.1.8.

Let A_n be the sub-Hopf algebra of the Steenrod algebra generated by Sq^1, Sq^2, ..., Sq^{2^n}, with Mitchell's A-module structure. For $k \geq 1$, let $J_k = A(P_k^0, P_{k+1}^0, \ldots)$.

First of all, Proposition 2.1.5, that $J_{n+2}A_n = 0$, is a restatement of a result of Mitchell's—see [Mit1], 3.14(a).

Proposition 2.1.4 *For each k, $1 \leq k \leq n+1$, there are short exact sequences of A-modules*

$$0 \to \Sigma^{|P_k^0|} A_n/J_k A_n \to A_n/J_{k+1}A_n \to A_n/J_k A_n \to 0.$$

Proof: The result will follow if we can prove two things:

(a) $P_k^0 \cdot - : A_n/J_{k+1}A_n \to A_n/J_{k+1}A_n$ is A-linear.

(b) $H(A_n/J_{k+1}A_n, P_k^0) = 0$.

Given these, then we have a short exact sequence

$$0 \to \text{im } P_k^0 \to A_n/J_{k+1}A_n \to \text{cok } P_k^0 \to 0$$

by A-linearity; im $P^0_k \cong$ cok P^0_k by exactness via multiplication by P^0_k; and cok $P^0_k \cong A_n/J_kA_n$ by simple algebra.

Part (a) follows from this lemma:

Lemma 2.2.1. *For any m, $[P^0_k, \mathrm{Sq}^m] = P^0_{k+1}\mathrm{Sq}^{m-2^k}$.*

Proof: This is a simple exercise in Milnor multiplication. □

Lemma 2.2.1 says that for any $a \in A$ and any $b \in A_n$, $[P^0_k, a](b) \in J_{k+1}A_n$. This is exactly what we need to prove part (a).

We prove part (b) by induction. We actually prove a slightly stronger result, that $H(A_n/J_{k+1}A_n, P^0_r) = 0$ for $1 \le r \le k$.

By Proposition 2.1.5, $A_n/J_{n+2}A_n = A_n$, and this is P^0_r-acyclic for $1 \le r \le n+1$. This starts the induction. We assume that $A_n/J_{k+2}A_n$ is P^0_r-acyclic, for $1 \le r \le k+1$; we want to show that $A_n/J_{k+1}A_n$ is, for $1 \le r \le k$.

Since this is the only remaining step in the proof of the proposition, then our inductive hypothesis gives us a short exact sequence of A-modules

$$0 \to \Sigma^{|P^0_{k+1}|}A_n/J_{k+1}A_n \to A_n/J_{k+2}A_n \to A_n/J_{k+1}A_n \to 0.$$

So we get a long exact sequence in P^0_r-homology (Proposition 1.2.5); by induction, $H(A_n/J_{k+2}A_n, P^0_r) = 0$ for $r \le k+1$, so we have

$$\begin{aligned} H_i(A_n/J_{k+1}A_n, P^0_r) &\cong H_{i+|P^0_r|}(\Sigma^{|P^0_{k+1}|}A_n/J_{k+1}A_n, P^0_r) \\ &\cong H_{i+|P^0_r|-|P^0_{k+1}|}(A_n/J_{k+1}A_n, P^0_r) \\ &\cong H_{i+j(|P^0_r|-|P^0_{k+1}|)}(A_n/J_{k+1}A_n, P^0_r) \end{aligned}$$

for all j. Since $A_n/J_{k+1}A_n$ is finite, then as long as $|P^0_r| \neq |P^0_{k+1}|$ (i.e., as long as $r \neq k+1$), we must have $H_i(A_n/J_{k+1}A_n, P^0_r) = 0$. This finishes the induction, and hence the proof of Proposition 2.1.4. □

Next, we prove

Proposition 2.1.8 *Given a bounded below A-module M and a short exact sequence of finite A-modules*

$$0 \to \Sigma^{|P|}K \to J \to K \to 0,$$

then

(a) Applying $\{M, -\}^{*,*}$ gives a Bockstein spectral sequence with $E_1 \cong \{M, J\}^{*,*} \otimes \mathbf{F}_2[v^{-1}]$, where $|v| = (1, |P|)$, abutting to $\{M, K\}^{*,*}$.

(b) If $M \simeq M\langle P^s_t, \infty\rangle$ and $|P| < |P^s_t|$, then the spectral sequence in part (a) converges.

Proof: Part (a) is standard. Part (b) follows from Remark 3.11 in [MRW]: Since M has no homology with respect to the differentials with degree less then $|P^s_t|$, then for any finite module J, in positive homological degrees $\{M, J\}^{*,*}$ has a vanishing line of slope $\frac{1}{|P^s_t|-1}$ (in the usual (internal − homological, homological)-coordinates—see 1.2.15). Since v acts with slope $\frac{1}{|P|-1} > \frac{1}{|P^s_t|-1}$, then $\{M, J\}^{*,*}$ is v-torsion. □

2.3 Jeff Smith's A-module structure on A_n and the proof of Proposition 2.1.6

The proof of Proposition 2.1.6 has several steps. First of all, we make some reductions. We want to show that there is a short exact sequence of A-modules

$$0 \to J_1 A_n \to A_n \to \Phi A_{n-1} \to 0;$$

it is enough to show that there is an A-linear surjection $p : A_n \to \Phi A_{n-1}$ (where ΦA_{n-1} is Mitchell's A_{n-1}, doubled) because the kernel of such a map p would contain $J_1 A_n$ since ΦA_{n-1} is evenly graded, and so for size reasons the kernel would have to equal this submodule. To construct p, first we construct $i : \Sigma^{b_n} \Phi A_{n-1} \hookrightarrow A_n$ with b_n chosen so that the top class of $\Sigma^{b_n} \Phi A_{n-1}$ gets mapped to the top class of A_n; then dualizing gives p, since Mitchell's A-module structures are self-dual (Theorem 2.1.2).

To construct i, we use Jeff Smith's description of the A-module structure on A_n as given by Mitchell (see [Mit2]): write $\mathbf{F}_2[x]$ (with $|x| = 1$) for the A-algebra $H^*(\mathbf{R}P^\infty; \mathbf{F}_2)$; let Z_n be the A-submodule of $\mathbf{F}_2[x]/x^{2^n}$ generated by x, so Z_n has basis $\{x, x^2, x^4, \ldots, x^{2^{n-1}}\}$. For any positive integer k, let Σ_k be the symmetric group on k letters, and let e_n be a certain idempotent in the group algebra $\mathbf{F}_2[\Sigma_{\binom{n}{2}}]$ (defined below). Then we have

Theorem 2.3.1 (Smith, [Mit2])**.** *As an A_{n-2}-module, $Z_n^{\otimes \binom{n}{2}} e_n$ is free of rank one, with generator in degree $2^n - 1 - n$.*

Theorem 2.3.2 ([Mit1])**.** *As A-modules, Mitchell's A_{n-2} is isomorphic to $Z_n^{\otimes \binom{n}{2}} e_n$.*

Smith's proof of this has not been published; since we will need one of his lemmas (Lemma 2.3.3), we may as well give a brief description of his argument. First, results from the theory of representations of the symmetric groups (see [Mit2] and 26.19 in [J],) give the following:

Lemma 2.3.3 (Smith). *Let V be a vector space over $\mathbf{F}_2$. Then*

$$\operatorname{rank}_{\mathbf{F}_2}(V^{\otimes\binom{n}{2}}e_n) = \begin{cases} 0 & \text{if } \operatorname{rank}_{\mathbf{F}_2}V < n-1 \\ 2^{\binom{n-1}{2}} & \text{if } \operatorname{rank}_{\mathbf{F}_2}V = n-1 \\ 2^{\binom{n}{2}} & \text{if } \operatorname{rank}_{\mathbf{F}_2}V = n \end{cases}$$

Thus the vector space rank of $Z_n^{\otimes\binom{n}{2}}e_n$ is the same as that of A_{n-2}. To show freeness, we use the P_t^s-Whitehead theorem (1.2.1) and this lemma:

Lemma 2.3.4 (Smith). *Let M be an A-module with* $\operatorname{rank}_{\mathbf{F}_2}H(M,P_t^s) < n-1$. *Then* $H(M^{\otimes\binom{n}{2}}e_n, P_t^s) = 0$.

The idea of the proof of Lemma 2.3.4 is to filter M so that the P_t^s-homology is unchanged and P_t^s acts primitively on the associated graded of $M^{\otimes\binom{n}{2}}$; then apply Lemma 2.3.3. A filtration that works in the case we're interested in, namely $M = Z_n$, is as follows: first put a decreasing filtration on the algebra $C = E[P_t^0, P_t^1, \ldots, P_t^{s-1}]$, defined by $C(i) = C_{\geq i|P_t^s|}$ (the subscript denotes dimension). Filter Z_n by $Z_n(i) = C(i)Z_n$, and filter $Z_n^{\otimes\binom{n}{2}}$ by the tensor filtration.

Anyway, to construct $i : \Sigma^{b_{n-2}}\Phi A_{n-3} \hookrightarrow A_{n-2}$, we note that there is an inclusion $\Phi Z_{n-1} \hookrightarrow Z_n$; we claim that the map i is given by the induced map

$$(\Phi Z_{n-1})^{\otimes\binom{n}{2}}e_n \to Z_n^{\otimes\binom{n}{2}}e_n.$$

In other words, we claim that the elements of $Z_n^{\otimes\binom{n}{2}}e_n$ which don't have any tensor factors of x in any of their terms form a submodule which is isomorphic to (a suspension of) ΦA_{n-3}.

Thus to prove Proposition 2.1.6, it is enough to show that $Z_{n-1}^{\otimes\binom{n}{2}}e_n$, the quotient A-module of $Z_n^{\otimes\binom{n}{2}}e_n$ (under the map $x^{2^{n-1}} \mapsto 0$), is isomorphic to A_{n-3}. Reindexing, we are reduced to proving the following:

Proposition 2.3.5. *As A-modules, $Z_n^{\otimes\binom{n}{2}}e_n \cong Z_n^{\otimes\binom{n+1}{2}}e_{n+1}$ (up to suspension).*

Corollary 2.3.6. *(a) There is an A-linear surjection $A_n \to A_{n-1}$; dually, there is an A-linear inclusion $\Sigma^{c_n}A_{n-1} \hookrightarrow A_n$ (where c_n is chosen so that the top class of A_{n-1} gets sent to the top class of A_n).*

(b) There is an A-linear surjection $A_n \to \Phi A_{n-1}$; dually, there is an A-linear inclusion $\Sigma^{b_n}\Phi A_{n-1} \hookrightarrow A_n$.

Part (a) of this corollary is originally due to Mitchell ([Mit1]).

Before we start the proof, we define e_n and establish some notation. First of all, write $Z = Z_n$. Think of a basis element of $Z^{\otimes \binom{n}{2}}$ as a triangular *Young diagram* of shape $(n-1, n-2, \ldots, 2, 1)$ (see [Mit2] or [J] for a description of Young diagrams), with $\binom{n}{2}$ total boxes, each box filled with a basis element. Call such a filled-in diagram a *tableau*. In other words, we are writing

$$Z^{\otimes \binom{n}{2}} = (Z^{\otimes n}) \otimes (Z^{\otimes n-1}) \otimes \cdots \otimes (Z^{\otimes 2}) \otimes Z.$$

An element of $Z^{\otimes \binom{n}{2}}$ is a sum of such tableaux. Note that this gives us a choice of inclusion $\Sigma_{\binom{n}{2}} \hookrightarrow \Sigma_{\binom{n+1}{2}}$—namely, $\Sigma_{\binom{n}{2}}$ acts on an $\binom{n+1}{2}$-element tableau by permuting everything except the first column.

The element $e_n \in \mathbf{F}_2[\Sigma_{\binom{n}{2}}]$ is defined as follows: let $R = R_n$ be the row-stabilizer subgroup of $\Sigma_{\binom{n}{2}}$ associated to the triangular Young diagram, and let $C = C_n$ be the column stabilizer subgroup (so, for example, an element of R acts on a tableau by permuting the elements of the first row with each other, permuting the elements of the second row with each other, and so on). Let $\overline{R}$ and $\overline{C}$ be the sums (in the group algebra $\mathbf{F}_2[\Sigma_{\binom{n}{2}}]$) of the elements in R and C, respectively. Then $e_n = \overline{R}\,\overline{C}$. This is an idempotent in $\mathbf{F}_2[\Sigma_{\binom{n}{2}}]$ ([Mit2], [J]).

Proof of Proposition 2.3.5: Define $\varphi : Z^{\otimes \binom{n}{2}} \to Z^{\otimes \binom{n+1}{2}}$ by

$$\varphi((a_{1,1} \otimes \cdots \otimes a_{1,n-1}) \otimes \cdots \otimes (a_{n-1,1})) = \\ \sum_{\sigma \in \Sigma_n} (x^{2^{\sigma(0)}} \otimes a_{1,1} \otimes \cdots \otimes a_{1,n-1}) \otimes \cdots \otimes (x^{2^{\sigma(n-2)}} \otimes a_{n-1,1}) \otimes (x^{2^{\sigma(n-1)}}).$$

(Here Σ_n is the group of permutations of $\{0, 1, \ldots, n-1\}$.) In other words, given a tableau a, φ returns a sum of tableaux where each term is obtained by adding a column of n boxes to the left side of a to make a larger triangular diagram, and filling those boxes with a permutation of the basis elements of Z.

φ is clearly monomorphic; φ is also A-linear: to see this, let $b = \sum_{\sigma \in \Sigma_n} x^{2^{\sigma(0)}} \otimes \cdots \otimes x^{2^{\sigma(n-1)}}$. By co-commutativity of A, φ is A-linear if and only if the map $\psi : a \mapsto b \otimes a$ is A-linear. Also, ψ is A-linear if and only if $\mathrm{Sq}^r(b) = 0$ for all $k > 0$. So we compute:

$$\mathrm{Sq}^r(b) = \sum \mathrm{Sq}^{j_0} x^{2^{\sigma(0)}} \otimes \cdots \otimes \mathrm{Sq}^{j_{n-1}} x^{2^{\sigma(n-1)}}$$

where the sum is over $\sigma \in \Sigma_n$ and $j_0 + \cdots + j_{n-1} = r$. This sum is 0, by symmetry.

Next, by Lemma 2.3.7 and Corollary 2.3.8 below, φ sends the generator of $Z^{\otimes\binom{n}{2}}e_n$ to an element of $Z^{\otimes\binom{n+1}{2}}e_{n+1}$—in the language of 2.3.7, φ sends y_n to y_{n+1}. Then since $Z^{\otimes\binom{n}{2}}e_n \cong \Sigma^{2^n-1-n}A_{n-2}$ is a cyclic A-module and $Z^{\otimes\binom{n+1}{2}}e_{n+1}$ is a submodule of $Z^{\otimes\binom{n+1}{2}}$, we have $\tilde{\varphi} : Z^{\otimes\binom{n}{2}}e_n \hookrightarrow Z^{\otimes\binom{n+1}{2}}e_{n+1}$ ($\tilde{\varphi}$ is monic since φ is). Lastly, since both of these modules have the same vector space dimension (by Lemma 2.3.3), $\tilde{\varphi}$ must be an isomorphism. $\square$

Lemma 2.3.7. *Define $w_n \in Z_n^{\otimes\binom{n}{2}}$ by*

$$w_n = \left(\begin{array}{ccccccccc} & x^{2^{n-2}} & \otimes & x^{2^{n-3}} & \otimes & \cdots & \otimes & x^2 & \otimes\ x^1 \\ \otimes & x^{2^{n-3}} & \otimes & x^{2^{n-4}} & \otimes & \cdots & \otimes & x^1 & \\ \otimes & \cdots & & & & & & & \\ & \vdots & & & & & & & \\ \otimes & x^2 & \otimes & x^1 & & & & & \\ \otimes & x^1 & & & & & & & \end{array} \right).$$

Then the generator of $Z_n^{\otimes\binom{n}{2}}e_n$ is $y_n = w_n\overline{C}_n$.

Proof: We prove this by induction on n: when $n = 2$ we have

$$\Sigma A_0 \cong Z_2^{\otimes 1}e_2 = Z_2 = \mathbf{F}_2\langle x, x^2\rangle$$

with A-generator $x = y_2$.

When $n > 2$, then we only have to show two things: first, that y_n lies in the right dimension—this is an easy calculation—and second, that y_n is in the image of e_n. More precisely, we claim that $w_n\overline{R}_n\overline{C}_n = w_n\overline{C_n} = y_n$. This follows from symmetry: if $r \in R_n$ does not fix the upper left hand corner of w_n, then the first column of $w_n r$ will have a repetition, and thus will be in ker $(\overline{C}_n)$. So

$$w_n\overline{R}_n\overline{C}_n = w_n(\sum_{r'} r')\overline{C_n},$$

where the sum is over all $r' \in R_n$ which fix the upper left corner. By an easy induction, we see that

$$w_n\overline{R}_n\overline{C}_n = w_n(\sum_{r''} r'')\overline{C}_n,$$

summed over $r'' \in R_n$ which fix the left column; this is the same as $w_n\overline{R}_{n-1}\overline{C}_n$, and this is just $w_n\overline{C}_n$, by induction on n. $\square$

Corollary 2.3.8. *y_{n+1} is nonzero in $Z_n^{\otimes\binom{n+1}{2}} e_{n+1}$.*

Proof: To see this, just note that no term in $y_{n+1} \in Z_{n+1}^{\otimes\binom{n+1}{2}} e_{n+1}$ has any tensor factors of x^{2^n} in it. $\square$

Chapter 3

The Margolis Adams spectral sequence

3.1 Introduction

We want to prove the following:

Theorem 3.1.1. *Let P^s_t be a differential in A; let A^s_t be the algebra of operations for P^s_t-homology. Let M be a bounded below finite type $\langle P^0_1, P^s_t\rangle$-colocal module, and let N be a bounded below finite $\langle P^s_t, \infty\rangle$-local module. Then there is a spectral sequence with*

$$E_2^{u,v,w} \cong \mathrm{Ext}_{A^s_t}^{u,w+v|P^s_t|}(H(M,P^s_t), H(N,P^s_t)) \otimes \mathbf{F}_2[v_{s,t}^{\pm 1}]_{(0,v,v|P^s_t|)},$$

where $|v_{s,t}| = (0,1,|P^s_t|)$ in (u,v,w)-coordinates (the subscript on $\mathbf{F}_2[v_{s,t}^{\pm 1}]$ is tridegree). The differential d_r has degree $(r, 1-r, 0)$. When $s = 0$, the spectral sequence converges to

$$\{M,N\}^{u+v,w} \cong \{M\langle P^s_t\rangle, N\}^{u+v,w} \cong \{M\langle P^s_t\rangle, N\langle P^s_t\rangle\}^{u+v,w}.$$

(The last two isomorphisms are a consequence of Proposition 1.2.9.)

This spectral sequence can be thought of as a version of the Adams spectral sequence with P^s_t-homology playing the role of a cohomology theory; we call it the *Margolis Adams spectral sequence*. The role of this spectral sequence is to determine stable map (i.e., Ext) information from homology; in our setup we use it to get the input for the sequence of Bockstein spectral sequences from something "computable" (P^s_t-homology).

In Section 3.2 we construct the spectral sequence and prove some simple results about convergence; in Section 3.3 we prove convergence for the primitive P^s_t's. In

Section 3.4 we discuss some possible approaches for the non-primitive case. The ideas in Sections 3.2 and 3.3 are primarily due to Margolis ([Ma2]).

3.2 Construction of the spectral sequence

Fix a differential P_t^s in the Steenrod algebra. Where there is no possible ambiguity, we will write $H(M)$ for $H(M, P_t^s)$.

The main tool in the construction of the spectral sequence is the co-representability of P_t^s-homology (Proposition 1.2.3). In particular, co-representability gives us this fact (see also the comments at the end of Section 1.1):

Proposition 3.2.1. *For any bounded below A-module M, there is an A-module Q which is a direct sum of suspensions of A/AP_t^s and free modules, and a map of A-modules $f : Q \to M$, such that f is surjective and $H(f)$ is surjective. If M is of finite type, then Q may be chosen to be of finite type as well.*

Using this, we can build a *Margolis Adams resolution*: given any M, construct Q and $f : Q \to M$ as in the proposition. Repeat this with the kernel of f, and proceed inductively. This gives a diagram as follows:

$$\begin{tikzcd} M = K_0 & Q_0 \arrow[l] & Q_1 \arrow[l] & Q_2 \arrow[l] & \dots \arrow[l] \\ & K_1 \arrow[u] & K_2 \arrow[u] & K_3 \arrow[u] & \end{tikzcd}$$

where each $0 \leftarrow K_i \leftarrow Q_i \leftarrow K_{i+1} \leftarrow 0$ is short exact, $0 \leftarrow M \leftarrow Q_0 \leftarrow Q_1 \leftarrow \cdots$ is exact, and $0 \leftarrow H(M) \leftarrow H(Q_0) \leftarrow H(Q_1) \leftarrow \cdots$ is exact. Note that a Margolis Adams resolution for M is essentially a "realization" of an A_t^s-free resolution of $H(M)$.

Applying $\{-, N\}^{*,*}$ to this diagram gives an exact couple:

$$\begin{tikzcd} \{K_*, N\}^* \arrow[rr, "{(-1,1)}"] & & \{K_*, N\}^* \arrow[dl, "{(0,0)}"] \\ & \{Q_*, N\}^* \arrow[ul, "{(1,0)}"] & \end{tikzcd}$$

(The numbers are bidegrees—the first coordinate is the subscript or homological degree, and the second is the loop degree. All the maps preserve internal degree, so it is suppressed here.) Hence we get a spectral sequence with $E_1 \cong \{Q_*, N\}^{*,*}$, and by Proposition 1.2.3 and Ω-periodicity of A/AP_t^s, this is a direct sum of copies of $H_*(N) \otimes \mathbf{F}_2[v_{s,t}^{\pm 1}]$ (where $v_{s,t}$ has tridegree $(0, 1, |P_t^s|)$). By construction, the exact sequence

$$0 \leftarrow H(M) \leftarrow H(Q_0) \leftarrow H(Q_1) \leftarrow \cdots$$

is an A_t^s-free resolution of $H(M)$, and the E_1-term here is just $\mathrm{Hom}^*_{A_t^s}(H(Q_*), H(N)) \otimes \mathbf{F}_2[v_{s,t}^{\pm 1}]$. Hence we have

$$E_2 \cong \mathrm{Ext}_{A_t^s}(H(M), H(N)) \otimes \mathbf{F}_2[v_{s,t}^{\pm 1}],$$

as desired. The degree of d_r can be calculated easily using the bidegrees as marked on the exact couple.

It is useful to reindex things a bit: assume that M is $\langle P_1^0, P_t^s \rangle$-colocal; then so is each K_i (since each Q_i is—see [Ma1], p. 408). Therefore each K_i is infinitely deloopable ([Ma1], 22.10); let $M_i = \Omega^{-i} K_i$. Using the triangulated structure of $\langle P_1^0, P_t^s \rangle$-colocal modules, we have cofibration sequences $M_{i-1} \to M_i \to \Omega^{-i} Q_i$, so the Margolis Adams resolution becomes a *Margolis Adams tower*:

$$\begin{tikzcd}
M = M_0 \arrow[r, "i_0"] & M_1 \arrow[r, "i_1"] \arrow[d] & M_2 \arrow[r, "i_2"] \arrow[d] & \cdots \\
& \Omega^{-1} Q_1 & \Omega^{-2} Q_2 &
\end{tikzcd}$$

with $H(i_k) = 0$ and the cofiber (or fiber) of i_k (stably) a direct sum of suspensions of A/AP_t^s. Conversely, of course, given such a tower, we can get a Margolis Adams resolution. Applying $\{-, N\}^{*,*}$ to this diagram gives the same exact couple as above, up to some reindexing in the loop degree.

We will discuss convergence from Boardman's point of view ([B]). Assume that M has a Margolis Adams tower; then we have a good choice for a filtered target group for our spectral sequence: we define a decreasing filtration on $\{M, N\}^{*,*} = \{M_0, N\}^{*,*}$ by $F^k = \mathrm{im}(\{M_k, N\}^{*,*} \to \{M_0, N\}^{*,*})$. Thus an element of F^k is a stable map $M \to N$ that factors through M_k. We have

$$\cdots \hookrightarrow F^2 \hookrightarrow F^1 \hookrightarrow F^0 = \{M, N\}^{*,*}.$$

Let $F^\infty = \lim F^k$.

We prove some simple results about Margolis Adams towers and our filtration.

Lemma 3.2.2. *Given* $L_0 \xrightarrow{j_0} L_1 \xrightarrow{j_1} L_2 \xrightarrow{j_2} \cdots$ *with* $H(j_k) = 0$ *for each* k, *a Margolis Adams tower* $M_0 \to M_1 \to M_2 \to \cdots$, *and a map* $h_0 : M_0 \to L_0$, *then for* *each k there is a map $h_k : M_k \to L_k$ so that the following diagram commutes up to stable equivalence:*

$$\begin{array}{ccccccc} M_0 & \to & M_1 & \to & M_2 & \to & \cdots \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \\ L_0 & \to & L_1 & \to & L_2 & \to & \cdots \end{array}$$

Proof: The proof is by induction.

We may assume that for each k we have

$$0 \to A/AP^s_t \otimes V_k \to M_k \oplus (A \otimes \tilde{V}_k) \to M_{k+1} \to 0.$$

(Throughout this chapter, U, V, and W will denote A-modules with trivial A-action—i.e., graded vector spaces; also, $\tilde{V}$ will denote $\Sigma^{-|P^s_t|}V$, so we have $A/AP^s_t \otimes V \hookrightarrow A \otimes \tilde{V}$—see [Ma1], 19.2.) Assume that we have

$$\begin{array}{ccccccc} M_0 & \to & M_1 & \to & \cdots & \to & M_k \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & & & \downarrow{\scriptstyle h_k} \\ L_0 & \to & L_1 & \to & \cdots & \to & L_k \end{array}$$

commuting up to stable equivalence. We need to construct $M_{k+1} \to L_{k+1}$ so that the appropriate diagram commutes. We have

$$\begin{array}{ccccc} A/AP^s_t \otimes V_k & \longrightarrow & M_k \oplus (A \otimes \tilde{V}_k) & \longrightarrow & M_{k+1} \\ & & \downarrow{\scriptstyle h_k} & & \\ & & L_k & \xrightarrow{j_k} & L_{k+1} \end{array}$$

Since $H(j_k) = 0$,

$$A/AP^s_t \otimes V_k \to M_k \to L_k \to L_{k+1}$$

is stably trivial; thus the map $M_k \to L_{k+1}$ factors through the cofiber of $A/AP^s_t \otimes V_k \to M_k$, namely M_{k+1}. □

Remark 3.2.3. *One can also prove a sort of uniqueness for the h_k's: if there are two maps $g, h : M_k \to L_k$ that make the appropriate diagrams commute, then $g \equiv h$ "modulo higher filtration." Since we don't need this result, we won't be any more precise about it.*

Corollary 3.2.4. *(a) The filtration defined above is independent of choice of Margolis Adams tower.*

(b) In fact, given $M = L_0 \xrightarrow{j_0} L_1 \xrightarrow{j_1} L_2 \xrightarrow{j_2} \cdots$ with $H(j_k) = 0$ for each k, and a Margolis Adams tower $M = M_0 \to M_1 \to \cdots$, if $h : M \to N$ factors through L_k for some k, then h factors through M_k.

3.3 Convergence of the spectral sequence

To prove convergence, we need to study $F^\infty = \lim F^k$ and $\lim^1 F^k$.

Lemma 3.3.1. $\lim^1 F^k = 0$.

Corollary 3.3.2 ([B])**.** *If M is finite type and infinitely deloopable, and N is finite, then the Margolis Adams spectral sequence converges conditionally to $\{M,N\}^*/F^\infty$.*

Proof of Lemma 3.3.1: Since M is of finite type, then each K_i is as well. Thus $\Omega^u K_i$ is of finite type for all u. (One can see this by an application of [Ma1] 21.1 or 22.10, for example—if K is finite type and $\langle P^0_1, P^s_t\rangle$-colocal, then there is an inclusion $K \hookrightarrow P$ with P finite type projective. Since $\Omega^{-1}K$ is a quotient of P, then it is finite type as well.) Since N is finite, then $\{K_i, N\}^{u,v} \cong \{M_i, N\}^{u+i,v}$ is finite for each (u,v). Therefore, in each bidegree we have an inverse system of finite groups. □

Notation 3.3.3. *If $h : M \to N$ is in F^∞, we say that h is of* infinite filtration*.*

From here on, we deal with the primitive differential, P^0_t, rather than with P^s_t; we also assume throughout that M is $\langle P^0_1, P^0_t\rangle$-colocal, and that N is $\langle P^0_t, \infty\rangle$-local.

Proof of Theorem 3.1.1: We need to show that $F^\infty = 0$. In other words, given a map $h : M \to N$ of infinite filtration, we show that it is stably trivial. We may assume that h is of degree 0. We inductively construct

$$M = M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$$

and maps $h_k : M_k \to N$ (stably) factoring h such that for some fixed c and for each k

- M_k is $\langle P^0_1, P^0_t\rangle$-colocal,
- $H(f_k, P^0_t) = 0$, and
- $|M_k| \geq c$.

Then h factors through $M_\infty = \operatorname{colim} M_k$; but M_∞ is bounded below and $\langle P^0_1, P^0_t - 1\rangle$-colocal (since P-homology commutes with colimits—see [Ma1], 19.15), so any map from M_∞ to $N = N\langle P^0_t, \infty\rangle$ is stably trivial (1.2.9). As we will see below, both the M_k's and c depend on $|N|$ in an explicit fashion.

We describe the induction. Let $d = \max \deg A(t-1)$, let $p = |P^0_t| = 2^t - 1$, and choose m so that $|N| - p > m$. Assume that we have

$$\begin{tikzcd} M_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & \cdots \arrow[r, "f_{k-1}"] & M_k \arrow[d, "h_k"] \\ N \arrow[r, equal] & N \arrow[r, equal] & \cdots \arrow[r, equal] & N \end{tikzcd}$$

commuting up to stable equivalence. We define M_{k+1} and f_k in two stages; we will define h_{k+1} later. Let K be the $A(t-1)$-submodule of M_k generated by elements of degree no more than $m-d$ (so that max deg $K \leq m < |N| - p$). Since $A(t-1)$ is a finite Hopf algebra, we can find V so that we have $K \hookrightarrow A(t-1) \otimes V$. Define an A-module M' by the short exact sequence

$$0 \to A \otimes_{A(t-1)} K \xrightarrow{g} M_k \oplus (A \otimes V) \to M' \to 0.$$

(To control $|M'|$, we will actually use a particular choice of V, described in the proof of Lemma 3.3.5.) Let $\{[y_i]\}$ be a basis for $H(M', P_t^0)$; let W be the sub-vector space of $H(M', P_t^0)$ generated by y_i with $|y_i| > m - d$. For each such y_i, let z_i be the corresponding basis element of $\tilde{W}$, i.e., "$z_i = \Sigma^{-p} y_i$." Define M_{k+1} by this short exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & A/AP_t^0 \otimes W & \to & M' \oplus (A \otimes \tilde{W}) & \to & M_{k+1} & \to & 0 \\ & & y_i & \mapsto & y_i + P_t^0 z_i & & & & \end{array}$$

Define f_k as the composite

$$M_k \hookrightarrow M_k \oplus (A \otimes V) \to M' \hookrightarrow M' \oplus (A \otimes \tilde{W}) \to M_{k+1}.$$

We will see below that in the first stage of this construction the low-dimensional P_t^0-homology is killed off (more precisely, the map $H_i(M_k) \to H_i(M')$ is 0 when $i \leq m - d$), and in the second stage the rest of the P_t^0-homology is killed off. By doing things this way, we can also keep control over the lower bounds of all of the modules in the construction.

We describe our induction more precisely. By induction we prove the following five things:

(a) M_{k+1} is $\langle P_1^0, P_t^0 \rangle$-colocal.

(b) $H(f_k, P_t^0) = 0$.

(c) There exists c independent of k so that (for a particular choice of V at each stage) $|M_{k+1}| \geq c$.

(d) $|H(M_{k+1}, P_v^u)| \geq m - d - p + 1$ for all P_v^u with $u < v$.

(e) There is an infinite filtration map $h_{k+1} : M_{k+1} \to N$ so that $h_{k+1} \circ f_k \simeq h_k$.

(We have discussed the need for parts (a), (b), and (c) earlier; we use part (d) to prove (c). We also clearly need the existence part of (e); the fact that h_k is of infinite filtration will let us construct h_{k+1}.)

The first two parts of our induction are easy: $A \otimes_{A(t-1)} K$ and $A/AP_t^0 \otimes W$ are $\langle P_1^0, P_t^0 \rangle$-colocal (see [Ma1], 19.21), and hence so are M' and M_{k+1}. Also, we have $H(f_k) = 0$: the map $H_i(M_k) \to H_i(M')$ is 0 when $i \leq m-d$, and the map $H_i(M') \to H_i(M_{k+1})$ is 0 when $i > m-d$. Why? The second of these is clear: $H_i(A/AP_t^0 \otimes W) \to H_i(M') \to H_i(M_{k+1})$ is exact, and the map $H_i(A/AP_t^0 \otimes W) \to H_i(M')$ is surjective when $i > m-d$. The first is just a little harder: we have an $A(t-1)$-linear map $j : K \to M_k$, and $H_i(j)$ is an isomorphism when $i \leq m-d$ (see Lemma 3.3.4). We have

$$\begin{array}{ccccc}
H_i(A \otimes_{A(t-1)} K) & \xrightarrow{H(g)} & H_i(M_k) & \longrightarrow & H_i(M') \\
\uparrow & & \uparrow{\scriptstyle H(j)} & & \\
H_i(K) & = & H_i(K) & &
\end{array}$$

where the top row is exact. So if $[z] \in H_i(M_k)$ with $i \leq m-d$, then $[z] \in \text{im } H(j)$ and hence $[z] \in \text{im } H(g)$; so by exactness $H(f_k)[z] = 0$.

Lemma 3.3.4. *Let $j : K \to M_k$ be the inclusion of $A(t-1)$-modules. Then for $P_v^u \in A(t-1)$, we have $|\text{ker } H(j, P_v^u)| > m-d+|P_v^u|$ and $|\text{cok } H(j, P_v^u)| > m-d$.*

Proof: If $0 \neq [a] \in \text{ker } H(j, P_v^u)$, then $ja = P_v^u b$ for some $b \in M_k - K$. Thus $|b| > m-d$, so $|a| > m-d+|P_v^u|$. If $[a] \notin \text{im } H(j, P_v^u)$, then $a \notin K$, so $|a| > m-d$. $\square$

We need to show that the M_k's are uniformly bounded below (parts (c) and (d) of our induction).

Lemma 3.3.5. *There exists a number c (independent of k) so that $|M_{k+1}| \geq c$.*

Proof: We prove this by induction. By Lemma 3.3.6 for M_k and Lemma 3.3.4, we have $|H(K, P_v^u)| \geq m-d-p+1$ for all $P_v^u \in A(t-1)$. By 19.7 in [Ma1], there exists a number e (independent of K) so that if $H_i(K, P_v^u) = 0$ when $i < r$, for all $P_v^u \in A(t-1)$, then $K = K_1 \oplus K_2$, where K_1 is $A(t-1)$-free and $|K_2| \geq r-e$. Thus, if $H_i(K, P_v^u) = 0$ when $i < r$, then we have $K \hookrightarrow K_1 \oplus (A(t-1) \otimes V')$ with $|V'| \geq r-e-d$.

Let $c = m-2d-p-e+1$. In our case, we have $K = K_1 \oplus K_2$ with K_1 free over $A(t-1)$ and (by induction on k) $|K_1| \geq c$; by the above arguments, $|K_2| \geq c+d$, so in all we have $K \hookrightarrow A(t-1) \otimes V$ with $|V| \geq c$. Therefore, since M' is a quotient of $M_k \oplus (A \otimes V)$ with $|V|$ and $|M_k|$ at least c, then the same bound holds for $|M'|$. Finally, M_{k+1} is a quotient of $M' \oplus (A \otimes \tilde{W})$, with $|\tilde{W}| > m-d-p > c$. $\square$

To finish parts (c) and (d) of the induction, we just need to prove

Lemma 3.3.6. *For all P_v^u and each k, $|H(M_{k+1}, P_v^u)| \geq m - d - p + 1$.*

Proof: By part (a) of our induction, we can assume that $P_v^u \in A(t-1)$, so $|P_v^u| \leq |P_t^0|$. Consider

$$\begin{tikzcd} A \otimes_{A(t-1)} K \arrow[r, "g"] & M_k \arrow[r] & M' \\ K \arrow[u, "b"] \arrow[r, equal] & K \arrow[u, "j"] & \end{tikzcd}$$

We want to show that $|H(M', P_v^u)| \geq m - d - p + 1$. We have a short exact sequence

$$0 \to \text{cok } H_i(g, P_v^u) \to H_i(M', P_v^u) \to \ker H_{i+|P_v^u|}(g, P_v^u) \to 0,$$

so we need information about $H(g, P_v^u)$. By induction, $|H(M_k, P_v^u)| \geq m - d - p + 1$, so $|\text{cok } H(g, P_v^u)| \geq m - d - p + 1$. What about $|\ker H(g, P_v^u)|$? There is a spectral sequence ([Ma1], 19.20) as follows: $(A \otimes_{A(t-1)} \mathbf{F}_2) \otimes H(K, P_v^u) \Rightarrow H(A \otimes_{A(t-1)} K, P_v^u)$. This spectral sequence comes from filtering $A \otimes_{A(t-1)} K$ and applying P_v^u-homology. Note that the filtration satisfies

$$F_s(A \otimes_{A(t-1)} K) = \begin{cases} 0 & s \leq 0 \\ 1 \otimes K & s = 1 \\ V_s \otimes K & s \geq 2 \ \text{ (where } V_s \text{ is a trivial } A\text{-module).} \end{cases}$$

Since $|H(K, P_v^u)| \geq m-d-p+1$ and $A \otimes_{A(t-1)} \mathbf{F}_2$ is zero between degrees 0 and $p+1$, then $|H(F_s/F_{s-1}, P_v^u)| \geq m - d + 2$ for $s \geq 2$. The differential d^r decreases filtration degree by r and increases internal degree by $|P_v^u|$; hence the differentials only affect $E^r_{s,t}$ for $s \geq 2$, $t \geq m - d + 2$. Thus $H_i(b, P_v^u) : H_i(K, P_v^u) \to H_i(A \otimes_{A(t-1)} K, P_v^u)$ is an isomorphism when $i < m - d + 1$. So we have

$$\begin{tikzcd} H(A \otimes_{A(t-1)} K, P_v^u) \arrow[r, "H(g, P_v^u)"] & H(M_k, P_v^u) \\ H(K, P_v^u) \arrow[u, "H(b, P_v^u)"] \arrow[r, equal] & H(K, P_v^u) \arrow[u, "H(j, P_v^u)"] \end{tikzcd}$$

Now, $H(b, P_v^u)$ is an isomorphism when $i < m-d+2$, and $H(j, P_v^u)$ is an isomorphism when $i < m - d + 1$ (by Lemma 3.3.4), so $H(g, P_v^u)$ is an isomorphism when $i < m - d + 1$. Thus

$$\begin{aligned} |H(M', P_v^u)| &= \min(|\text{cok } H_*(g, P_v^u)|, |\ker H_*(g, P_v^u)| - |P_v^u|) \\ &= m - d - |P_v^u| + 2 \\ &\geq m - d - p + 1. \end{aligned}$$

If $P^u_v \neq P^0_t$, then because A/AP^0_t is monochromatic we have $H(M', P^u_v) = H(M_{k+1}, P^u_v)$; so we're done.

If $P^u_v = P^0_t$, then we consider the exact sequence

$$\cdots \longrightarrow H_i(M') \longrightarrow H_i(M_{k+1}) \longrightarrow H_{i+p}(A/AP^0_t \otimes W) \xrightarrow{c} H_{i+p}(M') \longrightarrow \cdots,$$

where c is a monomorphism when $i + p < m - d + 1$. So

$$|H(M_{k+1})| \geq \min(|H(M')|, m - d - p + 1) = m - d - p + 1.$$

$\square$

To complete our induction, we need to construct $h_{k+1} : M_{k+1} \to N$ of infinite filtration, so that $h_{k+1} \circ f_k \simeq h_k$. We use the following lemma:

Lemma 3.3.7. *Let K be a finite $A(t-1)$-module; choose $m \geq \max\deg K$. If we have a short exact sequence of A-modules*

$$0 \to A \otimes_{A(t-1)} K \to M \to M' \to 0$$

and an A-module N with $|N| - p > m$, then any map $h : M \to N$ (of degree 0) factors stably through M':

$$\begin{array}{ccc} M & \to & M' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} \\ N & = & N \end{array}$$

Also, if h has infinite filtration, then so does h'.

(We will prove this later.) Applying the lemma gives

$$\begin{array}{ccc} M_k & \to & M' \\ \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h'} \\ N & = & N \end{array}$$

with h' of infinite filtration. Recall that $\{[y_i]\}$ is a basis for $H(M')$; let U be isomorphic to $H(M')$ as vector spaces, and give U the trivial A-action (so we have $W \hookrightarrow U$). Then we have

$$\begin{array}{ccccccccc} 0 & \to & A/AP^0_t \otimes W & \to & M' \oplus (A \otimes \tilde{W}) & \to & M_{k+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 0 & \to & A/AP^0_t \otimes U & \to & M' \oplus (A \otimes \tilde{U}) & \to & M'' & \to & 0 \end{array}$$

where all the vertical maps are split inclusions. Since M'' is the first stage in a Margolis Adams tower for M', and h' is of infinite filtration, we have

$$\begin{array}{ccc} M' & \rightarrow & M'' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h''} \\ N & = & N \end{array}$$

with h'' of infinite filtration. Define $h_{k+1} : M_{k+1} \rightarrow N$ by $h_{k+1} = h'' \circ j$. Since h'' is of infinite filtration, then so is h_{k+1}.

It remains to prove Lemma 3.3.7; this is somewhat lengthy. We do the easy part, then outline the rest, and then do the rest.

Proof of Lemma 3.3.7: The easy part is the existence of h': because max deg $K < |N|$, there are no maps (of degree 0) from $A \otimes_{A(t-1)} K$ to N; thus $h : M \rightarrow N$ induces $h' : M' \rightarrow N$.

To show that h' is of infinite filtration, the idea is to inductively construct a tower as in Corollary 3.2.4, part (b), and to factor h' through each stage of that tower. This is done by constructing a Margolis Adams tower for $A \otimes_{A(t-1)} K$ and building the tower for M' out of that and out of a tower for M. The tower for $A \otimes_{A(t-1)} K$ will be constructed so that (as in the preceding paragraph) there are no maps from each stage of that tower to N; thus at each stage the map to N from the tower for M will induce a map from the tower for M', which is what we want.

First, we construct the tower for $A \otimes_{A(t-1)} K$.

Lemma 3.3.8. *Given L, a finite $B = A(t-1)/A(t-1)P_t^0$-module with* max deg $L = m$, *then L has a B-free resolution*

$$0 \leftarrow L \leftarrow B_0 \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots$$

with $B_k = B \otimes V_k$, such that $\max_k(\text{max deg } V_k - kp) = m$.

In other words, $\mathrm{Ext}_B^{*,*}(L, \mathbf{F}_2)$ is zero below the line through the origin of slope $\frac{1}{p-1}$.

Corollary 3.3.9. *$H(A \otimes_{A(t-1)} L)$ has an A_t^0-resolution*

$$0 \leftarrow H(A \otimes_{A(t-1)} L) \leftarrow R_0 \leftarrow R_1 \leftarrow R_2 \leftarrow \cdots$$

where $R_k = A_t^0 \otimes V_k$, with V_k as above.

Proof of Corollary 3.3.9: By 19.26 in [Ma1], B is a sub-Hopf algebra of A_t^0 and $H(A \otimes_{A(t-1)} L) = A_t^0 \otimes_B H(L)$. Thus applying $A_t^0 \otimes_B -$ to the B-resolution for $H(L)$ gives us the desired A_t^0-resolution. □

Proof of Lemma 3.3.8: This is fairly standard: one has a B-resolution for $\mathbf{F}_2$ using a May-type resolution (see [Rav], Section 3.2), which as a vector space is isomorphic to the polynomial algebra $\mathbf{F}_2[h_{v,u} | P_v^u \in A(t-1), P_v^u \neq P_t^0]$, with $|h_{v,u}| = (1, |P_v^u|)$. So this resolution satisfies our conditions.

For an arbitrary module L one constructs the resolution by induction on m using the resolution for $\mathbf{F}_2$. □

We can realize the resolution in Corollary 3.3.9: let $M_0'' = A \otimes_{A(t-1)} K$, and define M_{k+1}'' by

$$0 \to \Sigma^{-kp} A/AP_t^0 \otimes V_k \to M_k'' \oplus (A \otimes \tilde{V}_k) \to M_{k+1}'' \to 0.$$

Applying Lemma 3.2.2 to $j : A \otimes_{A(t-1)} K \to M$ gives $j_k : M_k'' \to M_k$ for each $k \geq 0$. Using a simple inductive argument based on the fact that M_k'' is deloopable (M_k'' is $\langle P_1^0, P_t^0 \rangle$-colocal, so use [Ma1], 22.10), we may assume that each j_k is an inclusion. Define M_k' to be the cokernel of j_k; then the maps $M_k'' \to M_{k+1}''$ and $M_k \to M_{k+1}$ induce $M_k' \to M_{k+1}'$. So we have this commutative diagram (with exact columns):

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
M_0'' \arrow[r] \arrow[d, "j_0"] & M_1'' \arrow[r] \arrow[d, "j_1"] & M_2'' \arrow[r] \arrow[d, "j_2"] & \cdots \\
M_0 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & \cdots \\
M_0' \arrow[r] \arrow[d] & M_1' \arrow[r] \arrow[d] & M_2' \arrow[r] \arrow[d] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}$$

Applying $H(-)$ to this diagram gives

$$\begin{tikzcd}
 & H(M_{k+1}) \arrow[r] \arrow[d] & H(M_{k+2}) \arrow[d] \\
H(M_k') \arrow[r] \arrow[d, "\partial"] & H(M_{k+1}') \arrow[r] \arrow[d, "\partial"] & H(M_{k+2}') \\
H(M_k'') \arrow[r] & H(M_{k+1}'') &
\end{tikzcd}$$

with the middle column exact at $H(M_{k+1}')$. Since the top and bottom rows come from Margolis Adams towers, then both maps there are zero. So by a simple diagram chase we have $H(M_k') \xrightarrow{0} H(M_{k+2}')$.

Thus, once we have factored h' through the "tower" for M', we can apply part (b) of Corollary 3.2.4 to

$$M' = M'_0 \to M'_2 \to M'_4 \to \cdots$$

to conclude that h' is of infinite filtration.

Of course, we factor h' by induction. We have

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \Sigma^{-kp} A/AP_t^0 \otimes V_k & \to & M''_k & \to & M''_{k+1} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_k & \to & M_{k+1} & \xrightarrow{h_{k+1}} & N \\
 & & & & \downarrow & & \downarrow & & \| \\
 & & & & M'_k & \to & M'_{k+1} & \xrightarrow{h'_{k+1}} & N \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

To show that h'_{k+1} exists, we need to show that $h_{k+1}|M''_{k+1} = 0$. By induction, we show in fact that $\{M''_{k+1}, N\}^{0,0} = 0$. We have $\{M''_k, N\}^{0,0} = 0$, and by Corollary 3.3.9 we have $(\max \deg V_k) - kp \leq m$. Since $|N| - p > m$, then

$$\begin{aligned}
\{\Sigma^{-kp} A/AP_t^0, N\}^{1,0} &= \{\Sigma^{-kp} A/AP_t^0, N\}^{0,p} \\
&= \{\Sigma^{-kp} A/AP_t^0, \Sigma^{-p} N\}^{0,0} \\
&= 0.
\end{aligned}$$

So by the long exact sequence in $\{-,-\}^{*,*}$, we have $\{M''_{k+1}, N\}^{0,0} = 0$. Therefore $h_{k+1}|M''_{k+1} \simeq 0$, so $h_{k+1} : M_{k+1} \to N$ extends to a map from the cofiber. In other words, h'_{k+1} exists (so that the appropriate diagram commutes). $\square$

3.4 Comments on the non-primitive case

The salient feature of P_t^0 in Section 3.3 is that it is the largest degree differential in the sub-Hopf algebra $A(t-1)$—the fact that it is primitive is secondary. Thus one possible approach to proving convergence for non-primitive P_t^s's is to analyze the role that $A(t-1)$ plays in proving convergence, and then try to find a suitable substitute.

There are two place that using $A(t-1)$ is important: Lemma 3.3.6 for $H(-, P_v^u)$ depends on $A(t-1)$ agreeing with A through the degree of P_v^u; one may be able to prove this for sub-Hopf algebras that don't satisfy this, by a more careful analysis of the spectral sequence converging to $H(A\otimes_B K, P_v^u)$. Lemma 3.3.8 and Corollary 3.3.9 depend on other features of $A(t-1)$ that may be harder to deal with. The result of the corollary is the sticking point, since the present proof depends on some information about A_t^0; in particular, it uses the fact that A_t^0 is free over $A(t-1)/A(t-1)P_t^0$. If we can get a result like that for A_t^s, then building a resolution over some subalgebra of the Steenrod algebra should be easy, as in the lemma.

One route is to try $B = A(s+t-1)$. Then there is no problem in proving the analog of Lemma 3.3.6—as above, for each P_v^u this only depends on A and B agreeing through dimension $|P_v^u|$. Note that we lose part (a) of our induction; instead, we can only prove that M_k is $\langle P_1^0, P_{s+t}^0\rangle$-colocal. This is sufficient, though, because of the following diagram (where $M_\infty = \text{colim } M_k$):

$$\begin{array}{ccccc} M & \longrightarrow & M_\infty & \longrightarrow & N \\ \big\uparrow{\scriptstyle\simeq} & & \big\uparrow & & \\ M & \longrightarrow & M_\infty\langle P_1^0, P_t^s\rangle & = & M_\infty\langle P_t^0, P_t^s - 1\rangle \end{array}$$

So the map from M to $N = N\langle P_t^s, \infty\rangle$ factors through a $\langle P_1^0, P_t^s - 1\rangle$-module, and hence is trivial.

We do have to deal with Corollary 3.3.9, i.e., building a Margolis Adams tower for $A \otimes_B K$. And as expected, part of the difficulty here is understanding something about A_t^s when $s > 0$.

We need a correction of a result of Margolis. Let $C(P_t^s) = \{a \in A : [a, P_t^s] = 0\}$.

Proposition 3.4.1 (cf. [Ma1], 19.4). *Every element in A_t^s has a representative in A which commutes with P_t^s, so*

$$A_t^s \cong \frac{C(P_t^s)}{C(P_t^s) \cap (AP_t^s + P_t^s A)}.$$

Margolis' proof depends on showing that $C(P_t^s) \cap AP_t^s = C(P_t^s) \cap (AP_t^s + P_t^s A)$; however, he only shows that $C(P_t^s) \cap AP_t^s = C(P_t^s) \cap P_t^s A$. The desired equality does not hold in general; it fails, for example, in dimension 7 when $P_t^s = P_2^0$. □

Hence, if we can get some information on $C(P_t^s)$, then we may learn something about A_t^s. Here are some miscellaneous results (here $\text{Sq}(r_1, r_2, \ldots)$ is the Milnor basis element dual to $\xi_1^{r_1}\xi_2^{r_2}\cdots$):

Proposition 3.4.2. *(a) $[P_t^s, P_v^u] = 0$ if and only if $u < t$ and $v > s$.*

(b) Hence $C(P_t^s) \supseteq \text{alg}\,\{P_v^u : u < t, v > s\}$.

(c) $[P^s_t, \mathrm{Sq}(r_1, r_2, \ldots, r_{t-1})] = 0$ *if and only if* $r_i = 0$ *for all* i.

(d) *If* $a \in A$ *is a sum of Milnor basis elements of the form* $\mathrm{Sq}(r_1, r_2, \ldots, r_{t-1})$*, then* $[P^s_t, a] = 0$ *if and only if* $a = 0$ *or* $a = \mathrm{Sq}(0, 0, \ldots)$.

Proof: These are all exercises in Milnor multiplication. □

Corollary 3.4.3. *The connectivity of the augmentation ideal of* A^s_t *is* $|IA^s_t| = 2^{s+1} - 1$.

Proof: The connectivity of the augmentation ideal of $C(P^s_t)$ is $2^{s+1} - 1$, since the lowest degree non-trivial element in $C(P^s_t)$ is P^0_{s+1}. Also, $|AP^s_t + P^s_t A| = |P^s_t| = 2^{s+t} - 2^s > 2^{s+1} - 1$. □

Proposition 3.4.2 brings to mind some questions: for example, is the inclusion in part (b) an equality? If not, is $C(P^s_t)$ a sub-Hopf algebra of A? In trying to construct resolutions as in Corollary 3.3.9, one might also want to know: is $P^{s-r}_{t+r} \in C(P^s_t) \cap (AP^s_t + P^s_t A)$? This is a special case of the following: if we look at the algebra of operations for P^s_t-homology over $A(s+t-1)$ (call it B^s_t), is it true that if $|P^u_v| \geq |P^s_t|$, then $[P^u_v] = 0$ in B^s_t?

Unfortunately, the answer to all of these questions is no:

Example 3.4.4. *Consider* P^2_3*. It is easy to check that* $[P^2_3, \mathrm{Sq}(1, 0, 6)] = 0$*, while certainly* $\mathrm{Sq}(1, 0, 6) \notin \mathrm{alg}\{P^u_v : u < 2, v > 3\}$*. Also, the coproduct on* $\mathrm{Sq}(1, 0, 6)$ *has a term* $\mathrm{Sq}(1) \otimes \mathrm{Sq}(0, 0, 6)$ *in it, and* $\mathrm{Sq}(1) = \mathrm{Sq}^1 \notin C(P^2_3)$ *(since* $[P^2_3, \mathrm{Sq}^1] = \mathrm{Sq}(0, 0, 2, 1)$*). Lastly, one can easily check that* $P^1_4 \notin C(P^2_3) \cap (AP^2_3 + P^2_3 A)$*. (Note, though, that* $P^1_4 \in AP^2_3 A$*:* $P^1_4 = [P^2_3, \mathrm{Sq}^2] + \mathrm{Sq}^1 P^2_3 \mathrm{Sq}^1$*.)*

The problem of determining $C(P^s_t)$ in low degrees (say, below $|P^s_t|$) still may be accessible, and helpful. For example, a good description of $C(P^s_t) \cap A(s+t-1)$ as a sub-algebra of $C(P^s_t)$ would be a useful step in determining the structure of B^s_t as a sub-algebra of A^s_t.

Chapter 4

Computing $\{\mathbf{F}_2, \mathbf{F}_2\}_{A(1)}$

4.1 Introduction

In this chapter we apply the machinery that we have developed to the sub-Hopf algebra $A(1)$ of A. Unless otherwise indicated, all modules are $A(1)$-modules and all maps are $A(1)$-linear; in particular, we will leave the subscript $A(1)$ off of $\{-,-\}$.

Note that none of the results here are new. The purpose of this chapter is two-fold—the main reason is to give an example of how to apply our machinery (especially in a situation where we can actually do all of the calculations); another reason is to display some periodic behavior in $\mathrm{Ext}_{A(1)}$ that may not have been apparent before.

The sub-Hopf algebra $A(1)$ of the mod 2 Steenrod algebra contains two P^s_t's with square 0, namely $P^0_1 = Q_0 = \mathrm{Sq}^1$ and $P^0_2 = Q_1 = [\mathrm{Sq}^1, \mathrm{Sq}^2]$. Thus the Margolis chromatic spectral sequence collapses at E_2; i.e., it is a long exact sequence:

$$\cdots \leftarrow \{M\langle P^0_1\rangle, N\}^* \leftarrow \{M, N\}^* \leftarrow \{M\langle P^0_2\rangle, N\}^* \leftarrow \{M\langle P^0_1\rangle, N\}^{*-1} \leftarrow \cdots.$$

It is sometimes possible to find nice modules of types $M\langle P^0_1\rangle$ and $M\langle P^0_2\rangle$ and so compute $\{-, N\}$ with those; we can instead use the Margolis Adams spectral sequences and a Bockstein spectral sequence. We carry out both of these methods below, with $M = N = \mathbf{F}_2$.

Note 4.1.1. *In the figures in this chapter we use the following notation: when representing $A(1)$-modules, dots are rank 1 vector spaces, the short lines are Sq^1's, and the longer curves are Sq^2's; a number next to a dot shows the degree of that class. When describing maps between modules, the images of $A(1)$-generators are shown; if a generator hits the sum of two (or more) elements, then the elements in the image*

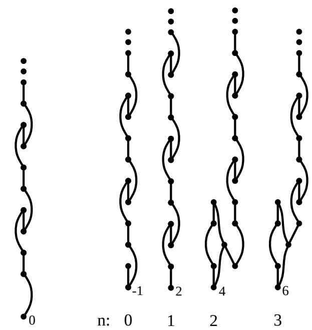

Figure 4.1: $\mathbf{F}_2\langle P_1^0\rangle$ and $\Omega^n\mathbf{F}_2\langle P_2^0\rangle$, $n = 0, 1, 2, 3$

are circled with a dotted line. Also, the pictures of $\mathrm{Ext}_{A(1)}(-,-)$ *and* $\{-,-\}_{A(1)}$ *are drawn with the usual axes—the vertical axis is homological or loop degree, and the horizontal axis is "*$t-s$*," the stem degree.*

4.2 Using models for $\mathbf{F}_2\langle P_1^0\rangle$ and $\mathbf{F}_2\langle P_2^0\rangle$

We let $M = N = \mathbf{F}_2$. We use modules of type $\mathbf{F}_2\langle P_1^0\rangle$ and $\mathbf{F}_2\langle P_2^0\rangle$ as described in the proof of Theorem 23.12 in [Ma1]; we have $\Omega\mathbf{F}_2\langle P_1^0\rangle \simeq \Sigma\mathbf{F}_2\langle P_1^0\rangle$ and $\Omega^4\mathbf{F}_2\langle P_2^0\rangle \simeq \Sigma^{12}\mathbf{F}_2\langle P_2^0\rangle$ (see Figure 4.1 for pictures of $\mathbf{F}_2\langle P_1^0\rangle$ and $\Omega^n\mathbf{F}_2\langle P_2^0\rangle$, $n = 0, 1, 2, 3$).

$\mathbf{F}_2\langle P_1^0\rangle$ has a single $A(1)$-generator every four dimensions, so we have

$$\mathrm{Hom}_{A(1)}(\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2) \cong \mathbf{F}_2\langle t_0, t_4, t_8, \ldots\rangle.$$

It is also clear that none of these is stably trivial; for example, $\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^2$ is zero on the generator in degree $4n$, so we can't factor the map t_{4n} through the only nonzero map $A(1) \to \mathbf{F}_2$. So,

$$\{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\}^0 \cong \mathbf{F}_2\langle t_0, t_4, t_8, \ldots\rangle.$$

Then, since $\Omega\mathbf{F}_2\langle P_1^0\rangle \simeq \Sigma\mathbf{F}_2\langle P_1^0\rangle$, we have

$$\{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\}^* \cong \mathbf{F}_2[b^{\pm 1}]\langle t_0, t_4, t_8, \ldots\rangle,$$

where $|b| = (1, 1)$; this is an isomorphism of vector spaces.

We claim that the b action is given by multiplication by $h_0 \in \{\mathbf{F}_2, \mathbf{F}_2\}^{1,1}$; to see this, we compute $h_0 t_{4n}$ by the composition $\Omega\mathbf{F}_2\langle P_1^0\rangle \xrightarrow{\Omega t_{4n}} \Sigma^{4n}\Omega\mathbf{F}_2 \xrightarrow{\Sigma^{4n}h_0} \Sigma^{4n+1}\mathbf{F}_2$ (see Figure 4.2); this is nonzero, so it must be bt_{4n}. This solves the extensions: we have

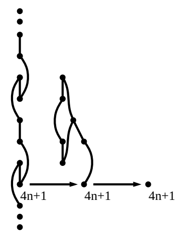

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Figure 4.2: $h_0 t_{4n} : \Omega\mathbf{F}_2\langle P_1^0\rangle \to \Sigma^{4n-1}\Omega\mathbf{F}_2 \to \Sigma^{4n}\mathbf{F}_2$

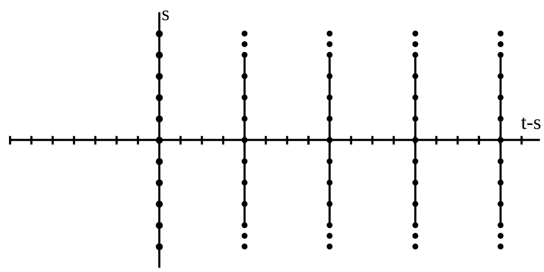

Figure 4.3: $\{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\}^{s,t}$

an isomorphism over $\{\mathbf{F}_2, \mathbf{F}_2\}^*$:

$$\{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\}^* \cong \mathbf{F}_2[h_0^{\pm 1}]\langle t_0, t_4, t_8, \ldots\rangle$$

(see Figure 4.3).

We compute $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^*$ in more or less the same way; since $\mathbf{F}_2\langle P_2^0\rangle$ is Ω-periodic with period 4, we only need to work out $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^u$ for $u = 0, 1, 2, 3$. Arguing as above, we get

$$\begin{aligned}
\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^0 &\cong \mathbf{F}_2\langle w_{-1}, w_3, w_7, w_{11}, \ldots\rangle, \\
\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^1 &\cong \mathbf{F}_2\langle x_2, x_4, x_8, x_{12}, \ldots\rangle, \\
\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^2 &\cong \mathbf{F}_2\langle y_4, y_5, y_9, y_{13}, \ldots\rangle, \\
\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^3 &\cong \mathbf{F}_2\langle z_6, z_{10}, z_{14}, z_{18}, \ldots\rangle.
\end{aligned}$$

To compute extensions we use $h_0, h_1 \in \{\mathbf{F}_2, \mathbf{F}_2\}^*$, $|h_0| = (1, 1)$, $|h_1| = (1, 2)$; via composition, we see that all of the extensions that are possible degree-wise actually occur; this finishes the computation of $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^*$ (see Figure 4.4 for a description).

Now we want to compute the boundary map in the Margolis chromatic spectral sequence. The boundary map is induced by $\Omega^n\mathbf{F}_2\langle P_2^0\rangle \to \Omega^{n-1}\mathbf{F}_2\langle P_1^0\rangle$, so we want to

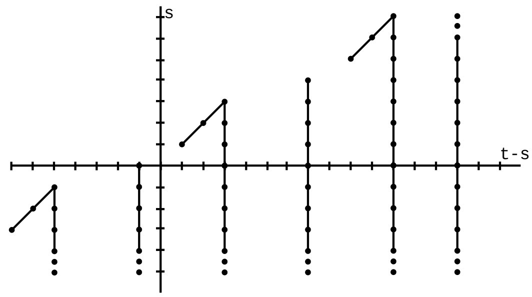

Figure 4.4: $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^{s,t}$

understand these maps (and it's enough to do so for $n = 1, 2, 3, 4$, using Ω-periodicity). For a pictorial description of these, see Figure 4.5. The idea is that $\Omega^n\mathbf{F}_2\langle P_2^0\rangle$ has a family of generators, one every four dimensions; these map bijectively to a similar family in $\Omega^{n-1}\mathbf{F}_2\langle P_1^0\rangle$. This means that we get all possible differentials. See Figure 4.6 for a picture of the spectral sequence and the differentials.

The extensions are automatic—the element in $\{\mathbf{F}_2, \mathbf{F}_2\}^{1,1}$ is by definition h_1, and the other h_1-extensions follow from periodicity. This computes $\{\mathbf{F}_2, \mathbf{F}_2\}^{**}$ (see Figure 4.7).

4.3 Using the Margolis Adams spectral sequence

Now we use the Margolis Adams spectral sequence and a Bockstein spectral sequence. To carry this out, we need to know the algebras of operations for P_1^0- and P_2^0-homology. We have

$$\begin{aligned} A(1)_1^0 &= \{A(1)/A(1)P_1^0, A(1)/A(1)P_1^0\} \\ &= H(A(1)/A(1)P_1^0; P_1^0) \\ &= E[z_5], \quad |z_5| = 5 \end{aligned}$$

(where z_5 is the image of $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2$), and

$$\begin{aligned} A(1)_2^0 &= H(A(1)/A(1)P_2^0; P_2^0) \\ &= H(E[\mathrm{Sq}^1, \mathrm{Sq}^2]; P_2^0) \\ &= E[\mathrm{Sq}^1, \mathrm{Sq}^2]. \end{aligned}$$

Also, the Bockstein spectral sequence arises from the short exact sequence

$$0 \to \Sigma\mathbf{F}_2 \to A(0) \to \mathbf{F}_2 \to 0,$$

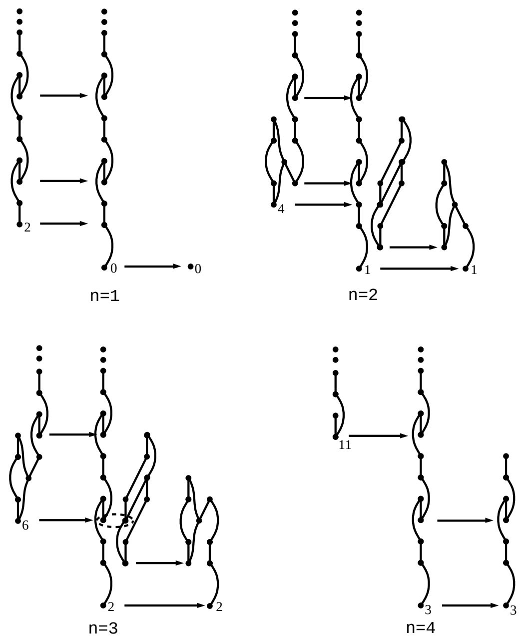

Figure 4.5: $0 \to \Omega^n \mathbf{F}_2\langle P_2^0\rangle \to \Omega^{n-1}\mathbf{F}_2\langle P_1^0\rangle \to \Omega^{n-1}\mathbf{F}_2 \to 0$, for $n = 1, 2, 3, 4$

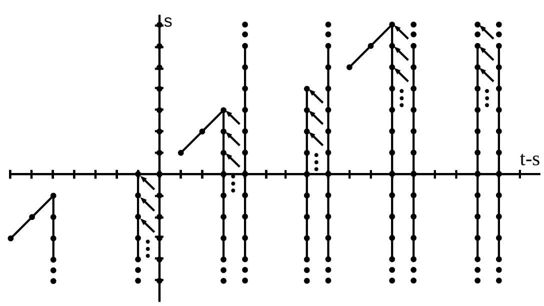

Figure 4.6: Margolis chromatic spectral sequence

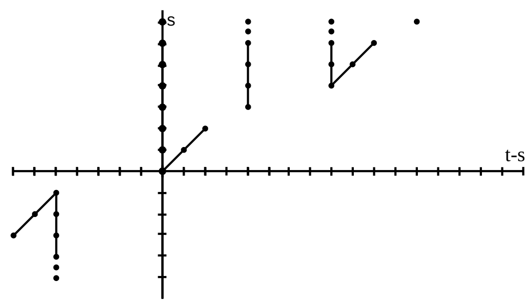

Figure 4.7: $\{\mathbf{F}_2, \mathbf{F}_2\}^{**}$

(here $A(0)$ is the sub-Hopf algebra of A generated by Sq^1, i.e., $A(0) \cong E[\mathrm{Sq}^1]$); for an arbitrary module N, we use this sequence tensored with N (using the diagonal action of $A(1)$):

$$0 \to \Sigma N \to A(0) \otimes N \to N \to 0.$$

The Margolis Adams spectral sequence for P_1^0 looks like

$$E_2 = \mathrm{Ext}_{E[z_5]}(\mathbf{F}_2, \mathbf{F}_2) \otimes \mathbf{F}_2[v_0^{\pm 1}] = \mathbf{F}_2[x_4, v_0^{\pm 1}] \Rightarrow \{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\},$$

where $|x_4| = (1, 5)$ (i.e., stem 4) and $|v_0| = (1, 1)$. There are no possible differentials, so the spectral sequence collapses, and we have

$$\{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\} \cong \mathbf{F}_2[x_4, v_0^{\pm 1}].$$

(See Figure 4.3 for $\{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\}$, with extensions included.)

For P_2^0 we have

$$E_2 = \mathrm{Ext}_{E[\mathrm{Sq}^1,\mathrm{Sq}^2]}(\mathbf{F}_2, A(0)) \otimes \mathbf{F}_2[v_1^{\pm 1}] \Rightarrow \{\mathbf{F}_2\langle P_2^0\rangle, A(0)\};$$

using a change-of-ring isomorphism, we get $E_2 \cong \mathbf{F}_2[h_1, v_1^{\pm 1}]$, where $|h_1| = (1, 2)$ and $|v_1| = (1, 3)$. To compute the differentials here we use a variant on the construction of the spectral sequence—we make an "injective" resolution of $A(0)$ using $E = E[\mathrm{Sq}^1, \mathrm{Sq}^2]$:

$$\begin{tikzcd} A(0) \arrow[r] & \Sigma^{-2}E \arrow[r] \arrow[d] & \Sigma^{-4}E \arrow[r] \arrow[d] & \Sigma^{-6}E \arrow[r] \arrow[d] & \cdots. \\ & \Sigma^{-2}A(0) & \Sigma^{-4}A(0) & \Sigma^{-6}A(0) & \end{tikzcd}$$

Applying $\{\mathbf{F}_2, -\}$ gives an exact couple (the numbers indicate the degrees of the maps—(loop, internal):

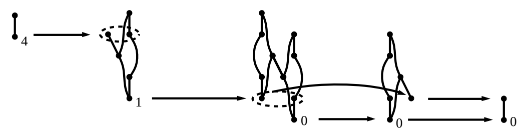

Figure 4.8: $\Sigma^4 A(0) \simeq \Sigma^{-8}\Omega^4 A(0) \to \Sigma^{-6}\Omega^3 A(0) \to \Sigma^{-4}\Omega^2 A(0) \to \Sigma^{-2}\Omega A(0) \to A(0)$

$$\begin{tikzcd}
\{\mathbf{F}_2, A(0)\} \arrow[rr, "(1{,}2)"] & & \{\mathbf{F}_2, A(0)\} \arrow[dl, "(0{,}-2)"] \\
& \{\mathbf{F}_2, E\} \arrow[ul, "(0{,}0)"] &
\end{tikzcd}$$

This leads to a Bockstein spectral sequence for the operation h_1. We have $E_1 \cong \{\mathbf{F}_2, E\}^{**} \cong \Sigma^{-3}\mathbf{F}_2[v_1^{\pm 1}]$, and the degree of the differential is $|d_r| = (1-r, -2r)$; thus the only possible nonzero differentials are $d_3 : v_1^{n+2} \mapsto v_1^n$. Now, E is Ω-periodic with period 1, and $A(0)$ has period 4; this implies that the whole spectral sequence is periodic, via multiplication by v_1^4. So we only have to compute d_3 on v_1^n for $n = 0$, 1, 2, and 3. To understand each of these, we need to understand the maps in

$$\cdots \to \Omega E \to \Omega A(0) \to \Sigma^2 A(0) \to E \to A(0) \to \Omega^{-1}\Sigma^{-2} A(0) \to \cdots;$$

these come from the maps in

$$\Sigma^4 A(0) \simeq \Sigma^{-8}\Omega^4 A(0) \to \Sigma^{-6}\Omega^3 A(0) \to \Sigma^{-4}\Omega^2 A(0) \to \Sigma^{-2}\Omega A(0) \to A(0)$$

and

$$0 \to \Sigma^2 \Omega^n A(0) \to \Omega^n E \to \Omega^n A(0) \to 0$$

for $n = 0, 1, 2, 3$; these are illustrated in Figures 4.8 and 4.9.

Now, v_1^n is given by

$$\mathbf{F}_2 \xrightarrow{v_1^n} \Sigma^{3n-3}\Omega^{-n} E \cong \Sigma^{-3} E;$$

and $d_3 v_1^n : \mathbf{F}_2 \to \Sigma^{3n-9}\Omega^{2-n} E$ is given by

$$\begin{tikzcd}
& & \Sigma^{3n-7}\Omega^{2-n} A(0) \arrow[r] \arrow[d] & \Sigma^{3n-9}\Omega^{2-n} E \\
& & \Sigma^{3n-5}\Omega^{1-n} A(0) \arrow[d] & \\
\mathbf{F}_2 \arrow[r] & \Sigma^{3n-3}\Omega^{-n} E \arrow[r] & \Sigma^{3n-3}\Omega^{-n} A(0) &
\end{tikzcd}$$

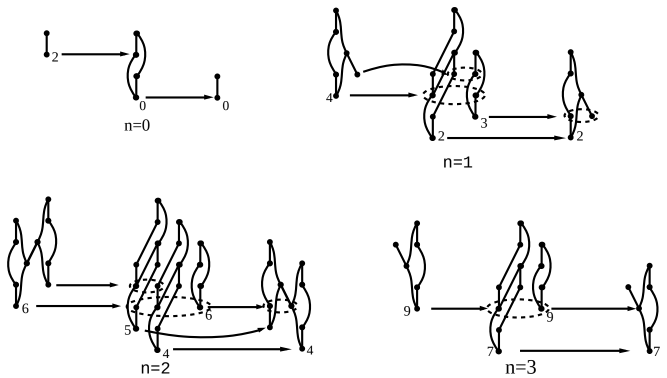

Figure 4.9: $0 \to \Sigma^2\Omega^n A(0) \to \Omega^n E \to \Omega^n A(0) \to 0$ for $n = 0, 1, 2, 3$

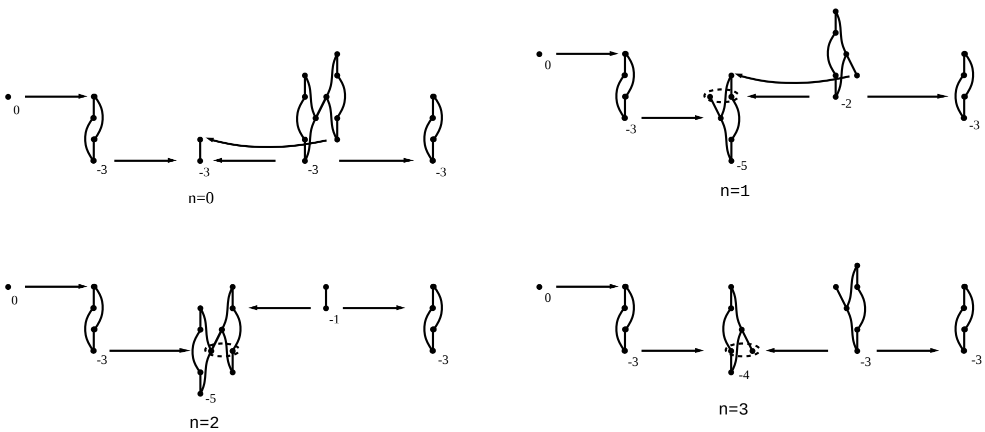

Figure 4.10: Computing $d_3 v_1^n$

In Figure 4.10 we compute these for $n = 0, 1, 2, 3$; we see that the nonzero differentials are $d_3 v_1^2 = v_1^0$ and $d_3 v_1^3 = v_1$.

These give us h_1-torsion of height 3 on v_1^0 and v_1; periodicity gives us the whole picture of v_1^4-periodic lightning flashes (see Figure 4.11).

To compute $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}$, we use the Bockstein spectral sequence

$$E_1 = \{\mathbf{F}_2\langle P_2^0\rangle, A(0)\} \otimes \mathbf{F}_2[h_0^{\pm 1}] \Rightarrow \{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}.$$

Now, $\{\mathbf{F}_2\langle P_2^0\rangle, A(0)\} \cong \{\mathbf{F}_2, A(0)\}$, so we represent elements in E_1 by maps $\mathbf{F}_2 \to A(0)$. This allows us to compute d_1, since d_1 is given by post-composition with the unique nonzero map $A(0) \to \mathbf{F}_2 \to \Sigma^{-1}A(0)$. As usual, by v_1^4-periodicity we only have to worry about homological degrees 0, 1, 2, and 3; we have $\mathbf{F}_2$-generators

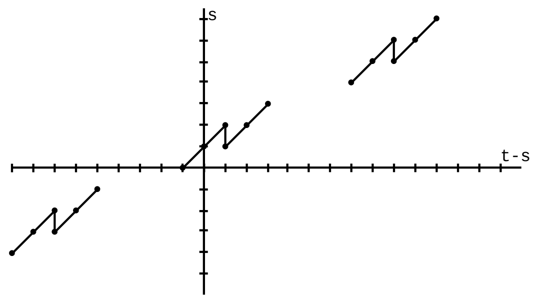

Figure 4.11: $\{\mathbf{F}_2\langle P_2^0\rangle, A(0)\}^{**}$

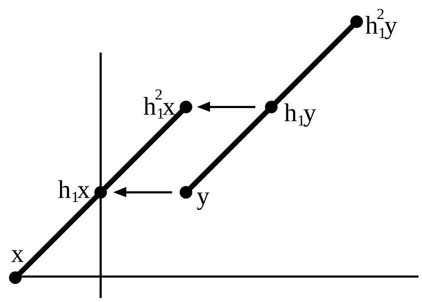

Figure 4.12: $\{\mathbf{F}_2, A(0)\}^s$ for $s = 0, 1, 2, 3$

$x \in \{\mathbf{F}_2, A(0)\}^{0,-1}$, $h_1x \in \{\mathbf{F}_2, A(0)\}^{1,1}$, $y \in \{\mathbf{F}_2, A(0)\}^{1,2}$, $h_1^2x \in \{\mathbf{F}_2, A(0)\}^{2,3}$, $h_1y \in \{\mathbf{F}_2, A(0)\}^{2,4}$, and $h_1^2y \in \{\mathbf{F}_2, A(0)\}^{3,6}$ (see Figure 4.12). We have representatives of these as stable maps (see Figure 4.13), and from these it is clear that $d_1y = h_1x$ and $d_1h_1y = h_1^2x$. Hence at E_2, we are left with x and h_1^2y, so there are no further differentials. Since $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}$ has a vanishing line of slope $\frac{1}{2}$, there are no convergence problems; thus we have computed $\{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}$, modulo extensions (see Figure 4.4). That the h_0 operation in the Bockstein spectral sequence is the same as $h_0 \in \{\mathbf{F}_2, \mathbf{F}_2\}^{1,1}$ follows easily from the construction of the spectral sequence.

Lastly, we want to understand the Margolis spectral sequence from this viewpoint.

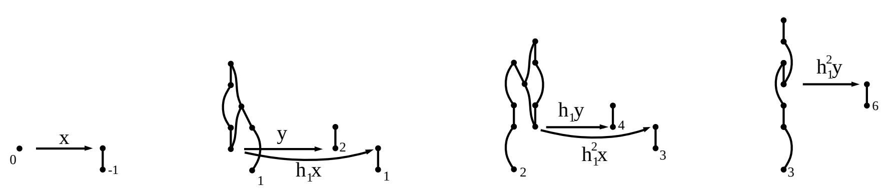

Figure 4.13: Stable map representatives for a basis for $\{\mathbf{F}_2, A(0)\}^u$, $u = 0, 1, 2, 3$

We have

$$\cdots \leftarrow \{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^{u+1} \xleftarrow{\delta} \{\mathbf{F}_2\langle P_1^0\rangle, \mathbf{F}_2\}^u \leftarrow \{\mathbf{F}_2, \mathbf{F}_2\}^u \leftarrow \{\mathbf{F}_2\langle P_2^0\rangle, \mathbf{F}_2\}^u \leftarrow \cdots.$$

This is a long exact sequence of $\{\mathbf{F}_2, \mathbf{F}_2\}$-modules, so in particular δ commutes with multiplication by h_0. Since we know that

$$\{\mathbf{F}_2, \mathbf{F}_2\}^{0,*} = \begin{cases} \mathbf{F}_2, & * = 0, \\ 0, & * \neq 0, \end{cases}$$

we have some differentials forced on us; extending using the h_0-structure gives the collection of differentials shown in Figure 4.6.

We won't worry about extensions here; we know that they all work out, giving us the picture in Figure 4.7.

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