Unstable Modules with Only the Top k Steenrod Operations

by

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Abstract

We study an abelian category of unstable modules with the top k Steenrod operations at the prime 2. We show that this category has homological dimension at most k. We establish forgetful functors, suspension functors, loop functors and Frobenius functors between such modules. The forgetful functors induce an inverse system of Ext groups, and the inverse system stabilizes when the covariant module is bounded above. We define an analogue of the Λ algebra in this context and verify that its cohomology computes Ext.

Thesis Supervisor: Haynes R. Miller
Title: Professor

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This thesis is dedicated to my mom, dad and husband. I want to thank my parents for their constant love and support. I want to thank my husband for accompanying me through this challenging and sometimes difficult period of my life.

Contents

Chapter 1

Introduction

The theme of this thesis is to investigate the category $\mathcal{U}_k$ of unstable modules at the prime 2 where only the top k Steenrod operations are allowed. In general, on a homogeneous element of degree n the top k Steenrod squares on it are $\mathrm{Sq}^n, \mathrm{Sq}^{n-1}, \dots, \mathrm{Sq}^{n-k+1}$.

Let A be the Steenrod algebra over the field $\mathbb{F}_2$. The cohomology of a topological space is naturally an unstable left A-module. The category $\mathcal{U}$ of such modules has been studied extensively (see e.g. [Sch94]). We refer to suspensions of the base field $\mathbb{F}_2$ as *(algebraic) spheres*. It is a basic problem in algebraic topology to compute the Ext groups between spheres in $\mathcal{U}$, as they coincide with the E_2 page of the unstable Adams spectral sequence.Unfortunately, those Ext groups are often difficult to compute and the category $\mathcal{U}$ is not of finite homological dimension. Our category $\mathcal{U}_k$ is easier to work with than $\mathcal{U}$ in the sense that it is of finite homological dimension. Computation of Ext groups in $\mathcal{U}_k$ in turn contributes to computation of Ext groups in $\mathcal{U}$, because by Theorem 6.0.1, when N, a module in $\mathcal{U}$, is bounded above degree $k+1$, the Ext groups into N in those two categories agree with each other.

This thesis consists of seven chapters. We review the theory of ringoids in Chapter 2. In Chapter 3, we formally describe the category $\mathcal{U}_k$ in the language of ringoids. The category $\mathcal{U}$ is re-described as the category of modules over a certain ringoid in Definition 3.2.4. We similarly describe the new category $\mathcal{U}_k$ as the category of modules over a certain ringoid in Definition 3.2.5. We further analyze the structure of free modules in $\mathcal{U}_k$ in Section 3.3 and describe the symmetric monoidal structure of $\mathcal{U}_k$ in

Section 3.4.

Chapter 4 is devoted to various functors between different $\mathcal{U}_k$'s and $\mathcal{U}$. The forgetful functors $\mathcal{U} \to \mathcal{U}_k$ and $\mathcal{U}_{k+1} \to \mathcal{U}_k$ are discussed in Section 4.1. As in the category $\mathcal{U}$, there are suspension functors Σ, Frobenius functors Φ and loop functors Ω between categories $\mathcal{U}_k$ with different subscripts k, as described in Sections 4.2 and 4.3.

We abbreviate $\mathrm{Ext}^*_{\mathcal{U}_k}(M,N)$ to $\mathrm{Ext}^*_k(M,N)$. The main result of Chapter 5 is that the homological dimension of $\mathcal{U}_k$ is at most k, or equivalently:

Theorem 5.3.2. $\mathrm{Ext}^s_k(-,-) = 0$ *for all* $s > k \geq 0$.

For any modules M and N in $\mathcal{U}$, the forgetful functors $\mathcal{U}_{k+1} \to \mathcal{U}_k$ induce an inverse system $\cdots \to \mathrm{Ext}^s_2(uM,uN) \to \mathrm{Ext}^s_1(uM,uN) \to \mathrm{Ext}^s_0(uM,uN)$. The main theorem in Chapter 6 is:

Theorem 6.0.1. *If N is bounded above, then the inverse system above stabilizes and the limit is equal to* $\mathrm{Ext}^s_{\mathcal{U}}(M,N)$.

The cohomology of the Λ complex is the Ext group into spheres in $\mathcal{U}$ [BCK+66]. In Chapter 7 we introduce its analogue in the category $\mathcal{U}_k$. We construct a functor Λ_k from $\mathcal{U}_k$ to cochain complexes of $\mathbb{N}$-graded $\mathbb{F}_2$-vector spaces in Section 7.3. This functor Λ_k enjoys the following property:

Theorem 7.4.1. *For any module M in $\mathcal{U}_k$, the cochain complex $\Lambda_k(M)$ is of length at most k and $H^{s,a}(\Lambda_k(M)) \cong \mathrm{Ext}^s_k(M, S_k(a))$ for all s, a. Here $S_k(a)$ is the sphere module in $\mathcal{U}_k$ concentrated in degree a.*

Our arguments also give an alternative proof of the standard fact that the cohomology of the Λ complex computes the Ext groups into spheres in $\mathcal{U}$.

Chapter 2

Ringoids

In this chapter, we briefly review the theory of ringoids, which is not a novelty of this thesis. Informally, a ringoid is a ring with several objects. Ringoids have subringoids, ideals and quotients, extending these ring-theoretic notions. We point interested readers to more extensive literature e.g. [Mit72].

2.1 Ringoids and modules

Definition 2.1.1 (Preadditive category)**.** A category $\mathcal{A}$ is called preadditive if each morphism set $\mathcal{A}(x, y)$ is endowed with the structure of an abelian group in such a way that the compositions

$$\mathcal{A}(x, y) \times \mathcal{A}(y, z) \to \mathcal{A}(x, z)$$

are bilinear.

Definition 2.1.2 (Additive functor)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is said to be additive if

$$F : \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$$

is a homomorphism of abelian groups for all objects x and y in $\mathcal{A}$.

Definition 2.1.3 (Ringoid)**.** A ringoid is a small preadditive category and a morphism of ringoids is an additive functor.

Denote the category of ringoids by **Ringoid**. A ring (with identity) is just a special ringoid — a ringoid with a single object. Many statements in ring theory can be generalized to ringoids. Our main examples of ringoids in this thesis will have the following connectivity property: objects are $\mathrm{Ob}(\mathcal{A}) = \mathbb{Z}$, morphism sets $\mathcal{A}(m, n) = 0$ if $m > n$, whereas $\mathcal{A}(m, m) = \mathbb{F}_2$, generated by the identity morphism.

Definition 2.1.4 (Left module over a ringoid)**.** A left module over a ringoid $\mathcal{A}$ is a covariant additive functor from $\mathcal{A}$ to the category **Ab** of abelian groups.

Denote the category of those modules by $\mathcal{A}$**Mod**. Since **Ab** is a complete and cocomplete abelian category which satisfies *Ab5*, the same is true of $\mathcal{A}$**Mod**. See [Gro57] for more details on the axiom *Ab5*. From now on, by an $\mathcal{A}$-module we mean a left $\mathcal{A}$-module.

Remark 2.1.5 (Module $\mathcal{A}(x, -)$)**.** For any object x in a ringoid $\mathcal{A}$, the covariant functor $\mathcal{A}(x, -) : \mathcal{A} \to \mathbf{Ab}$ is an $\mathcal{A}$-module. By the Yoneda lemma, for any $\mathcal{A}$-module M, the following is an isomorphism of abelian groups

$$\mathcal{A}\mathbf{Mod}(\mathcal{A}(x, -), M) \cong M(x).$$

What's more, the isomorphism is natural in both x and M. The $\mathcal{A}$-module $\mathcal{A}(x, -)$ is projective because the covariant functor $\mathcal{A}\mathbf{Mod}(\mathcal{A}(x, -), -) : \mathcal{A}\mathbf{Mod} \to \mathbf{Ab}$ is exact.

Definition 2.1.6 (Free module)**.** An $\mathcal{A}$-module M is said to be free if it's isomorphic to $\bigoplus_{i \in I} \mathcal{A}\left(x(i), -\right)$ for some index set I.

We present the proposition below without proof because it is standard.

Proposition 2.1.7. A module over a ringoid is projective if and only if it's a retract of some free module.

For future use, we give the following definition.

Definition 2.1.8 (Ringoid adjoining a null object)**.** Given any ringoid $\mathcal{A}$, we can adjoin a null object $+$ to it and get a new ringoid $\mathcal{A}^+$ as follows:

- $\mathrm{Ob}(\mathcal{A}^+) = \{+\} \sqcup \mathrm{Ob}(\mathcal{A})$,
- $\mathcal{A}^+(x,y) = \mathcal{A}(x,y)$ for all $x, y \in \mathrm{Ob}(\mathcal{A})$,
- $\mathcal{A}^+(+,+) = \mathcal{A}^+(+,x) = \mathcal{A}^+(x,+) = 0$ for all $x \in \mathrm{Ob}(\mathcal{A})$.

A left $\mathcal{A}^+$-module M always has $M(+) = 0$ because $\mathcal{A}^+(+,+) = 0$. So the two categories $\mathcal{A}^+\mathbf{Mod}$ and $\mathcal{A}\mathbf{Mod}$ are equivalent.

2.2 Subringoids and ideals

A subring of a ring (with identity) is an abelian subgroup that is closed under multiplication and contains the multiplicative identity of the ring. We generalize this to the following definition of a subringoid.

Definition 2.2.1 (Subringoid)**.** A subringoid is a wide preadditive subcategory of the ringoid.

In other words, a subringoid $\mathcal{B}$ of a ringoid $\mathcal{A}$ is a subcategory of $\mathcal{A}$ such that

- $\mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\mathcal{A})$,
- $\mathcal{B}(x,y)$ is an abelian subgroup of $\mathcal{A}(x,y)$ for all objects x, y.

An ideal in a ring is an abelian subgroup that is closed under multiplication with elements in the ring from both the left and the right. We generalize this to the following definition of an ideal in a ringoid.

Definition 2.2.2 (Ideal and quotient)**.** An ideal $\mathcal{I}$ in a ringoid $\mathcal{A}$ consists of an abelian subgroup $\mathcal{I}(x,y)$ of $\mathcal{A}(x,y)$ for each pair of objects $x, y \in \mathrm{Ob}(\mathcal{A})$ such that for all objects $x, y, z \in \mathrm{Ob}(\mathcal{A})$,

- the image of composition $\mathcal{I}(x,y) \times \mathcal{A}(y,z) \to \mathcal{A}(x,z)$ lies in $\mathcal{I}(x,z)$,

- the image of composition $\mathcal{A}(x,y) \times \mathcal{I}(y,z) \to \mathcal{A}(x,z)$ lies in $\mathcal{I}(x,z)$.

Given a ringoid $\mathcal{A}$ and an ideal $\mathcal{I}$ in it, we can form the quotient ringoid $\mathcal{Q} = \mathcal{A}/\mathcal{I}$.

More precisely, given a ringoid $\mathcal{A}$ and an ideal $\mathcal{I}$ in it, we define their quotient $\mathcal{Q}$ as the category with

- $\mathrm{Ob}(\mathcal{Q}) = \mathrm{Ob}(\mathcal{A})$,
- $\mathcal{Q}(x,y) = \mathcal{A}(x,y)/\mathcal{I}(x,y)$ for all objects x, y.

There is an obvious ringoid map $\mathcal{A} \to \mathcal{Q}$ given by the quotient maps $\mathcal{A}(x,y) \to \mathcal{Q}(x,y)$ on morphisms.

The following lemma extends standard results in ring theory.

Lemma 2.2.3 (Intersection)**.** Let $\mathcal{A}$ be a ringoid. Then

- any intersection of subringoids of $\mathcal{A}$ is again a subringoid of $\mathcal{A}$,
- any intersection of ideals in $\mathcal{A}$ is again an ideal in $\mathcal{A}$,
- the intersection of an ideal $\mathcal{I}$ in $\mathcal{A}$ and a subringoid $\mathcal{B}$ is an ideal in the subringoid $\mathcal{B}$.

Definition 2.2.4 (Subringoid generated by a set of morphisms)**.** Let $\mathcal{A}$ be a ringoid and $\mathcal{M}$ be a set of morphisms in it. Then the subringoid of $\mathcal{A}$ generated by $\mathcal{M}$ is defined to be the smallest subringoid (intersection of all the subringoids) of $\mathcal{A}$ containing all the morphisms in $\mathcal{M}$.

More explicitly, the subringoid of $\mathcal{A}$ generated by $\mathcal{M}$ is the ringoid $\mathcal{B}$ such that $\mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\mathcal{A})$, and for any two objects x and y,

$$\mathcal{B}(x,y) = \left\{ \sum_{i=1}^{n} a_i m_i \right\},$$

where $n \geq 0, a_i \in \mathbb{Z}$ and $m_i : x \to y$ is a composition of morphisms in $\mathcal{M}$. Note that we allow the composition to be empty and m_i to be the identity morphism $\mathrm{id} : x \to x$. It is easy to see that $\mathcal{B}$ is a subringoid of $\mathcal{A}$ and it is the smallest one containing $\mathcal{M}$.

Definition 2.2.5 (Ideal generated by a set of morphisms)**.** Let $\mathcal{A}$ be a ringoid and $\mathcal{M}$ be a set of morphisms in it. Then the ideal in $\mathcal{A}$ generated by $\mathcal{M}$ is defined to be the smallest ideal (intersection of all the ideals) in $\mathcal{A}$ containing all the morphisms in $\mathcal{M}$.

More explicitly, the ideal in $\mathcal{A}$ generated by $\mathcal{M}$ is $\mathcal{I}$ with

$$\mathcal{I}(x, y) = \left\{ \sum_{i=1}^{n} a_i (f_i \circ m_i \circ g_i) \right\},$$

where $n \geq 0, a_i \in \mathbb{Z}, g_i \in \mathcal{A}(x, x_i), m_i \in \mathcal{M}(x_i, y_i), f_i \in \mathcal{A}(y_i, y)$. It is easy to see that $\mathcal{I}$ is an ideal in $\mathcal{A}$ and that it is the smallest one containing $\mathcal{M}$.

2.3 Adjunction between quivers and ringoids

In this section, we will develop a pair of adjoint functors between the category of quivers and the the category of ringoids. Then we will use this to give an alternative definition of subringoid generated by a set of morphisms.

Definition 2.3.1 (Quiver)**.** A quiver G consists of two sets E and V and two functions $s, t : E \rightrightarrows V$. If $G = (E, V, s, t)$ and $G' = (E', V', s', t')$ are two quivers, a morphism $g : G \to G'$ is a pair of morphisms $g_0 : V \to V'$ and $g_1 : E \to E'$ such that $g_0 \circ s = s' \circ g_1$ and $g_0 \circ t = t' \circ g_1$. Denote the category of quivers by **Quiver**. Intuitively, E consists of oriented edges and V consists of vertices.

Denote the category of categories by **Cat**.

Definition 2.3.2 (Free functor from **Quiver** to **Cat**)**.** We define the free functor F : **Quiver** $\to$ **Cat** to be the left adjoint to the forgetful functor u : **Cat** $\to$ **Quiver**. Below we give one construction of the free functor. Given a quiver $G = (E, V)$, we construct a category $\mathcal{C} = FG$ such that

- $\mathrm{Ob}(\mathcal{C}) = V$,

- the morphism set $\mathcal{C}(x, y)$ is the set of finite paths from x to y in the quiver G, where a path is defined as a finite sequence of composable edges and an "empty path" constitutes the identity morphisms of $\mathcal{C}$,

- the composition law of $\mathcal{C}$ follows from concatenation of paths in the quiver G.

Definition 2.3.3 (Free functor from **Cat** to **Ringoid**). We define the free functor $\mathbb{Z} : \mathbf{Cat} \to \mathbf{Ringoid}$ to be the left adjoint to the forgetful functor $u : \mathbf{Ringoid} \to \mathbf{Cat}$. Here is one construction of the free functor $\mathbb{Z}$. Given a category $\mathcal{C}$, we construct a ringoid $\mathcal{A} = \mathbb{Z}\mathcal{C}$ such that

- $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{C})$,

- the morphism set $\mathcal{A}(x, y)$ is the free abelian group generated by $\mathcal{C}(x, y)$, where x, y are any two objects in $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{C})$,

- the composition $\mathcal{A}(x, y) \times \mathcal{A}(y, z) \to \mathcal{A}(x, z)$ sends

$$\left(\sum_{i=1}^{m} a_i f_i, \sum_{j=1}^{n} b_j g_j \right) \mapsto \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j (g_j \circ f_i),$$

  where a_i, b_j are integers and $f_i \in \mathcal{C}(x, y), g_j \in \mathcal{C}(y, z)$.

Composing those two pairs of adjoint functors above, we get a pair of adjoint functors between **Quiver** and **Ringoid**. Next, we will use that adjunction to give an alternative definition of the subringoid generated by a set of morphisms.

In general, the image of a functor is not necessarily a subcategory. But as we will see in the following lemma, when the functor is "nice", the image will be a subcategory.

Lemma 2.3.4. If $F : \mathcal{A} \to \mathcal{B}$ is a functor of categories such that $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ is bijective, then the image $\mathcal{C} := F\mathcal{A}$ is a wide subcategory of $\mathcal{B}$.

Proof. We know $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\mathcal{A})$. We also know that the morphism set $\mathcal{C}(x, y)$ is equal to the image of $F : \mathcal{A}(x, y) \to \mathcal{B}(x, y)$ for all objects x and y. So the identify morphisms exist in $\mathcal{C}$.

The composition law in $\mathcal{C}$ follows the composition law in $\mathcal{B}$. We only need to check that if $f \in \mathcal{C}(x, y)$ and $g \in \mathcal{C}(y, z)$, then $g \circ f \in \mathcal{C}(x, z)$. Since $f = F(f')$ for some $f' \in \mathcal{A}(x, y)$ and $g = F(g')$ for some $g' \in \mathcal{A}(y, z)$, we have $g \circ f = F(g') \circ F(f') = F(g' \circ f')$ with $g' \circ f' \in \mathcal{A}(x, z)$. Therefore, $g \circ f \in \mathcal{C}(x, z)$. □

Furthermore, the following lemma will give us a condition on when the image of a morphism of ringoids is a subringoid.

Lemma 2.3.5. If $F : \mathcal{A} \to \mathcal{B}$ is a morphism of ringoids such that $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ is bijective, then the image $\mathcal{C} := F\mathcal{A}$ is a subringoid of $\mathcal{B}$.

Proof. By the lemma above, $\mathcal{C}$ is a wide subcategory of $\mathcal{B}$. Since $F : \mathcal{A}(x, y) \to \mathcal{B}(x, y)$ is a homomorphism of abelian groups for all objects x and y, we know that $\mathcal{C}(x, y)$ is an abelian subgroup of $\mathcal{B}(x, y)$. Therefore, by definition $\mathcal{C}$ is a subringoid of $\mathcal{B}$. □

Definition 2.3.6 (Subringoid generated by a set of morphisms)**.** Let $\mathcal{A}$ be a ringoid and $\mathcal{M}$ be a set of morphisms in it. Denote by $\mathcal{M}'$ the union of $\mathcal{M}$ and $\{\mathrm{id} : x \to x, \forall x \in \mathrm{Ob}(\mathcal{A})\}$. Then we get a morphism of quivers $\mathcal{M}' \to u\mathcal{A}$ and by adjunction, this gives arise to a morphism of ringoids $F\mathcal{M}' \to \mathcal{A}$. Since this morphism of ringoids is bijective on objects, its image is a subringoid of $\mathcal{A}$ by the lemma above. We define this subringoid as the subringoid of $\mathcal{A}$ generated by morphisms in $\mathcal{M}$.

Chapter 3

Unstable modules with only the top k Steenrod operations

3.1 Steenrod algebra and unstable modules over it

Definition 3.1.1 (Steenrod algebra)**.** The mod 2 Steenrod algebra A is the quotient of the free unital graded $\mathbb{F}_2$-algebra generated by the elements Sq^i of degree i by the ideal generated by

$$\mathrm{Sq}^0 = 1, \mathrm{Sq}^i = 0 \quad \text{when } i < 0$$

and the Adem relations

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{t=0}^{\lfloor i/2 \rfloor} \binom{j-t-1}{i-2t} \mathrm{Sq}^{i+j-t} \mathrm{Sq}^t \quad \text{when } 0 < i < 2j. \tag{3.1}$$

We shall denote by $\mathcal{M}$ the category of graded left A-modules, whose morphisms are A-linear maps of degree zero. From now on, by an A-module we mean a module in the category $\mathcal{M}$.

The following is a standard fact:

Lemma 3.1.2. A basis of the Steenrod algebra A as a graded vector space over $\mathbb{F}_2$ is given by the admissible squares $\mathrm{Sq}^{i(1)}\mathrm{Sq}^{i(2)} \cdots \mathrm{Sq}^{i(m)}$ with $i(1) \geq 2i(2), i(2) \geq 2i(3), \ldots, i(m-1) \geq 2i(m), i(m) > 0$.

Notation 3.1.3 (Lower squares)**.** If x is a homogeneous element in an A-module, then we denote

$$\mathrm{Sq}_i x = \mathrm{Sq}^{|x|-i} x.$$

Proposition 3.1.4 (Adem relations in lower squares)**.** Take any A-module M. Let $i, j, n \in \mathbb{Z}$ satisfy $n > j, 2n > i + j$. Then for any homogeneous element x of degree n in M,

$$\mathrm{Sq}_i \mathrm{Sq}_j x = \sum_{s=\lceil (i+j)/2 \rceil}^{n} \binom{s-j-1}{2s-i-j} \mathrm{Sq}_{i+2j-2s} \mathrm{Sq}_s x.$$

Proof. Compute:

$$\begin{aligned}
\mathrm{Sq}_i \mathrm{Sq}_j x &= \mathrm{Sq}^{2n-i-j} \mathrm{Sq}^{n-j} x \\
&= \sum_{t=0}^{\lfloor n-(i+j)/2 \rfloor} \binom{n-j-t-1}{2n-i-j-2t} \mathrm{Sq}^{3n-i-2j-t} \mathrm{Sq}^t x \\
&= \sum_{t=0}^{\lfloor n-(i+j)/2 \rfloor} \binom{n-j-t-1}{2n-i-j-2t} \mathrm{Sq}_{-2n+i+2j+2t} \mathrm{Sq}_{n-t} x \\
&= \sum_{s=\lceil (i+j)/2 \rceil}^{n} \binom{s-j-1}{2s-i-j} \mathrm{Sq}_{i+2j-2s} \mathrm{Sq}_s x.
\end{aligned}$$

The second equality comes from the Adem relations in upper squares. The last equality comes from the substitution $s = n - t$. $\square$

Remark 3.1.5. When $i > j$, all the terms of the summation on the right hand side satisfy $i+2j-2s \leq s$. The reason is $s \geq (i+j)/2 \geq j$ and thus $(i+j-2s)+(j-s) \leq 0$. So whenever there is a $\mathrm{Sq}_i \mathrm{Sq}_j$ with $i > j$, one can rewrite it as of a sum of $\mathrm{Sq}_{i'} \mathrm{Sq}_{j'}$'s such that $i' \leq j'$. This observation agrees with Proposition 3.3.1 in a later section, which says whenever there is a sequence of lower squares, one can always rewrite it as a sum of $\mathrm{Sq}_{i(1)} \mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}$'s such that $i(1) \leq i(2) \leq \cdots \leq i(m)$.

Remark 3.1.6. When $i, j < k$, all the terms of the summation on the right hand side satisfy $i+2j-2s, s < k$. The reason is $i+2j-2s \leq (i+2j-2s)+(2s-i-j) = j < k$. The binomial coefficient implies $s - j - 1 \geq 2s - i - j$ and thus $s \leq i - 1 < k$. So applying the Adem relations to any $\mathrm{Sq}_i \mathrm{Sq}_j$ with $i, j < k$ leads to a sum of $\mathrm{Sq}_{i'} \mathrm{Sq}_{j'}$'s with $i', j' < k$.

Definition 3.1.7 (Unstable A-module)**.** An A-module is said to be unstable if $\mathrm{Sq}^i x = 0$ for any homogeneous element x and any $i > |x|$.

In other words, M is unstable if $\mathrm{Sq}_i M = 0$ for all $i < 0$.

Note that if M is an unstable A-module, then $M^n = 0$ for all $n < 0$ because $x = \mathrm{Sq}^0 x = 0$ for any homogeneous element x in M of degree $n < 0$. We shall denote by $\mathcal{U}$ the full subcategory of $\mathcal{M}$ with objects the unstable ones. It is a well-known category and more details about it can be found in [Sch94]. From now on, by an unstable A-module we mean a module in the category $\mathcal{U}$.

Example 3.1.8 (Sphere module $S(n)$)**.** For any integer n, we define the sphere module $S(n)$ to be the A-module with the degree n part equal to $\mathbb{F}_2$ and the other parts equal to zero. The sphere module is defined for any integer n, but it is unstable only when $n \geq 0$.

3.2 Intuition and formal definition

Let k be any natural number. We will define a modification of the category $\mathcal{U}$, where the only Steenrod operations allowed are Sq_i with $i < k$ and their compositions. Those graded vector spaces are no longer modules over the Steenrod algebra, as illustrated in Remark 3.2.10. We require M to be unstable, i.e. $\mathrm{Sq}_i = 0$ when $i < 0$ and thus $M^n = 0$ when $n < 0$.

For example, when $k = 0$, none nontrivial Steenrod operation is defined and what we get is exactly $\mathbb{N}$-graded $\mathbb{F}_2$-vector spaces. When $k = 1$, there is one nontrivial Steenrod operation Sq_0 which doubles the degree. Note that Sq_0 is equal to identity on degree zero.

For another example, when $k = 3$, there are three nontrivial Steenrod operations $\mathrm{Sq}_2, \mathrm{Sq}_1, \mathrm{Sq}_0$. The Steenrod operations Sq_2 and Sq_1 are not available in all degrees — Sq_2 acts only on degrees ≥ 2 and Sq_1 acts only on degrees ≥ 1.

We shall denote by $\mathcal{U}_k$ the category of vector spaces with such additional structure. To make this idea clear, we are going to use the language of ringoids. We are interested in

- $\mathcal{M}$, the category of A-modules,
- $\mathcal{U}$, the category of unstable A-modules,
- $\mathcal{U}_k$, the category of unstable graded vector spaces over $\mathbb{F}_2$ with only the top k Steenrod operations.

Notation 3.2.1. Our category $\mathcal{U}_k$ should not be confused with the Krull filtration of the category $\mathcal{U}$ despite the conflict of notation.

Although $\mathcal{U}$ is not the category of modules over any graded ring, it is the category of modules over a certain ringoid. In fact, we can formulate each of the three categories above as the category of modules over a ringoid.

Definition 3.2.2 (Ringoid $\mathcal{R}$)**.** Let $\mathcal{R}$ be the ringoid such that

- the objects are all the integers,
- for any $a, b \in \mathbb{Z}$, the morphism set $\mathcal{R}(a, b)$ is the $\mathbb{F}_2$-vector space generated by the set of finite sequences of integers $(c_1, c_2, \ldots, c_m)$ such that $a = c_1 < c_2 < \cdots < c_m = b$,
- the composition of morphisms is the concatenation of two sequences.

We write the sequence

$$(a = c_1, c_2, \ldots, c_m = b)$$

as

$$\mathrm{Sq}^{c_m - c_{m-1}} \cdots \mathrm{Sq}^{c_3 - c_2} \mathrm{Sq}^{c_2 - c_1}.$$

For example, the morphism set $\mathcal{R}(-1, 2)$ is a four-dimensional $\mathbb{F}_2$-vector space generated by

$$(-1, 0, 1, 2), (-1, 0, 2), (-1, 1, 2), (-1, 2)$$

or equivalently

$$\mathrm{Sq}^1\mathrm{Sq}^1\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^1, \mathrm{Sq}^1\mathrm{Sq}^2, \mathrm{Sq}^3.$$

The identity morphism $n \to n$ is also written as Sq^0.

Definition 3.2.3 (Ringoid $\mathcal{A}$)**.** Let $\mathcal{I}$ be the ideal in $\mathcal{R}$ generated by the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j - \sum_{t=0}^{\lfloor i/2 \rfloor} \binom{j-t-1}{i-2t} \mathrm{Sq}^{i+j-t}\mathrm{Sq}^t \in \mathcal{R}(n, n+i+j) \text{ with } 0 < i < 2j.$$

Define $\mathcal{A}$ as the quotient ringoid of $\mathcal{R}$ by $\mathcal{I}$.

This new ringoid $\mathcal{A}$ is the same as the ringoid such that

- the objects are all the integers,
- for any $a, b \in \mathbb{Z}$, the morphism set $\mathcal{A}(a, b)$ is the degree $(b-a)$ part of the Steenrod algebra A.

The category of left modules over the ringoid $\mathcal{A}$ is exactly $\mathcal{M}$, the category of modules over the Steenrod algebra A.

Definition 3.2.4 (Ringoid $\mathcal{Q}$)**.** Let $\mathcal{J}$ be the ideal in $\mathcal{R}$ generated by the Adem relations (as in Definition 3.2.3) and the instability conditions

$$\mathrm{Sq}^i : n \to n+i \text{ with } i > n.$$

Define $\mathcal{Q}$ as the quotient ringoid of $\mathcal{R}$ by $\mathcal{J}$.

Then the category of left modules over the ringoid $\mathcal{Q}$ is exactly $\mathcal{U}$, the category of unstable modules over the Steenrod algebra A. Note that $\mathcal{Q}$ can also be seen as the quotient ringoid of $\mathcal{A}$ by the ideal $\mathcal{L}$ generated by the instability conditions alone.

Definition 3.2.5 (Ringoid $\mathcal{Q}_k$)**.** Let $k \geq 0$. Let $\mathcal{Q}_k$ be the subringoid of $\mathcal{Q}$ generated by Sq_i with $0 \leq i < k$.

Denote the category of left modules over the ringoid $\mathcal{Q}_k$ by $\mathcal{U}_k$ and call it the category of unstable modules with only the top k Steenrod operations.

Example 3.2.6 (Sphere module $S_k(n)$)**.** The sphere module $S_k(n)$ is defined to be the module in $\mathcal{U}_k$ with the degree n part equal to $\mathbb{F}_2$ and the other parts equal to zero. Note that n cannot be negative because the negative degree parts of a module in $\mathcal{U}_k$ must be zero.

Example 3.2.7 (Category $\mathcal{U}_0$)**.** The modules in $\mathcal{U}_0$ allow no Steenrod operation. The category $\mathcal{U}_0$ is just the category of $\mathbb{N}$-graded $\mathbb{F}_2$-vector spaces.

Example 3.2.8 (Category $\mathcal{U}_1$)**.** The modules in $\mathcal{U}_1$ allow the top Steenrod operation Sq_0. The category $\mathcal{U}_1$ is just the category of $\mathbb{N}$-graded $\mathbb{F}_2$-vector spaces with operation Sq_0 doubling the degree and equal to the identity map on degree zero.

Example 3.2.9 (Category $\mathcal{U}_2$)**.** The modules in $\mathcal{U}_2$ allow the top two Steenrod operations $\mathrm{Sq}_0, \mathrm{Sq}_1$. The operation Sq_1 doubles the degree and then reduces the degree by one. There is one relation $\mathrm{Sq}_1\mathrm{Sq}_0 = 0$. The category $\mathcal{U}_2$ is just the category of $\mathbb{N}$-graded $\mathbb{F}_2$-vector spaces with operations $\mathrm{Sq}_0, \mathrm{Sq}_1$ changing degree accordingly such that

- Sq_0 is equal to the identity map on degree zero,
- Sq_1 is not defined on degree zero,
- $\mathrm{Sq}_1\mathrm{Sq}_0 = 0$.

Remark 3.2.10. A module in $\mathcal{U}_k$ is not in general a module over the Steenrod algebra. Consider this module M in category $\mathcal{U}_2$: $M^i = \mathbb{F}_2$ if $i = 2, 3, 6$ and $M^i = 0$ otherwise. Say the nontrivial elements in each degree are $\iota_2, \iota_3, \iota_6$, and the Steenrod operations on them are $\mathrm{Sq}_1\iota_2 = \iota_3$ and $\mathrm{Sq}_0\iota_3 = \iota_6$. This is a module in $\mathcal{U}_2$ but not a module over the Steenrod algebra, because if all Steenrod operations were allowed, then $\mathrm{Sq}_2\mathrm{Sq}_0 = \mathrm{Sq}_0\mathrm{Sq}_1$ would lead to $\iota_6 = \mathrm{Sq}_0\mathrm{Sq}_1\iota_2 = \mathrm{Sq}_2\mathrm{Sq}_0\iota_2 = 0$.

Remark 3.2.11. The work [PW00] introduced a bigraded bialgebra $\mathcal{K}$ over $\mathbb{F}_2$ and its sub-bialgebra $\mathcal{K}(k)$. The category $\mathcal{U}$ is equivalent to the category of unstable modules over $\mathcal{K}$, and the category $\mathcal{U}_k$ is equivalent to the category of unstable modules over $\mathcal{K}(k)$. The element D_i in $\mathcal{K}$ corresponds to the lower Steenrod square Sq_i.

3.3 Example: free modules

Recall the concept of free modules over a ringoid from Definition 2.1.6. The following are free modules over ringoids defined in the last section:

- $\mathcal{A}(n, -)$ is a free $\mathcal{A}$-module for any integer n,
- $F(n) := \mathcal{Q}(n, -)$ is a free $\mathcal{Q}$-module for any integer n,
- $F_k(n) := \mathcal{Q}_k(n, -)$ is a free $\mathcal{Q}_k$-module for any integers n and $k \geq 0$.

Note that $F(n) = 0$ and $F_k(n) = 0$ for all $n < 0$.

From now on, we use ι_n to denote the universal element of degree n.

Proposition 3.3.1. A basis of $\mathcal{A}(n, -)$ as a graded vector space over $\mathbb{F}_2$ is

$$\mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$$

with

$$i(1) \leq i(2) \leq \cdots \leq i(m) < n.$$

Proof. The admissible basis of the Steenrod algebra A is

$$\mathrm{Sq}^{j(1)}\mathrm{Sq}^{j(2)} \cdots \mathrm{Sq}^{j(m)}$$

with

$$j(s) > 0, j(s) \geq 2j(s+1).$$

Therefore, a basis of $\mathcal{A}(n, -)$ is

$$\mathrm{Sq}^{j(1)}\mathrm{Sq}^{j(2)} \cdots \mathrm{Sq}^{j(m)}\iota_n$$

with

$$j(s) > 0, j(s) \geq 2j(s+1).$$

This translates to

$$\mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$$

with

$$i(1) \leq i(2) \leq \cdots \leq i(m) < n. \qquad \square$$

We will write down the corresponding basis of $F(n)$ and $F_k(n)$ as vector spaces over $\mathbb{F}_2$ explicitly. Before that, we present a lemma about the structure of $\mathcal{L}$, the ideal in the ringoid $\mathcal{A}$ generated by $\mathrm{Sq}^i : n \to n+i$ with $i > n$. Remember that the ringoid $\mathcal{Q}$ is equal to the quotient of $\mathcal{A}$ by $\mathcal{L}$.

Lemma 3.3.2. A basis of $\mathcal{L}(n, a)$ as a vector space over $\mathbb{F}_2$ is given by the admissible Steenrod monomials

$$\mathrm{Sq}^{j(1)}\mathrm{Sq}^{j(2)} \cdots \mathrm{Sq}^{j(m)} : n \to a,$$

where there exists at least one $s \in [1, m]$ such that

$$j(s) > n + \sum_{t=s+1}^{m} j(t).$$

Proof. By definition, $\mathcal{L}(n, a)$ as a vector space is generated by

$$\mathrm{Sq}^{j(1)}\mathrm{Sq}^{j(2)} \cdots \mathrm{Sq}^{j(m)} : n \to a,$$

where there exists at least one $s \in [1, m]$ such that

$$j(s) > n + \sum_{t=s+1}^{m} j(t).$$

Using the Adem relations (3.1), we can rewrite the morphism above as a sum of some admissible Steenrod monomials. Say we are applying the Adem relations on $\mathrm{Sq}^i\mathrm{Sq}^j : b \to b+i+j$ and get a sum of $\mathrm{Sq}^{i'}\mathrm{Sq}^{j'} : b \to c$. If $i > j + b$, then $i' > j' + b$ because the Adem relations increase the first upper index and decrease the second one. If $j > b$, then $i' > j' + b$ because $i' = i + j - j'$ and $i \geq 2j'$ together imply $i' \geq j' + j > j' + b$. Therefore, for each admissible summand

$$\mathrm{Sq}^{j'(1)}\mathrm{Sq}^{j'(2)} \cdots \mathrm{Sq}^{j'(m')} : n \to \ell$$

on the right hand side of the Adem relations, there exists at least one $s \in [1, m']$ such that

$$j'(s) > n + \sum_{t=s'+1}^{m'} j'(t).$$

So those admissible monomials generate the $\mathcal{L}(n, a)$. $\square$

Proposition 3.3.3. When $n \geq 0$, a basis of $F(n)$ as a graded vector space over $\mathbb{F}_2$ is

$$\mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$$

with

$$0 \leq i(1) \leq i(2) \leq \cdots \leq i(m) < n.$$

Proof. Recall $F(n) = \mathcal{Q}(n, -)$ and $F(n)^a = \mathcal{Q}(n, a) = \mathcal{A}(n, a)/\mathcal{L}(n, a)$. By Lemma 3.3.2, a basis of $\mathcal{Q}(n, -)$ as a vector space over $\mathbb{F}_2$ is given by

$$\mathrm{Sq}^{j(1)}\mathrm{Sq}^{j(2)} \cdots \mathrm{Sq}^{j(m)}\iota_n$$

with

$$j(s) > 0, j(s) \geq 2j(s+1), j(s) \leq n + \sum_{t=s+1}^{m} j(t).$$

This translates to

$$\mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$$

with

$$0 \leq i(1) \leq i(2) \leq \cdots \leq i(m) < n. \qquad \square$$

Proposition 3.3.4. When $n \geq 0$, a basis of $F_k(n)$ as a graded vector space over $\mathbb{F}_2$ is

$$\mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$$

with

$$0 \leq i(1) \leq i(2) \leq \cdots \leq i(m) < \min(n, k).$$

Proof. Recall $F_k(n) = \mathcal{Q}_k(n, -)$ and $F_k(n)^a = \mathcal{Q}_k(n, a)$. For all integers n and a, $\mathcal{Q}_k(n, a)$ is a sub $\mathbb{F}_2$-vector space of $\mathcal{Q}(n, a)$ generated by the compositions of

$\mathrm{Sq}_0, \ldots, \mathrm{Sq}_{k-1}$. Remark 3.1.6 tells us that application of the Adem relations on any composition of $\mathrm{Sq}_0, \ldots, \mathrm{Sq}_{k-1}$ leads to a sum of some compositions of $\mathrm{Sq}_0, \ldots, \mathrm{Sq}_{k-1}$.

Proposition 3.3.3 gives a basis of $\mathcal{Q}(n, a)$ as a $\mathbb{F}_2$-vector space. Therefore, a basis of $\mathcal{Q}_k(n, -)$ is

$$\mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$$

with

$$0 \leq i(1) \leq i(2) \leq \cdots \leq i(m) < \min(n, k). \qquad \square$$

Proposition 3.3.4 implies that when $k \geq n$, $F_k(n)$ is obtained by applying the forgetful functor to $F(n)$. See Definition 4.1.1 for forgetful functors.

Remark 3.3.5. All projective modules in $\mathcal{M}, \mathcal{U}, \mathcal{U}_k$ are free. The reason is as follows: Proposition 2.1.7 says that they are retracts of free modules. The three propositions above in this section imply:

1. a retract of $\mathcal{A}(n, -)$ is either zero or $\mathcal{A}(n, -)$ itself,
2. a retract of $F(n)$ is either zero or $F(n)$ itself,
3. a retract of $F_k(n)$ is either zero or $F_k(n)$ itself.

Remark 3.3.6 (Locally Noetherian)**.** The category $\mathcal{U}$ is locally Noetherian (see [Sch94] Chapter 1, Section 8). But the categories $\mathcal{U}_k$ are not locally Noetherian in general. For example, $\mathcal{U}_2$ is not locally Noetherian. Although $F_2(2)$ is finitely generated, it has a submodule M which is not finitely generated: let M be the submodule generated by $\mathrm{Sq}_0(\mathrm{Sq}_1)^i\iota_2$ with $i \geq 0$. We do not pursue the injective cogenerators of the category $\mathcal{U}_k$ because it is not locally Noetherian.

3.4 Symmetric monoidal category

We will construct a functor $\otimes : \mathcal{U}_k \times \mathcal{U}_k \to \mathcal{U}_k$, and then prove that $\mathcal{U}_k$ is a symmetric monoidal category with this tensor product and that $S_k(0)$ is the unit object with respect to this tensor product. Before constructing the tensor product, we propose an alternative construction of the ringoid $\mathcal{Q}_k$ in terms of generators and relations.

Recall that $\mathcal{Q} = \mathcal{R}/\mathcal{J}$ and $\mathcal{Q}_k$ is the subringoid of $\mathcal{Q}$ generated by the top k squares. In other words, $\mathcal{Q}_k$ is a subringoid of a quotient ringoid of $\mathcal{R}$. We will present $\mathcal{Q}_k$ as a quotient ringoid of a subringoid of $\mathcal{R}$, after the following lemma.

Lemma 3.4.1. Let $\mathcal{A}$ be a ringoid. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$ and $\mathcal{M}$ be a set of morphisms in $\mathcal{A}$. Denote by $\mathcal{B}$ the subringoid of $\mathcal{A}$ generated by $\mathcal{M}$. Denote by $\mathcal{C}$ the subringoid of $\mathcal{A}$ generated by $\mathcal{M}$ and $\mathcal{I}$. Then the following three ringoids are isomorphic:

- $\mathcal{C}/\mathcal{I}$,
- $\mathcal{B}/(\mathcal{B}\cap\mathcal{I})$,
- the subringoid $\mathcal{D}$ of $\mathcal{A}/\mathcal{I}$ generated by the image of $\mathcal{M}$ in $\mathcal{A}/\mathcal{I}$.

Proof. The bijection on objects is easy to see. We need to construct a bijection between the morphism sets. Observe that

$$\frac{\mathcal{C}}{\mathcal{I}}(x,y) = \frac{\mathcal{C}(x,y)}{\mathcal{I}(x,y)} = \frac{\mathcal{B}(x,y)+\mathcal{I}(x,y)}{\mathcal{I}(x,y)}.$$

Finish by noting

$$\mathcal{D}(x,y) = \frac{\mathcal{B}(x,y)+\mathcal{I}(x,y)}{\mathcal{I}(x,y)}$$

and

$$\frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{I}}(x,y) = \frac{\mathcal{B}(x,y)}{\mathcal{B}(x,y)\cap\mathcal{I}(x,y)} = \frac{\mathcal{B}(x,y)+\mathcal{I}(x,y)}{\mathcal{I}(x,y)}.$$

□

Proposition 3.4.2. Let $k \geq 0$. Let $\mathcal{R}_k$ be the subringoid of $\mathcal{R}$ generated by

$$\mathcal{M} := \{\mathrm{Sq}_i \text{ with } 0 \leq i < k\}.$$

Then $\mathcal{R}_k/(\mathcal{R}_k\cap\mathcal{J})$ is isomorphic to the ringoid $\mathcal{Q}_k$.

Proof. Recall that $\mathcal{Q}_k$ is the subringoid of $\mathcal{R}/\mathcal{J}$ generated by the image of $\mathcal{M}$ in $\mathcal{R}/\mathcal{J}$. According to Lemma 3.4.1, it is isomorphic to $\mathcal{R}_k/(\mathcal{R}_k\cap\mathcal{J})$. □

After describing the ringoid $\mathcal{Q}_k$ in terms of generators and relations, we are now ready to define the tensor product structure on category $\mathcal{U}_k$.

Proposition 3.4.3. $(\mathcal{U}_k, \otimes, S_k(0))$ is a symmetric monoidal category with

$$(M \otimes N)^n := \bigoplus_{i+j=n} M^i \otimes N^j$$

and the top k Steenrod squares are given on $x \otimes y$ by the Cartan formula

$$\mathrm{Sq}_n(x \otimes y) := \sum_{i+j=n} (\mathrm{Sq}_i x) \otimes \left(\mathrm{Sq}_j y\right).$$

Proof. The right hand side in the definition of $(M \otimes N)^n$ is a finite direct sum because $M^i = 0$ when $i < 0$ and similarly for N. The right hand side in the definition of $\mathrm{Sq}_n(x \otimes y)$ only involves the top k Steenrod operations because $j \geq 0$ implies $i = n - j \leq n < k$ and similarly for j. This check uses the instability condition $i, j \geq 0$.

We need to verify that this makes $M \otimes N$ is a module over $\mathcal{Q}_k$. According to Proposition 3.4.2, it suffices to verify that $M \otimes N$ is a left $\mathcal{R}_k$-module with $f(x \otimes y) = 0$ for any morphism $f \in \mathcal{R}_k \cap \mathcal{J}$ and any two nonzero homogeneous elements x, y in M, N. Since there is a similar tensor product structure on $\mathcal{U}$ and $\mathcal{Q} = \mathcal{R}/\mathcal{J}$, we know that $f(x \otimes y) = 0$ for any morphism $f \in \mathcal{J}$. This completes our proof that $M \otimes N$ is indeed a module over $\mathcal{Q}_k$.

This tensor product $M \otimes N$ is functorial in both M and N, so we get the tensor product functor $\otimes : \mathcal{U}_k \times \mathcal{U}_k \to \mathcal{U}_k$. The sphere module $S_k(0)$ is the unit object with respect to this tensor product because $(M \otimes S_k(0))^n = M^n$ and $\mathrm{Sq}_n(x \otimes y) = (\mathrm{Sq}_n x) \otimes y$, where x is any homogeneous element in M and y is the only nontrivial element in $S_k(0)$. □

Chapter 4

Functors between categories $\mathcal{U}$ and $\mathcal{U}_k$

4.1 Forgetful functor

An object of $\mathcal{U}_k$ is not in general an object of $\mathcal{U}$ (see Remark 3.2.10), but an object of $\mathcal{U}$ is an object of $\mathcal{U}_k$ if we forget some Steenrod operations.

Definition 4.1.1 (Forgetful functor)**.** The inclusion morphisms of ringoids

$$\mathcal{Q}_0 \to \mathcal{Q}_1 \to \cdots \to \mathcal{Q}_{k-1} \to \mathcal{Q}_k \to \cdots \to \mathcal{Q}$$

induce forgetful functors

$$\mathcal{U}_0 \leftarrow \mathcal{U}_1 \leftarrow \cdots \leftarrow \mathcal{U}_{k-1} \leftarrow \mathcal{U}_k \leftarrow \cdots \leftarrow \mathcal{U}.$$

We denote all forgetful functors by u since it doesn't cause any ambiguity.

Remark 4.1.2. The forgetful functors are additive and exact. The forgetful functors are compatible with tensor products in the sense of the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{U} \times \mathcal{U} & \xrightarrow{\otimes} & \mathcal{U} \\ {\scriptstyle u\times u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{U}_k \times \mathcal{U}_k & \xrightarrow{\otimes} & \mathcal{U}_k \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{k+1} \times \mathcal{U}_{k+1} & \xrightarrow{\otimes} & \mathcal{U}_{k+1} \\ {\scriptstyle u\times u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{U}_k \times \mathcal{U}_k & \xrightarrow{\otimes} & \mathcal{U}_k \end{array}$$

Remark 4.1.3. The forgetful functor $u : \mathcal{U}_{k+1} \to \mathcal{U}_k$ has a left adjoint that takes

$F_k(n)$ to $F_{k+1}(n)$. Similarly, the forgetful functor $u : \mathcal{U} \to \mathcal{U}_k$ has a left adjoint that takes $F_k(n)$ to $F(n)$.

Proposition 4.1.4. The forgetful functors preserve free modules.

Proof. It suffices to prove that $u(F(n))$ and $u(F_{k+1}(n))$ are free modules in $\mathcal{U}_k$. By Propositions 3.3.3 and 3.3.4, $u(F(n))$ is a direct sum of $F_k(|x|)$ where $x = \mathrm{Sq}_{i(1)}\mathrm{Sq}_{i(2)} \cdots \mathrm{Sq}_{i(m)}\iota_n$ with $k \leq i(1) \leq \cdots \leq i(m) < n$. Similarly, $u(F_{k+1}(n))$ is a direct sum of $F_k(|x|)$ where $x = (\mathrm{Sq}_k)^m\iota_n$ with $m \geq 0$ if $k < n$ and $x = \iota_n$ if $k \geq n$. □

4.2 Suspension functor

The suspension functor in this section and the Frobenius and loop functors to be introduced in the next section are analogues of the standard results for $\mathcal{U}$ described in Chapter 1 of [Sch94].

Definition 4.2.1 (Suspension morphism)**.** We define the suspension morphism $\sigma : \mathcal{A} \to \mathcal{A}$ of ringoids as

$$\sigma(n) = n - 1, \quad \sigma\left(\mathrm{Sq}^i\right) = \mathrm{Sq}^i.$$

Lemma 4.2.2. The suspension morphism $\sigma : \mathcal{A} \to \mathcal{A}$ induces a suspension morphism of the quotient ringoids $\sigma' : \mathcal{Q} \to \mathcal{Q}$, and the suspension morphism $\sigma' : \mathcal{Q} \to \mathcal{Q}$ further induces a suspension morphism of subringoids $\sigma_k : \mathcal{Q}_{k+1} \to \mathcal{Q}_k$.

Proof. The first statement is due to the inclusion $\sigma(\mathcal{I}(n, n+i)) \subseteq \mathcal{I}(n-1, n+i-1)$. The second statement is true because σ sends $\mathrm{Sq}_i = \mathrm{Sq}^{n-i} \in \mathcal{A}(n, 2n-i)$ with $i < k+1$ to $\mathrm{Sq}_{i-1} = \mathrm{Sq}^{n-i} \in \mathcal{A}(n-1, 2n-i-1)$ with $i-1 < k$. □

Definition 4.2.3 (Suspension functor)**.** The suspension morphism $\sigma' : \mathcal{Q} \to \mathcal{Q}$ of ringoids induces a suspension functor $\Sigma : \mathcal{U} \to \mathcal{U}$. Similarly, the suspension morphism $\sigma_k : \mathcal{Q}_{k+1} \to \mathcal{Q}_k$ of ringoids induces a suspension functor $\Sigma : \mathcal{U}_k \to \mathcal{U}_{k+1}$.

We denote all suspension functors by Σ since it doesn't cause any ambiguity. The suspension functors are exact, because they just shift the underlying sets and maps.

Let M be any module in $\mathcal{U}_k$. Then the underlying sets and the Steenrod operations of ΣM are

$$(\Sigma M)^{n+1} = M^n \quad \forall n$$

and

$$\mathrm{Sq}_{i+1}(\Sigma x) = \Sigma(\mathrm{Sq}_i x) \quad \forall i < k,$$

where Σx denotes the element in $(\Sigma M)^{n+1}$ corresponding to a homogeneous x in M^n. The operation Sq_0 is zero on Σx because $\mathrm{Sq}_0(\Sigma x) = \Sigma(\mathrm{Sq}_{-1} x) = 0$.

Remark 4.2.4. The suspension functors commute with the forgetful functors:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Sigma} & \mathcal{U} \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow \\ \mathcal{U}_k & \xrightarrow{\Sigma} & \mathcal{U}_{k+1} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{U}_{k+1} & \xrightarrow{\Sigma} & \mathcal{U}_{k+2} \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow \\ \mathcal{U}_k & \xrightarrow{\Sigma} & \mathcal{U}_{k+1} \end{array}$$

because the corresponding diagrams of ringoids are commutative.

Proposition 4.2.5. The suspension functor $\Sigma : \mathcal{U}_k \to \mathcal{U}_{k+1}$ restricts to an equivalence between $\mathcal{U}_k$ and the full subcategory of $\mathcal{U}_{k+1}$ with objects the ones with $\mathrm{Sq}_0 = 0$.

Proof. Denote the full subcategory by $\mathcal{C}$. It suffices to find a functor $F : \mathcal{C} \to \mathcal{U}_k$ such that $\Sigma F = 1$ and $F\Sigma = 1$. Here is the construction of functor F. Given a module M in $\mathcal{C}$, we observe that $M^0 = 0$ because $\mathrm{Sq}_0 = 0$ is the identity map on M^0. Construct FM as the one-degree downward shift of M. More precisely, let

$$(FM)^n = M^{n+1} \quad \forall n \geq 0$$

and

$$\begin{array}{ccc} (FM)^n & \xrightarrow{\mathrm{Sq}^i} & (FM)^{n+i} \\ {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\ M^{n+1} & \xrightarrow{\mathrm{Sq}^i} & M^{n+i+1} \end{array} \qquad \forall 0 \leq i \leq n, i > n-k.$$

It is easy to check that FM is indeed a module in $\mathcal{U}_k$ and F is a functor from $\mathcal{C}$ to $\mathcal{U}_k$. It is also easy to check that $\Sigma F = 1$ and $F\Sigma = 1$. □

4.3 Frobenius functor and loop functor

We are going to introduce a functor $\Phi : \mathcal{U}_k \to \mathcal{U}_{2k}$, which is a analogue of the standard Frobenius functor $\Phi : \mathcal{U} \to \mathcal{U}$. To do that, we will adjoin a null object to our ringoids $\mathcal{A}, \mathcal{Q}, \mathcal{Q}_k$ and work with $\mathcal{A}^+, \mathcal{Q}^+, \mathcal{Q}_k^+$. Recall that

$$\mathcal{M} = \mathcal{A}\mathbf{Mod} = \mathcal{A}^+\mathbf{Mod},$$
$$\mathcal{U} = \mathcal{Q}\mathbf{Mod} = \mathcal{Q}^+\mathbf{Mod},$$
$$\mathcal{U}_k = \mathcal{Q}_k\mathbf{Mod} = \mathcal{Q}_k^+\mathbf{Mod}.$$

We are going to construct Frobenius morphisms

$$\mathcal{A}^+ \to \mathcal{A}^+, \quad \mathcal{Q}^+ \to \mathcal{Q}^+, \quad \mathcal{Q}_{2k}^+ \to \mathcal{Q}_k^+,$$

and construct Frobenius functors based on them.

Lemma 4.3.1. There is a unique morphism of ringoids $\phi : \mathcal{A}^+ \to \mathcal{A}^+$ satisfying

- $\phi(2n) = n, \phi(2n+1) = +, \phi(+) = +,$
- $\phi\left(\mathrm{Sq}^{2i}\right) = \mathrm{Sq}^i.$

We call this morphism the *Frobenius morphism*.

Proof. It suffices to prove that the Frobenius morphism sends

$$\mathrm{Sq}^i\mathrm{Sq}^j - \sum_{t=0}^{\lfloor i/2\rfloor} \binom{j-t-1}{i-2t}\mathrm{Sq}^{i+j-t}\mathrm{Sq}^t \tag{4.1}$$

to zero. If $i+j$ is odd, every summing term is sent to zero. So we assume that $i+j$ is even. If both i and j are odd, then we need to prove that

$$\binom{j-t-1}{i-2t} \equiv 0 \mod 2 \quad \text{when } t \text{ is even.}$$

It is true by Lucas's theorem. From now on, we assume that both i and j are even.

The Frobenius morphism sends (4.1) to

$$\mathrm{Sq}^{i/2}\mathrm{Sq}^{j/2} - \sum_{s=0}^{\lfloor i/4 \rfloor} \binom{j-2s-1}{i-4s} \mathrm{Sq}^{i/2+j/2-s}\mathrm{Sq}^{s}.$$

We know that

$$0 = \mathrm{Sq}^{i/2}\mathrm{Sq}^{j/2} - \sum_{s=0}^{\lfloor i/4 \rfloor} \binom{j/2-s-1}{i/2-2s} \mathrm{Sq}^{i/2+j/2-s}\mathrm{Sq}^{s}.$$

So it suffices to prove

$$\binom{j-2s-1}{i-4s} \equiv \binom{j/2-s-1}{i/2-2s} \mod 2,$$

or equivalently

$$\binom{2a+1}{2b} \equiv \binom{a}{b} \mod 2.$$

It is true again by Lucas's theorem. □

Lemma 4.3.2. The Frobenius morphism $\mathcal{A}^+ \to \mathcal{A}^+$ induces morphisms $\mathcal{Q}^+ \to \mathcal{Q}^+$ and $\mathcal{Q}^+_{2k} \to \mathcal{Q}^+_k$. They are also called *Frobenius morphisms.*

Proof. The Frobenius morphism $\mathcal{A}^+ \to \mathcal{A}^+$ induces a morphism $\mathcal{Q}^+ \to \mathcal{Q}^+$ because ϕ sends $\mathrm{Sq}^i \in \mathcal{A}^+(n, n+i)$ with $i < n$ to $\mathrm{Sq}^{i/2} \in \mathcal{A}^+(n/2, n/2+i/2)$ with $i/2 < n/2$ if both n and i are even, and zero otherwise. Furthermore, the morphism $\mathcal{Q}^+ \to \mathcal{Q}^+$ induces a morphism $\mathcal{Q}^+_{2k} \to \mathcal{Q}^+_k$ because ϕ sends $\mathrm{Sq}_i = \mathrm{Sq}^{n-i} \in \mathcal{A}^+(n, 2n-i)$ with $i < 2k$ to $\mathrm{Sq}_{i/2} = \mathrm{Sq}^{n/2-i/2} \in \mathcal{A}^+(n/2, n-i/2)$ with $i/2 < k$ if both n and i are even, and zero otherwise. □

Definition 4.3.3 (Frobenius functors)**.** We define the Frobenius functor $\Phi : \mathcal{U} \to \mathcal{U}$ to be the functor induced by the Frobenius morphism $\mathcal{Q}^+ \to \mathcal{Q}^+$ of ringoids. Similarly, we call the functor $\Phi : \mathcal{U}_k \to \mathcal{U}_{2k}$ induced by the Frobenius morphism $\mathcal{Q}^+_{2k} \to \mathcal{Q}^+_k$ of ringoids the Frobenius functor as well.

Remark 4.3.4. The Frobenius functors commute with the forgetful functors:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi} & \mathcal{U} \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow \\ \mathcal{U}_k & \xrightarrow{\Phi} & \mathcal{U}_{2k} \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{k+1} & \xrightarrow{\Phi} & \mathcal{U}_{2(k+1)} \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow \\ \mathcal{U}_k & \xrightarrow{\Phi} & \mathcal{U}_{2k} \end{array}$$

because the corresponding diagrams of ringoids are commutative.

Remark 4.3.5. The Frobenius functors are exact. If M is a module in $\mathcal{U}_k$, then ΦM is a module in $\mathcal{U}_{2k}$ with

$$(\Phi M)^{2n} = M^n, \quad (\Phi M)^{\mathrm{odd}} = 0.$$

We denote the element in $(\Phi M)^{2n}$ corresponding to x in M^n by Φx. The Steenrod operations on ΦM are

$$\mathrm{Sq}_{2i}(\Phi x) = \Phi(\mathrm{Sq}_i x), \quad \mathrm{Sq}_{\mathrm{odd}}(\Phi x) = 0.$$

For any $k > 0$, there is a natural transformation $\phi u \to \mathrm{id}$ between morphisms of ringoids $\mathcal{Q}_k^+ \to \mathcal{Q}_k^+$, where u is the forgetful morphism $\mathcal{Q}_k^+ \to \mathcal{Q}_{2k}^+$ and ϕ is the Frobenius morphism $\mathcal{Q}_{2k}^+ \to \mathcal{Q}_k^+$. The natural transformation is given by

$$\begin{aligned} &\mathrm{Sq}_0 : \phi u(2n) = n \to 2n, \\ &0 : \phi u(2n+1) = + \to 2n+1, \\ &0 : \phi u(+) = + \to +. \end{aligned}$$

This natural transformation gives rise to another natural transformation $\lambda : u\Phi \to \mathrm{id}$ between functors $\mathcal{Q}_k^+\mathbf{Mod} \to \mathcal{Q}_k^+\mathbf{Mod}$, i.e. between functors $\mathcal{U}_k \to \mathcal{U}_k$. The map $\lambda_M : u\Phi M \to M$ of modules in $\mathcal{U}_k$ sends Φx to $\mathrm{Sq}_0 x$. The kernel and cokernel of $\lambda_M : u\Phi M \to M$ are suspensions because Sq_0 acts trivially on both of them. We define functors $\Omega, \Omega_1 : \mathcal{U}_k \to \mathcal{U}_{k-1}$ such that $\Sigma\Omega M$ is the the cokernel of λ_M and

$\Sigma\Omega_1 M$ is the kernel of λ_M. So the following is an exact sequence in $\mathcal{U}_k$

$$0 \to \Sigma\Omega_1 M \to u\Phi M \to M \to \Sigma\Omega M \to 0.$$

Remark 4.3.6. The functors Ω and Ω_1 commute with the forgetful functors:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Omega} & \mathcal{U} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{U}_{k+1} & \xrightarrow{\Omega} & \mathcal{U}_k \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{k+2} & \xrightarrow{\Omega} & \mathcal{U}_{k+1} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{U}_{k+1} & \xrightarrow{\Omega} & \mathcal{U}_k \end{array}$$

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Omega_1} & \mathcal{U} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{U}_{k+1} & \xrightarrow{\Omega_1} & \mathcal{U}_k \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{k+2} & \xrightarrow{\Omega_1} & \mathcal{U}_{k+1} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{U}_{k+1} & \xrightarrow{\Omega_1} & \mathcal{U}_k \end{array}$$

because the forgetful functor u is exact and commutes with Φ, Σ.

We state the following proposition without proof, since the proof is analogous to the proof of its analogue in the category $\mathcal{U}$ (see e.g. Proposition 1.7.5 in [Sch94]).

Proposition 4.3.7. Let k be any positive integer. The loop functor $\Omega : \mathcal{U}_k \to \mathcal{U}_{k-1}$ is the left adjoint to the suspension functor $\Sigma : \mathcal{U}_{k-1} \to \mathcal{U}_k$. The functor $\Omega_1 : \mathcal{U}_k \to \mathcal{U}_{k-1}$ is the first left derived functor of Ω, and all higher derived functors are trivial.

Example 4.3.8. The functors Ω_1 and Ω act on free modules as:

$$\Omega F_k(n) = \begin{cases} F_{k-1}(n-1) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

$$\Omega_1 F_k(n) = 0$$

Remark 4.3.9. Let M be any module in $\mathcal{U}_k$ with $k > 0$. Then

$$(\Omega M)^{2m} = M^{2m+1}$$
$$(\Omega M)^{2m+1} = \operatorname{coker}(\mathrm{Sq}_0 : M^{m+1} \to M^{2m+2})$$
$$(\Omega_1 M)^{2m} = 0$$
$$(\Omega_1 M)^{2m+1} = \ker(\mathrm{Sq}_0 : M^{m+1} \to M^{2m+2})$$

Example 4.3.10. The functors Ω_1 and Ω act on sphere modules as:

$$\Omega_1 S_k(n) = \begin{cases} S_{k-1}(2n-1) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

$$\Omega S_k(n) = \begin{cases} S_{k-1}(n-1) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

Chapter 5

The homological dimension of the category $\mathcal{U}_k$

In this chapter, we shall prove that the homological dimension of the category $\mathcal{U}_k$ is at most k. Our goal is to prove that $\mathrm{Ext}_k^s(M, N) = 0$ for all $s > k \geq 0$. Our strategy is to first prove it for N a sphere module by induction on k, then for N bounded above, and finally for N a general module.

5.1 Algebraic EHP sequence

The following lemma follows, as does the standard algebraic EHP sequence in $\mathcal{U}$, from analyzing the composite functor spectral sequence associated to the composite $\mathcal{U}_{k-1}(\Omega(-), N) = \mathcal{U}_k(-, \Sigma N)$.

Lemma 5.1.1. Let M be any module in $\mathcal{U}_k$ and N any module in $\mathcal{U}_{k-1}$. Then the following is a long exact sequence of $\mathbb{F}_2$-vector spaces

$$\cdots \to \mathrm{Ext}^s_{k-1}(\Omega M, N) \longrightarrow \mathrm{Ext}^s_k(M, \Sigma N) \longrightarrow \mathrm{Ext}^{s-1}_{k-1}(\Omega_1 M, N) \to$$

$$\mathrm{Ext}^{s+1}_{k-1}(\Omega M, N) \longrightarrow \mathrm{Ext}^{s+1}_k(M, \Sigma N) \longrightarrow \mathrm{Ext}^{s}_{k-1}(\Omega_1 M, N) \to \cdots$$

Proposition 5.1.2. $\mathrm{Ext}^s_k(-, S_k(n)) = 0$ for all $s > k \geq 0$.

Proof. We will proceed by induction on k. When $k = 0$, the functor $\mathcal{U}_0(-, S_0(n))$ is exact so $\mathrm{Ext}^s_0(-, S_0(n)) = 0$ for all $s > k = 0$. Now assume $k > 0$. Take $N = S_{k-1}(n-1)$ in the lemma above and we get a long exact sequence. When $s > k$, both $\mathrm{Ext}^s_{k-1}(\Omega M, N)$ and $\mathrm{Ext}^{s-1}_{k-1}(\Omega_1 M, N)$ are zero by the induction hypothesis. Thus $\mathrm{Ext}^s_k(M, \Sigma N) = 0$. $\square$

5.2 Bounded above modules

Proposition 5.2.1. If N is bounded above, then $\mathrm{Ext}^s_k(-, N) = 0$ for all $s > k \geq 0$.

Proof. Assume $N \neq 0$ and say the highest degree in which N is nontrivial is equal to n. We will proceed by induction on n. When $n = 0$, N is equal to a direct sum of sphere modules and everything follows. Now assume $n > 0$. Below is a short exact sequence of modules in $\mathcal{U}_k$

$$0 \to N' \to N \to N'' \to 0,$$

where N' is the degree n part of N and N'' is the degree $< n$ part of N. This short exact sequence induces a long exact sequence of Ext groups

$$\begin{array}{ccccc} & & & & \cdots \\ \mathrm{Ext}_k^s(M,N') & \longrightarrow & \mathrm{Ext}_k^s(M,N) & \longrightarrow & \mathrm{Ext}_k^s(M,N'') \\ \mathrm{Ext}_k^{s+1}(M,N') & \longrightarrow & \mathrm{Ext}_k^{s+1}(M,N) & \longrightarrow & \mathrm{Ext}_k^{s+1}(M,N'') \\ \cdots & & & & \end{array}$$

Since N' is a direct sum of spheres, Proposition 5.1.2 implies $\mathrm{Ext}_k^s(M, N') = 0$. We also know that $\mathrm{Ext}_k^s(M, N'') = 0$ by the induction hypothesis. Therefore, $\mathrm{Ext}_k^s(M, N) = 0$. $\square$

5.3 General modules

In this section, we are going to prove $\mathrm{Ext}_k^s(-, -) = 0$ for all $s > k \geq 0$. Preparing for that proof, we present without proof the following lemma on the Milnor exact sequence for Ext groups.

Lemma 5.3.1 (Milnor exact sequence, see e.g. Theorem 3.5.8 in [Wei94])**.** Let M be any module in $\mathcal{U}_k$. Let $N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \cdots$ be an inverse system of modules in $\mathcal{U}_k$ such that all maps are surjective. Denote its inverse limit by N. Then below is a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Ext}_k^{s-1}(M, N_i) \to \mathrm{Ext}_k^s(M, N) \to \varprojlim \mathrm{Ext}_k^s(M, N_i) \to 0.$$

Theorem 5.3.2. $\mathrm{Ext}_k^s(-, -) = 0$ *for all* $s > k \geq 0$.

Proof. Let M and N be any two modules in $\mathcal{U}_k$. We are going to prove $\mathrm{Ext}_k^s(M, N) = 0$ for all $s > k \geq 0$. For any $i \geq 0$, define N_i to be the module in $\mathcal{U}_k$ with the degree $\leq i$ part equal to N and the degree $> i$ part zero. That is, $N_i^j = N^j$ if $j \leq i$ and $N_i^j = 0$ if $j > i$. The Steenrod operations on N_i follow from N. Then we get a surjective inverse system

$$N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \cdots$$

and the inverse limit of the inverse system is exactly our module N. So we get the Milnor exact sequence of Ext groups

$$0 \to \varprojlim{}^1 \mathrm{Ext}_k^{s-1}(M, N_i) \to \mathrm{Ext}_k^s(M, N) \to \varprojlim \mathrm{Ext}_k^s(M, N_i) \to 0.$$

Since each N_i is bounded above, we know $\mathrm{Ext}_k^s(M, N_i) = 0$ for all i by Proposition 5.2.1 and thus the right term in the short exact sequence is zero.

It remains to prove that the left term in the short exact sequence is zero. For all $i \geq 0$, $0 \to K \to N_{i+1} \to N_i \to 0$ is a short exact sequence with K equal to a direct sum of sphere modules. That short exact sequence leads to a long exact sequence, part of which looks like

$$\cdots \to \mathrm{Ext}_k^{s-1}(M, N_{i+1}) \to \mathrm{Ext}_k^{s-1}(M, N_i) \to \mathrm{Ext}_k^s(M, K) \to \cdots .$$

We know that $\mathrm{Ext}_k^s(M, K) = 0$ by Proposition 5.1.2. Therefore, the map $\mathrm{Ext}_k^{s-1}(M, N_{i+1}) \to \mathrm{Ext}_k^{s-1}(M, N_i)$ is a surjection for all i. The left term is thus zero because it is the $\varprojlim{}^1$ of a surjective inverse system (see e.g. Proposition 3.5.7 in [Wei94]). $\square$

Chapter 6

Inverse system of Ext groups

Since the forgetful functor $u : \mathcal{U}_{k+1} \to \mathcal{U}_k$ preserves projectives by Proposition 4.1.4 and is exact by Proposition 4.1.2, it induces a map of Ext groups $\mathrm{Ext}^s_{k+1}(M, N) \to \mathrm{Ext}^s_k(uM, uN)$, where M and N are any two modules in $\mathcal{U}_{k+1}$. Therefore, given any two modules M and N in $\mathcal{U}$, there is an inverse system of Ext groups

$$\cdots \to \mathrm{Ext}^s_2(uM, uN) \to \mathrm{Ext}^s_1(uM, uN) \to \mathrm{Ext}^s_0(uM, uN).$$

In this chapter, we will study this inverse system and its inverse limit. The main result of this chapter is summarized in the following theorem.

Theorem 6.0.1. *Let M and N be any two nonzero modules in the category $\mathcal{U}$. Let s be any non-negative integer. If N is bounded above, then the inverse system*

$$\cdots \to \mathrm{Ext}^s_2(uM, uN) \to \mathrm{Ext}^s_1(uM, uN) \to \mathrm{Ext}^s_0(uM, uN)$$

stabilizes and the limit is equal to $\mathrm{Ext}^s_{\mathcal{U}}(M, N)$. More specifically, if $N^n = 0$ for $n > k + 1$, then the maps

$$\mathrm{Ext}^s_{k+1}(uM, uN) \to \mathrm{Ext}^s_k(uM, uN)$$

and

$$\mathrm{Ext}^s_{\mathcal{U}}(M, N) \to \mathrm{Ext}^s_k(uM, uN)$$

are isomorphisms.

Before proving this theorem, we take some time defining the truncated categories and relating them to our familiar categories $\mathcal{U}$ and $\mathcal{U}_k$.

Definition 6.0.2 (Truncated categories)**.** For any non-negative integer n, we let $\mathcal{U}^{\leq n}$ be the full subcategory of $\mathcal{U}$ whose objects are M with $M^i = 0$ for any $i > n$. Similarly, we let $\mathcal{U}_k^{\leq n}$ be the full subcategory of $\mathcal{U}_k$ whose objects are M with $M^i = 0$ for any $i > n$.

Remark 6.0.3. There are obvious truncation functors $T : \mathcal{U} \to \mathcal{U}^{\leq n}$ and $T : \mathcal{U}_k \to \mathcal{U}_k^{\leq n}$. They are the left adjoints to the inclusion functors. The forgetful functor $\mathcal{U} \to \mathcal{U}_k$ restricts to $\mathcal{U}^{\leq n} \to \mathcal{U}_k^{\leq n}$, and the forgetful functor $\mathcal{U}_{k+1} \to \mathcal{U}_k$ restricts to $\mathcal{U}_{k+1}^{\leq n} \to \mathcal{U}_k^{\leq n}$. In other words, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{u} & \mathcal{U}_k \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{U}^{\leq n} & \xrightarrow{u} & \mathcal{U}_k^{\leq n} \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{k+1} & \xrightarrow{u} & \mathcal{U}_k \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{U}_{k+1}^{\leq n} & \xrightarrow{u} & \mathcal{U}_k^{\leq n} \end{array}$$

Lemma 6.0.4. (i) When $n \leq k + 1$, the forgetful functors

$$\mathcal{U}^{\leq n} \to \mathcal{U}_k^{\leq n} \quad \text{and} \quad \mathcal{U}_{k+1}^{\leq n} \to \mathcal{U}_k^{\leq n}$$

are both isomorphisms.

(ii) When $N^i = 0$ for any $i > n$, the truncation functors lead to isomorphisms

$$\mathcal{U}(M, N) \to \mathcal{U}^{\leq n}(TM, TN)$$

and

$$\mathcal{U}_k(M, N) \to \mathcal{U}_k^{\leq n}(TM, TN).$$

(iii) When $n \leq k + 1$ and $N^i = 0$ for any $i > n$, the forgetful functors lead to isomorphisms

$$\mathcal{U}(M, N) \to \mathcal{U}_k(uM, uN)$$

and

$$\mathcal{U}_{k+1}(M,N) \to \mathcal{U}_k(uM,uN).$$

Proof. (i) On a module in $\mathcal{U}^{\leq n}$ or a module in $\mathcal{U}_k^{\leq n}$, the only nontrivial lower Steenrod operations are $\mathrm{Sq}_0, \mathrm{Sq}_1, \ldots, \mathrm{Sq}_{n-2}$. The forgetful functors $\mathcal{U}^{\leq n} \to \mathcal{U}_k^{\leq n}$ and $\mathcal{U}_{k+1}^{\leq n} \to \mathcal{U}_k^{\leq n}$ forget operations Sq_i with $i \geq k$. They are not really forgetting anything because $n-2 < k$.

(ii) When $i > n$, any map $M \to N$ would send M^i to 0 because $N^i = 0$.

(iii) The following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}(M,N) & \longrightarrow & \mathcal{U}_k(uM,uN) \\ \downarrow & & \downarrow \\ \mathcal{U}^{\leq n}(TM,TN) & \longrightarrow & \mathcal{U}_k^{\leq n}(uTM,uTN) \end{array}$$

The bottom horizontal map is an isomorphism by part (i). Both vertical maps are isomorphisms by part (ii). So the top horizontal map is an isomorphism. Similarly, one can prove that $\mathcal{U}_{k+1}(M,N) \to \mathcal{U}_k(uM,uN)$ is an isomorphism. □

Theorem 6.0.1 follows immediately from part (iii) of Lemma 6.0.4.

Chapter 7

Λ-complex for modules in $\mathcal{U}_k$

In this chapter, we will introduce a contravariant functor Λ_k from the category of unstable modules with only the top k Steenrod operations to the category of cochain complexes of graded vector spaces over $\mathbb{F}_2$, namely $\Lambda_k : \mathcal{U}_k^{\mathrm{op}} \to \mathrm{Ch}^*(\mathrm{Gr}(\mathbb{F}_2\mathbf{Mod}))$. Note that we use the upper index to emphasize the *cochain* complex. The cohomological degree is denoted by s and the degree in the graded vector space is denoted by a. To motivate its study, we list two nice properties of this functor here:

- The cohomology $H^{s,a}(\Lambda_k(M))$ is naturally isomorphic to $\mathrm{Ext}_k^s(M, S_k(a))$ for all s, a, according to Theorem 7.4.1.

- The cochain complex $\Lambda_k(M)$ is relatively small and easy to compute. To be more concrete, $\Lambda_k^s(M) = 0$ for all $s < 0$ or $s > k$. Furthermore, when M is bounded above and is of finite dimension in each degree, so is each module in its cochain complex $\Lambda_k(M)$.

7.1 Recall: Λ algebra and Λ functor

In this section, we provide a brief recollection of our knowledge on the Λ algebra and the Λ functor. The Λ algebra was first introduced in [BCK+66].

Formally, Λ is an associative differential bigraded $\mathbb{F}_2$-algebra with generators $\lambda_i \in$

$\Lambda^{1,i+1}$ for $i \geq 0$ and relations

$$\lambda_i \lambda_{2i+1+j} = \sum_{t \geq 0} \binom{j-t-1}{t} \lambda_{i+j-t} \lambda_{2i+1+t} \quad \text{for } i, j \geq 0 \tag{7.1}$$

with differential

$$d(\lambda_i) = \sum_{j \geq 1} \binom{i-j}{j} \lambda_{i-j} \lambda_{j-1}.$$

We refer to the first grading in Λ as the cohomological degree s and the second as the internal degree t. The differential d in Λ increases s by one and preserves t. The Λ algebra is the Koszul dual of the Steenrod algebra A (see [Pri70]). We will systematically use the PBW basis, or equivalently the admissible monomials, of the Λ algebra:

Definition 7.1.1 (Admissible monomials)**.** A monomial

$$\lambda_I := \lambda_{I(1)} \lambda_{I(2)} \cdots \lambda_{I(s)} \in \Lambda$$

is said to be admissible if

$$2I(r) \geq I(r+1) \quad \text{for all } 1 \leq r < s.$$

The excess of λ_I is defined as

$$\text{excess}(I) := \sum_{r=1}^{s-1} (2I(r) - I(r+1)).$$

Proposition 7.1.2 (Theorem 7.11 in [Cur71])**.** The admissible monomials form an additive basis for Λ.

Proposition 7.1.3 (Propositions 2.4 and 2.6 in [BCK+66])**.** The internal degree t part of the s-th cohomology of Λ is naturally isomorphic to the s-th Ext group in the category of $\mathcal{M}$ from $S(0)$ to $S(t)$, i.e.

$$H^{s,t}(\Lambda) \cong \text{Ext}^s_{\mathcal{M}}(S(0), S(t)).$$

Definition 7.1.4 (Subcomplex $\Lambda(m)$)**.** $\Lambda(m)$ is defined to be the sub-bigraded vector space of Λ spanned by the admissible monomials λ_I with $I(1) < m$. The trivial monomial 1 lives in all $\Lambda(m)$.

The lemma below implies that $\Lambda(m)$ is a subcomplex of Λ.

Lemma 7.1.5 (Proposition 1.8.3 in [Wan67] and Proposition 5.1 in [Sin75])**.**

$$d(\Lambda(m)) \subseteq \Lambda(m), \quad \Lambda^{s,t}(m)\Lambda(m+t) \subseteq \Lambda(m).$$

Proposition 7.1.6 (Theorem 7.12 in [Cur71])**.** The internal degree t part of the s-th cohomology of $\Lambda(m)$ is naturally isomorphic to the s-th Ext group in the category of $\mathcal{U}$ from $S(m)$ to $S(m+t)$, i.e.

$$H^{s,t}(\Lambda(m)) \cong \operatorname{Ext}^s_{\mathcal{U}}(S(m), S(m+t)).$$

Notation 7.1.7 (Bi-suspension $\Sigma^{s,t}$)**.** Given a cochain complex C of graded vector spaces, we denote by $\Sigma^{s,t}C$ the bi-suspension of C where we suspend the cohomological degree by s and the vector space grading by t.

Proposition 7.1.8 (11.1 in [Cur71])**.** Let e denote the inclusion $\Lambda(m) \subseteq \Lambda(m+1)$. Then for each $m \geq 0$, there is a short exact sequence

$$0 \to \Lambda(m) \xrightarrow{e} \Lambda(m+1) \xrightarrow{h} \Sigma^{1,m+1}\Lambda(2m+1) \to 0.$$

The map h drops λ_m if the admissible monomial starts with λ_m and sends all other admissible basis vectors to zero.

As in 3.2 in [BC70], one can generalize $\Lambda(m)$ to $\Lambda(M)$ for any $M \in \mathcal{U}$ in such a way that $\Lambda(m) = \Lambda(S(m))$ and the cohomology of $\Lambda(M)$ is naturally isomorphic to the Ext group from M to spheres.

Definition 7.1.9 ($\Lambda(M)$)**.** Given any $M \in \mathcal{U}$, we construct a cochain complex

$$\Lambda(M) = \bigoplus_m \Lambda(m) \otimes \operatorname{Hom}(M^m, \mathbb{F}_2)$$

with differential

$$d(\lambda_I \otimes x_m) = d(\lambda_I) \otimes x_m + \sum_{i \geq 1} \lambda_{i-1}\lambda_I \otimes x_m \mathrm{Sq}^i.$$

The Steenrod operation Sq^i acts from the right on $x_m \in \mathrm{Hom}(M^m, \mathbb{F}_2)$. We denote the dual of a vector space V by $V^\vee$. The subspace $\Lambda(M)$ of $\Lambda \otimes (M^\vee)$ is closed under the differential by Lemma 7.1.5. The relations (7.1) in the Λ algebra and the ones (3.1) in the Steenrod algebra A play so well with each other that one can check $d^2(\lambda_I \otimes x_m) = 0$.

The cochain complex $\Lambda(M)$ is bigraded: the first grading is the cohomological degree s and the second grading is the *absolute* internal degree a. The absolute internal degree a of $\lambda_I \otimes x_m$ is equal to m plus the internal degree of λ_I. The differential d in $\Lambda(M)$ increases s by one and preserves a.

Proposition 7.1.10 (Theorem 3.3 in [BC70])**.** The absolute internal degree a part of the s-th cohomology of $\Lambda(M)$ is naturally isomorphic to the s-th Ext group in the category of $\mathcal{U}$ from M to $S(a)$, i.e.

$$H^{s,a}(\Lambda(M)) \cong \mathrm{Ext}^s_{\mathcal{U}}(M, S(a)).$$

We can view Λ as a contravariant functor from $\mathcal{U}$ to $\mathrm{Ch}^*(\mathrm{Gr}(\mathbb{F}_2\mathbf{Mod}))$.

7.2 Cochain complex $\Lambda_k(m)$

Definition 7.2.1. For all $m, k \geq 0$, $\Gamma(m, k)$ is defined to be the sub-bigraded vector space of $\Lambda(m)$ spanned by the admissible monomials λ_I with $I(1) < m$ and

$$\mathrm{excess}(I) + (s-1) > I(1) - (m-k).$$

The trivial monomial 1 does not live in any $\Gamma(m, k)$.

The lemma below implies that $\Gamma(m, k)$ is a subcomplex of $\Lambda(m)$.

Notation 7.2.2. We denote by $t(I)$ the internal degree of λ_I.

Lemma 7.2.3. $\Gamma(m,k)$ is closed under the differential.

Proof. Since $\mathrm{excess}(I) = t(I) - s + I(1) - 2I(s)$, the conditions $\mathrm{excess}(I) + (s-1) > I(1) - (m-k)$ and $t(I) + m - k - 1 > 2I(s)$ are equivalent. Since the differential d does not change $t(I), m$ and k, it suffices to prove that the differential d does not increase the last subscript. In other words, we need to prove that $d(\lambda_I)$ can be written as a sum of admissible monomials λ_J with $J(s+1) \leq I(s)$. This is true because neither the differential formula nor the relations in Λ increase the second subscript. □

Observe that $\Gamma(m, k+1)$ is a subcomplex of $\Gamma(m,k)$.

Definition 7.2.4. For all $m, k \geq 0$, $\Lambda_k(m)$ is defined to be the quotient cochain complex of $\Lambda(m)$ by its subcomplex $\Gamma(m,k)$. The differentials in $\Lambda_k(m)$ follow from those in $\Lambda(m)$.

Observe that all nontrivial admissible monomials λ_I in $\Lambda_k(m)$ have $m-k \leq I(1) \leq m-1$, because $I(1) < m$ and $0 \leq \mathrm{excess}(I) + (s-1) \leq I(1) - (m-k)$. Note that $\Lambda_k^s(m) = 0$ when $s > k$ or $s < 0$. When $s > k$, $\mathrm{excess}(I) + s - 1 < I(1) - m + s$ and thus $\mathrm{excess}(I) \leq I(1) - m < 0$. No admissible monomial λ_I can have $\mathrm{excess}(I) < 0$.

In a later section, we are going to prove Theorem 7.4.1, a special case of which is that the cohomology $H^{s,t}(\Lambda_k(m))$ is equal to $\mathrm{Ext}_k^s(S_k(m), S_k(m+t))$ for all s, t. This result is the analogue of Proposition 7.1.6 in the world of $\mathcal{U}_k$.

Proposition 7.2.5. The following is a short exact sequence:

$$0 \to \Lambda_k(m) \xrightarrow{e} \Lambda_{k+1}(m+1) \xrightarrow{h} \Sigma^{1,m+1}\Lambda_k(2m+1) \to 0.$$

Proof. Since

$$\Gamma(m+1, k+1) \cap \Lambda(m) = \Gamma(m,k)$$

and

$$h\left(\Gamma(m+1,k+1)\right) = \Sigma^{1,m+1}\Gamma(2m+1,k),$$

the short exact sequence in Proposition 7.1.8

$$0 \to \Lambda(m) \xrightarrow{e} \Lambda(m+1) \xrightarrow{h} \Sigma^{1,m+1}\Lambda(2m+1) \to 0$$

induces a new short exact sequence

$$0 \to \frac{\Lambda(m)}{\Gamma(m,k)} \xrightarrow{e} \frac{\Lambda(m+1)}{\Gamma(m+1,k+1)} \xrightarrow{h} \frac{\Sigma^{1,m+1}\Lambda(2m+1)}{\Sigma^{1,m+1}\Gamma(2m+1,k)} \to 0. \qquad \square$$

The short exact sequence above recovers the algebraic EHP sequence in Lemma 5.1.1 when M and N are sphere modules:

$$\begin{array}{l} \cdots \xrightarrow{P} \mathrm{Ext}_k^s(m,n) \xrightarrow{E} \mathrm{Ext}_{k+1}^s(m+1,n+1) \xrightarrow{H} \mathrm{Ext}_k^{s-1}(2m+1,n) \\ \xrightarrow{P} \mathrm{Ext}_k^{s+1}(m,n) \xrightarrow{E} \mathrm{Ext}_{k+1}^{s+1}(m+1,n+1) \xrightarrow{H} \mathrm{Ext}_k^{s-1}(2m+1,n) \xrightarrow{P} \cdots \end{array}$$

Here $\mathrm{Ext}_k^s(m,n)$ is an abbreviation for $\mathrm{Ext}_k^s(S_k(m), S_k(n))$.

Example 7.2.6. We write down the structure of several $\Lambda_k(m)$'s explicitly.

1. $\Lambda_0(m)$ has trivial differentials and additive basis $\{1\}$ for all $m \geq 0$.

2. $\Lambda_1(m)$ has trivial differentials and the following additive basis

$$\begin{cases} \{1\} & \text{if } m = 0 \\ \{1, \lambda_{m-1}\} & \text{if } m \geq 1 \end{cases}$$

3. $\Lambda_2(m)$ has trivial differentials and the following additive basis

$$\begin{cases} \{1\} & \text{if } m = 0 \\ \{1, \lambda_0, \lambda_0\lambda_0\} & \text{if } m = 1 \\ \{1, \lambda_{m-2}, \lambda_{m-1}, \lambda_{m-1}\lambda_{2m-2}\} & \text{if } m \geq 2 \end{cases}$$

4. $\Lambda_3(m)$ has trivial differentials and the following additive basis

$$\begin{cases} \{1\} & \text{if } m = 0 \\ \{1, \lambda_0, \lambda_0\lambda_0, \lambda_0\lambda_0\lambda_0\} & \text{if } m = 1 \\ \{1, \lambda_0, \lambda_1, \lambda_0\lambda_0, \lambda_1\lambda_1, \lambda_1\lambda_2, \lambda_1\lambda_2\lambda_4\} & \text{if } m = 2 \\ \{1, \lambda_{m-3}, \lambda_{m-2}, \lambda_{m-1}, \lambda_{m-2}\lambda_{2m-4}, \lambda_{m-1}\lambda_{2m-3}, & \\ \quad \lambda_{m-1}\lambda_{2m-2}, \lambda_{m-1}\lambda_{2m-2}\lambda_{4m-4}\} & \text{if } m \geq 3 \end{cases}$$

5. $\Lambda_5(3)$ has the following additive basis

$$\begin{aligned}
&1, \\
&\lambda_0, \lambda_1, \lambda_2, \\
&\lambda_0\lambda_0, \lambda_1\lambda_0, \lambda_1\lambda_1, \lambda_1\lambda_2, \lambda_2\lambda_1, \lambda_2\lambda_2, \lambda_2\lambda_3, \lambda_2\lambda_4, \\
&\lambda_0\lambda_0\lambda_0, \lambda_1\lambda_1\lambda_2, \lambda_1\lambda_2\lambda_3, \lambda_1\lambda_2\lambda_4, \\
&\lambda_2\lambda_2\lambda_4, \lambda_2\lambda_3\lambda_5, \lambda_2\lambda_3\lambda_6, \lambda_2\lambda_4\lambda_6, \lambda_2\lambda_4\lambda_7, \lambda_2\lambda_4\lambda_8, \\
&\lambda_1\lambda_2\lambda_4\lambda_8, \lambda_2\lambda_3\lambda_6\lambda_{12}, \lambda_2\lambda_4\lambda_7\lambda_{14}, \lambda_2\lambda_4\lambda_8\lambda_{15}, \lambda_2\lambda_4\lambda_8\lambda_{16}
\end{aligned}$$

The only nontrivial differential in this complex is $d(\lambda_2) = \lambda_1\lambda_0$.

7.3 Functor $\Lambda_k : \mathcal{U}_k^{\text{op}} \to \text{Ch}^*(\text{Gr}(\mathbb{F}_2\mathbf{Mod}))$

Definition 7.3.1. Given any $M \in \mathcal{U}_k$, we construct a cochain complex

$$\Lambda_k(M) = \bigoplus_m \Lambda_k(m) \otimes \text{Hom}(M^m, \mathbb{F}_2)$$

with differentials

$$d(\lambda_I \otimes x_m) = d(\lambda_I) \otimes x_m + \sum_{i \geq 1, m-2i<k} \lambda_{i-1}\lambda_I \otimes x_m \text{Sq}^i.$$

Note that the differential is similar to Definition 7.1.9 except for the condition

$m - 2i < k$. Here we require $m - 2i < k$ because $x_m \mathrm{Sq}^i = x_m \mathrm{Sq}_{m-2i}$ and M as a module in $\mathcal{U}_k$ only allows operations $\mathrm{Sq}_0, \ldots, \mathrm{Sq}_{k-1}$.

Lemma 7.3.2. The differential d is well-defined.

Proof. It suffices to prove $\lambda_{i-1}\lambda_I \in \Gamma(m-i,k)$ if $\lambda_I \in \Gamma(m,k)$ when $k \geq 0, i \geq 1$ and $m \geq 2i$. By Lemma 7.1.5, $2i \leq m$ implies that $\lambda_{i-1}\Lambda_I$ lives in $\Lambda(m-i)$. Note that the condition for an admissible monomial $\lambda_I \in \Lambda(m)$ to be in $\Gamma(m,k)$ is $\mathrm{excess}(I) + (s-1) > I(1) - (m-k)$, which is equivalent to $t(I) - 2I(s) > k - m + 1$. Say $\lambda_{i-1}\lambda_I = \sum_J \lambda_J$ with λ_J being admissible monomials. Then $t(J) = i + t(I)$ and $t(J) - 2J(s+1) = i + t(I) - 2J(s+1) \geq i + t(I) - 2I(s) > i + k - m + 1$. So $\lambda_J \in \Gamma(m-i,k)$. □

Theorem 7.3.3. $d^2 = 0$ *in* $\Lambda_k(M)$.

Proposition 7.3.4. The cochain complex $\Lambda_k(uM)$ is a quotient of $\Lambda(M)$.

We first prove two lemmas which will come in handy when proving Theorem 7.3.3 and Proposition 7.3.4. Lemma 7.3.5 is about the cochain complex $\Lambda_k(m)$ and Lemma 7.3.6 is about the standard Λ algebra.

Lemma 7.3.5. If $i+1 \leq m-k$ and λ_I is any admissible monomial in $\Lambda(m+i+1)$, then $\lambda_i\lambda_I = 0 \in \Lambda_k(m)$.

Proof. By Lemma 7.1.5, $\lambda_i\lambda_I$ lives in $\Lambda(m)$. If $\lambda_I = 1$, then λ_i is trivial in $\Lambda_k(m)$ because $i < m-k$. From now on, assume the length of λ_I is $s \geq 1$. Say $\lambda_i\lambda_I = \sum_J \lambda_J$ where each λ_J is admissible of length $s+1$ in $\Lambda(m)$. It suffices to prove $\mathrm{excess}(J)+s > J(1) - (m-k)$ or equivalently, $J(1) + \cdots + J(s) + s + (m-k) > J(s+1)$. Since $i+1 \leq m-k$, it suffices to prove $J(1) + \cdots + J(s) + i + (s+1) > J(s+1)$, which follows from Lemma 7.3.6 □

Lemma 7.3.6. Let λ_I be any admissible monomial of length $s \geq 1$. Write $\lambda_i\lambda_I$ as a sum of admissible monomials λ_J's. Then $i + J(1) + \cdots + J(s) \geq J(s+1)$.

Proof. We will proceed by induction on s. Let's consider the base case $s = 1$. When $\lambda_i\lambda_{I(1)}$ is admissible, $i + J(1) \geq J(2)$ is true because $J(1) = i, J(2) = I(1)$ and

$2i \geq I(1)$. When $\lambda_i \lambda_{I(1)}$ is not admissible, we write $I(1) = 2i + 1 + j$ with $j \geq 0$ and then apply relation (7.1), whose right hand side is admissible. It's still true that $i + J(1) \geq J(2)$ because $J(1) = i + j - t, J(2) = 2i + 1 + t$ and the binomial coefficient $\binom{j-t-1}{t}$ in relation (7.1) requires $j - t - 1 \geq t$.

Assume $s > 1$. Write $\lambda_i \lambda_{I(1,2,\ldots,s-1)}$ as a sum of admissible monomials λ_P. Then by the case $s-1$, $i + P(1) + \cdots P(s-1) \geq P(s)$. Now consider $\lambda_P \lambda_{I(s)} = \lambda_{P(1,\ldots,s-1)} \lambda_{P(s)} \lambda_{I(s)}$. It is equal to a sum of $\lambda_{P(1,\ldots,s-1)} \lambda_{Q(1,2)}$ where both $\lambda_{P(1,\ldots,s-1)}$ and $\lambda_{Q(1,2)}$ are admissible monomials. By the base case $s = 1$, $P(s) + Q(1) \geq Q(2)$. Adding those two inequalities up leads to $i + P(1) + \cdots + P(s-1) + Q(1) \geq Q(2)$. Further applying relations (7.1) to $\lambda_{P(1,\ldots,s-1)} \lambda_{Q(1,2)}$ will either preserve both sides or increase the left hand side while decreasing the right hand side. □

Proof of Theorem 7.3.3. In this proof, we declare $\binom{a}{b} = 0$ if $a < 0$ or $b < 0$. Let x be any element in M_m.

$$\begin{aligned} d^2(\lambda_I \otimes x) =& d^2(\lambda_I) \otimes x \\ &+ \sum_{n \geq 1, m-2n<k} \lambda_{n-1} d(\lambda_I) \otimes x\mathrm{Sq}^n \\ &+ \sum_{n \geq 1, m-2n<k} d(\lambda_{n-1}\lambda_I) \otimes x\mathrm{Sq}^n \\ &+ \sum_{i,j \geq 1, m-2i<k, m-i-2j<k} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^i\mathrm{Sq}^j \\ =& \sum_{n \geq 2, m-2n<k} d(\lambda_{n-1})\lambda_I \otimes x\mathrm{Sq}^n \\ &+ \sum_{i,j \geq 1, m-2i<k, m-i-2j<k} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^i\mathrm{Sq}^j \end{aligned}$$

For convenience, define $S(m, k)$ to be the set of indices $(i, j) \in \mathbb{Z}^2_{>0}$ satisfying

$$m - 2i < k, m - i - 2j < k.$$

Define A, B, C as

$$\begin{aligned} A &:= \sum_{(i,j)\in S(m,k), i\geq 2j} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^i\mathrm{Sq}^j, \\ B &:= \sum_{(i,j)\in S(m,k), i<2j} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^i\mathrm{Sq}^j, \\ C &:= \sum_{n\geq 2, m-2n<k} d(\lambda_{n-1})\lambda_I \otimes x\mathrm{Sq}^n. \end{aligned}$$

Therefore, $d^2(\lambda_I \otimes x) = A + B + C$. In A, the composite $\mathrm{Sq}^i\mathrm{Sq}^j$ is admissible but $\lambda_{j-1}\lambda_{i-1}$ is not. Applying relations (7.1) we get

$$\begin{aligned} A &= \sum_{(i,j)\in S(m,k), i\geq 2j} \sum_{t\geq 0} \binom{i-2j-t-1}{t} \lambda_{i-j-t-1}\lambda_{2j+t-1}\lambda_I \otimes x\mathrm{Sq}^i\mathrm{Sq}^j \\ &= \sum_{(i,j)\in S(m,k), i\geq 2j} \sum_{v\geq 2j} \binom{i-v-1}{v-2j} \lambda_{i+j-v-1}\lambda_{v-1}\lambda_I \otimes x\mathrm{Sq}^i\mathrm{Sq}^j \\ &= \sum_{(i',j',s,t)\in A(m,k)} \binom{s-i'-1}{i'-2t} \lambda_{j'-1}\lambda_{i'-1}\lambda_I \otimes x\mathrm{Sq}^s\mathrm{Sq}^t, \end{aligned}$$

where $A(m, k)$ is defined as the set of indices $(i, j, s, t) \in \mathbb{Z}^4_{>0}$ satisfying

$$i < 2j, s \geq 2t, i + j = s + t, m - s - 2t < k.$$

In the second equality, we used the substitution $v = 2j + t$. In the third equality, we used the substitution $i' = v, j' = i + j - v, s = i, t = j$. It is straightforward to check the third equality and the correspondence of index sets. In B, the composite

$\lambda_{j-1}\lambda_{i-1}$ is admissible but $\mathrm{Sq}^i\mathrm{Sq}^j$ is not. Applying the Adem relations, we get

$$\begin{aligned}
B &= \sum_{(i,j)\in S(m,k), i<2j} \sum_{t\geq 0} \binom{j-t-1}{i-2t} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^{i+j-t}\mathrm{Sq}^t \\
&= \sum_{(i,j)\in S(m,k), i<2j} \sum_{t\geq 1} \binom{j-t-1}{i-2t} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^{i+j-t}\mathrm{Sq}^t \\
&\quad + \sum_{(i,j)\in S(m,k), i<2j} \binom{j-1}{i} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^{i+j} \\
&= D + E,
\end{aligned}$$

where D, E are defined as

$$\begin{aligned}
D &:= \sum_{(i,j,s,t)\in D(m,k)} \binom{s-i-1}{i-2t} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^s\mathrm{Sq}^t, \\
E &:= \sum_{(i,j)\in E(m,k)} \binom{j-1}{i} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^{i+j},
\end{aligned}$$

$D(m,k)$ is defined to be the set of indices $(i,j,s,t) \in \mathbb{Z}^4_{>0}$ satisfying

$$i < 2j, s \geq 2t, i+j = s+t, m-2i < k$$

and $E(m,k)$ is defined to be the set of indices $(i,j) \in \mathbb{Z}^2_{>0}$ satisfying

$$i < 2j, m-2i < k.$$

Therefore, $d^2(\lambda_I \otimes x) = A + D + E + C$. Applying the differential formula to $d(\lambda_{n-1})$, we get

$$\begin{aligned}
C &= \sum_{n\geq 2, m-2n<k} \sum_{i\geq 1} \binom{n-i-1}{i} \lambda_{n-i-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^n \\
&= \sum_{(i,j)\in C(m,k)} \binom{j-1}{i} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^{i+j},
\end{aligned}$$

where $C(m,k)$ is defined as the set of indices $(i,j) \in \mathbb{Z}^2_{>0}$ satisfying

$$i < 2j, m - 2(i+j) < k.$$

In the second equality, we used the substitution $j = n - i$. Observe that A and D are summations of the same expression over slightly different index set.The same for E and C.

Take any $(i,j,s,t) \in D(m,k)$. Then $(m-2i) - (m-s-2t) = s + 2t - 2i \geq 1$ because the binomial coefficient leads to $s - i - 1 \geq i - 2t$. So $(i,j,s,t) \in A(m,k)$ and $D(m,k)$ is a subset of $A(m,k)$. The difference of those two index sets consist of $(i,j,s,t) \in A(m,k)$ satisfying $m - s - 2t = k - 1$ and $s + 2t - 2i = 1$. Any (i,j,s,t) in the difference $A(m,k) - D(m,k)$ satisfies $m - 2i = k$.

Take any $(i,j) \in E(m,k)$. Then $(m - 2i - 2j) - (m - 2i) = -2j < 0$. So $(i,j) \in C(m,k)$ and $E(m,k)$ is a subset of $C(m,k)$. The difference of those two index sets consists of $(i,j) \in C(m,k)$ satisfying $m - 2i \geq k$.

Therefore,

$$A + D = \sum_{(i,j,s,t) \in A(m,k) - D(m,k)} \binom{s-i-1}{i-2t} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^s\mathrm{Sq}^t$$

and

$$E + C = \sum_{(i,j) \in C(m,k) - E(m,k)} \binom{j-1}{i} \lambda_{j-1}\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^{i+j}.$$

The index (i,j,s,t) in $A(m,k) - D(m,k)$ and the index (i,j) in $C(m,k) - E(m,k)$ both satisfy $m - 2i \geq k$. By Lemma 7.3.5, $\lambda_{i-1}\lambda_I = 0$ when $m - 2i \geq k$ and thus $A + D = 0, E + C = 0$. Adding them up we get $d^2(\lambda_I \otimes x) = A + D + E + C = 0$. □

Proof of Proposition 7.3.4. There is an obvious quotient map from $\Lambda(M)$ to $\Lambda_k(uM)$. It suffices to prove that the quotient maps and the differentials commute:

$$\begin{array}{ccc} \Lambda(M) & \xrightarrow{d} & \Lambda(M) \\ {\scriptstyle\text{quotient}}\downarrow & & \downarrow{\scriptstyle\text{quotient}} \\ \Lambda_k(uM) & \xrightarrow{d} & \Lambda_k(uM) \end{array}$$

According to the definitions of differentials in $\Lambda(M)$ and $\Lambda_k(uM)$ respectively, we only need to prove $\lambda_{i-1}\lambda_I = 0 \in \Lambda_k(m-i)$ if $m-2i \geq k$ and λ_I is an admissible monomial in $\Lambda(m)$. That is exactly the statement of Lemma 7.3.5 with an appropriate variable substitution. ☐

Proposition 7.3.7. The contravariant functor

$$\Lambda_k : \mathcal{U}_k^{\mathrm{op}} \to \mathrm{Ch}^*(\mathrm{Gr}(\mathbb{F}_2\mathbf{Mod}))$$

is exact.

Proof. It follows immediately from Definition 7.3.1. ☐

7.4 Cohomology of the cochain complex $\Lambda_k(M)$

Bousfield and Curtis proved Proposition 7.1.10 in Section 3 of [BC70]. Theorem 7.4.1, our main result in this section, is the analogue of that proposition in the world of $\mathcal{U}_k$. Our proof of this theorem is inspired by the Bousfield-Curtis approach.

Theorem 7.4.1. *For any module $M \in \mathcal{U}_k$, the cochain complex $\Lambda_k(M)$ is of length at most k and there is a natural isomorphism*

$$H^{s,a}(\Lambda_k(M)) \cong \mathrm{Ext}_k^s(M, S_k(a)) \quad \textit{for all } s, a.$$

Proof. Since the functor Λ_k is exact by Proposition 7.3.7, it suffices to prove this theorem only when M is a free module in $\mathcal{U}_k$. That is, we only need to prove

$$H^{s,a}(\Lambda_k F_k(n)) = \begin{cases} \mathbb{F}_2 & \text{if } s=0, a=n \\ 0 & \text{otherwise} \end{cases}$$

We will prove this by induction on k and n. When $k=0$ or $n=0$, the complex $\Lambda_k F_k(n)$ is $\mathbb{F}_2$ at degree $s=0, a=n$ and zero at the other degrees. So its cohomology is equal to $\mathbb{F}_2$ if $s=0, a=n$ and zero otherwise. For the induction step, assume $k>0$

and $n > 0$. Recall from Example 4.3.8 that $\Omega F_k(n) = F_{k-1}(n-1)$ and $\Omega_1 F_k(n) = 0$. Then Proposition 7.4.3 leads to $H^{s,a}\Lambda_k F_k(n) \cong H^{s,a-1}\Lambda_{k-1}F_{k-1}(n-1)$, which by induction is equal to $\mathbb{F}_2$ at $s = 0, a = n$ and zero otherwise. □

Remark 7.4.2. Let M be any module in $\mathcal{U}_{k+1}$ and denote $\mathrm{Hom}(M^m, \mathbb{F}_2)$ by M_m. Recall the exact sequence $0 \to \Sigma\Omega_1 M \to u\Phi M \to M \to \Sigma\Omega M \to 0$ from Section 4.3. Its dual is still exact. So we get a natural inclusion $i : (\Omega M)_m \to M_{m+1}$ and a natural projection $p : M_{m+1} \to (\Omega_1 M)_{2m+1}$. By the exactness of the dual, the image of $i : (\Omega M)_m \to M_{m+1}$ is equal to the kernel of $\mathrm{Sq}_0 : M_{m+1} \to M_{(m+1)/2}$ if m is odd and M_{m+1} if m is even. Similarly, the kernel of $p : M_{m+1} \to (\Omega_1 M)_{2m+1}$ is equal to the image of $\mathrm{Sq}_0 : M_{2m+2} \to M_{m+1}$. The Steenrod operations act on $i(x)$ and $p(x)$ as $i(x)\mathrm{Sq}^i = i(x\mathrm{Sq}^i)$, $p(x)\mathrm{Sq}^{2i} = p(x\mathrm{Sq}^i)$ and $p(x)\mathrm{Sq}^{\mathrm{odd}} = 0$.

Proposition 7.4.3. Let M be any module in $\mathcal{U}_{k+1}$. Then there is a natural long exact sequence

$$\begin{array}{l} \cdots \to H^{s,a}\Lambda_k(\Omega M) \longrightarrow H^{s,a+1}\Lambda_{k+1}(M) \longrightarrow H^{s-1,a}\Lambda_k(\Omega_1 M) \\ \to H^{s+1,a}\Lambda_k(\Omega M) \longrightarrow H^{s+1,a+1}\Lambda_{k+1}(M) \longrightarrow H^{s,a}\Lambda_k(\Omega_1 M) \to \cdots \end{array}$$

Proof. Recall the two maps $e : \Lambda_k(m) \hookrightarrow \Lambda_{k+1}(m+1)$ and $h : \Lambda_{k+1}(m+1) \twoheadrightarrow \Sigma^{1,m+1}\Lambda_k(2m+1)$ in the short exact sequence in Proposition 7.2.5. We construct the following two cochain maps

$$f : \Sigma^{0,1}\Lambda_k(\Omega M) \to \Lambda_{k+1}(M)$$
$$\lambda_I \otimes x_m \mapsto e(\lambda_I) \otimes i(x_m)$$

and

$$
\begin{aligned}
g : \Lambda_{k+1}(M) &\to \Sigma^{1,1}\Lambda_k(\Omega_1 M) \\
\lambda_I \otimes x_{m+1} &\mapsto h(\lambda_I) \otimes p(x_{m+1})
\end{aligned}
$$

where the maps i and p are introduced in Remark 7.4.2. See Lemmas 7.4.6 and 7.4.7 for why f and g are cochain maps. Since both e and i are injective, the cochain map f is injective. Since both h and p are surjective, the cochain map g is surjective too. The composition $g \circ f$ is zero because the composition $h \circ e$ is zero. Denote the kernel of g by $F_1(M)$ and the image of f by $F_2(M)$. We get filtered complexes $F_2(M) \subseteq F_1(M) \subseteq \Lambda_{k+1}(M)$. The quotient complex $F_1(M)/F_2(M)$ is acyclic by Lemma 7.4.8. So

$$H^{s,a}F_1(M) \cong H^{s,a}F_2(M) \cong H^{s,a-1}\Lambda_k(\Omega M).$$

Then the short exact sequence $0 \to F_1(M) \to \Lambda_{k+1}(M) \to \Sigma^{1,1}\Lambda_k(\Omega_1 M) \to 0$ leads to the long exact sequence in the statement of the proposition. □

Remark 7.4.4. When $m - 2i \geq 0$, this diagram commutes

$$
\begin{array}{ccc}
\Lambda(m) & \xrightarrow{\quad e \quad} & \Lambda(m+1) \\
\downarrow{\scriptstyle \lambda_{i-1}\cdot(-)} & & \downarrow{\scriptstyle \lambda_{i-1}\cdot(-)} \\
\Lambda(m-i) & \xrightarrow{\quad e \quad} & \Lambda(m+1-i)
\end{array}
$$

Lemma 7.4.5. When $m - 2i \geq 0$, this diagram commutes

$$
\begin{array}{ccc}
\Lambda(m+1) & \xrightarrow{\quad h \quad} & \Lambda(2m+1) \\
\downarrow{\scriptstyle \lambda_{i-1}\cdot(-)} & & \downarrow{\scriptstyle \lambda_{2i-1}\cdot(-)} \\
\Lambda(m-i+1) & \xrightarrow{\quad h \quad} & \Lambda(2m-2i+1)
\end{array}
$$

Proof. Let λ_I be any admissble word in $\Lambda(m+1)$. We know the first index $I(1) \leq m$. When $I(1) < m$, $h(\lambda_I) = 0$ so it suffices to prove $h(\lambda_{i-1}\lambda_I) = 0$. It is true because the conditions $\lambda_I \in \Lambda(m)$ and $\lambda_{i-1} \in \Lambda^i(m-i)$ lead to $\lambda_{i-1}\lambda_I \in \Lambda(m-i)$ by Lemma

7.1.5. When $I(1)=m$, we need to prove $h(\lambda_{i-1}\lambda_m\lambda_{I(2)}\dots\lambda_{I(s)}) = \lambda_{2i-1}\lambda_{I(2)}\dots\lambda_{I(s)}$. Since $m-2i\geq 0$, we apply the Adem relations (7.1) and get

$$\lambda_{i-1}\lambda_m = \sum_t \binom{m-2i-t}{t}\lambda_{m-i-t}\lambda_{2i+t-1}.$$

When $t>0$, $\lambda_{m-i-t}\lambda_{2i+t-1}\in\Lambda^{2,m+i+1}(m-i-t+1)\subseteq\Lambda^{2,m+i+1}(m-i)$. That together with $\lambda_{I(2)}\dots\lambda_{I(s)}\in\Lambda(2m+1)$ leads to $\lambda_{m-i-t}\lambda_{2i+t-1}\lambda_{I(2)}\dots\lambda_{I(s)}\in\Lambda(m-i)$ by Lemma 7.1.5. So the bottom h map sends $\lambda_{m-i-t}\lambda_{2i+t-1}\lambda_{I(2)}\dots\lambda_{I(s)}$ to zero. When $t=0$, we get $\lambda_{m-i}\lambda_{2i-1}$ and the admissible form of $\lambda_{m-i}\lambda_{2i-1}\lambda_{I(2)}\dots\lambda_{I(s)}$ always has the first index equal to $m-i$. So the bottom h map sends $\lambda_{m-i}\lambda_{2i-1}\lambda_{I(2)}\dots\lambda_{I(s)}$ to $\lambda_{2i-1}\lambda_{I(2)}\dots\lambda_{I(s)}$. □

Lemma 7.4.6. $f:\Sigma^{0,1}\Lambda_k(\Omega M)\to\Lambda_{k+1}(M)$ is a cochain map.

Proof. We need to verify $fd(\lambda_I\otimes x_m) = df(\lambda_I\otimes x_m)$. The former is equal to

$$\begin{aligned}
&ed(\lambda_I)\otimes i(x_m) + \sum_{0\leq m-2i<k} e(\lambda_{i-1}\lambda_I)\otimes i(x_m\mathrm{Sq}^i)\\
=&de(\lambda_I)\otimes i(x_m) + \sum_{0\leq m-2i<k} e(\lambda_{i-1}\lambda_I)\otimes i(x_m)\mathrm{Sq}^i
\end{aligned}$$

because $ed=de$ and $i(x_m\mathrm{Sq}^i) = i(x_m)\mathrm{Sq}^i$. The latter is equal to

$$\begin{aligned}
&de(\lambda_I)\otimes i(x_m) + \sum_{0\leq m+1-2i<k+1} \lambda_{i-1}e(\lambda_I)\otimes i(x_m)\mathrm{Sq}^i\\
=&de(\lambda_I)\otimes i(x_m) + \sum_{0\leq m-2i<k} e(\lambda_{i-1}\lambda_I)\otimes i(x_m)\mathrm{Sq}^i
\end{aligned}$$

because $i(x_m)\mathrm{Sq}_0=0$ and Remark 7.4.4. □

Lemma 7.4.7. $g:\Lambda_{k+1}(M)\to\Sigma^{1,1}\Lambda_k(\Omega_1 M)$ is a cochain map.

Proof. We need to verify $gd(\lambda_I\otimes x_{m+1}) = dg(\lambda_I\otimes x_{m+1})$. The former is equal to

$$\begin{aligned}
&hd(\lambda_I)\otimes p(x) + \sum_{0\leq m+1-2i<k+1} h(\lambda_{i-1}\lambda_I)\otimes p(x\mathrm{Sq}^i)\\
=&hd(\lambda_I)\otimes p(x) + \sum_{0\leq m-2i<k} \lambda_{2i-1}h(\lambda_I)\otimes p(x\mathrm{Sq}^i)
\end{aligned}$$

because $p(x\mathrm{Sq}_0) = 0$ and Lemma 7.4.5. The latter is equal to

$$
\begin{aligned}
& dh(\lambda_I) \otimes p(x) + \sum_{0 \leq 2m+1-2i < k} \lambda_{i-1} h(\lambda_I) \otimes p(x)\mathrm{Sq}^i \\
=& hd(\lambda_I) \otimes p(x) + \sum_{0 \leq m-2i < (k-1)/2} \lambda_{2i-1} h(\lambda_I) \otimes p(x\mathrm{Sq}^i)
\end{aligned}
$$

because $dh = hd$, $p(x\mathrm{Sq}^i) = p(x)\mathrm{Sq}^{2i}$ and $p(x)\mathrm{Sq}^{\mathrm{odd}} = 0$. So it suffices to prove

$$
\lambda_{2i-1} h(\lambda_I) = 0 \in \Lambda_k(2m - 2i + 1) \text{ when } m - 2i \geq \frac{k-1}{2},
$$

which follows directly from Lemma 7.3.5. □

Lemma 7.4.8. The quotient complex $F_1(M)/F_2(M)$ is acyclic.

Proof. We have $F_1(M)/F_2(M) = A \oplus B$ where

$$
A = \bigoplus_m \frac{\Lambda_{k+1}(m+1)}{\ker(h)} \otimes \ker(p), \quad B = \bigoplus_m \ker(h) \otimes \frac{M_{m+1}}{\mathrm{image}(i)}.
$$

Note that here the maps h, p, i are implicitly indexed by m:

$$
\begin{aligned}
& h : \Lambda_{k+1}(m+1) \to \Sigma^{1,m+1}\Lambda_k(2m+1) \\
& p : M_{m+1} \to (\Omega_1 M)_{2m+1} \\
& i : (\Omega M)_m \to M_{m+1}
\end{aligned}
$$

Construct an additive map $r : A \to B$ by sending $\lambda_I \otimes x\mathrm{Sq}_0$ to $eh(\lambda_I) \otimes x$. It is easy to verify that this map is well-defined. Extend r to be zero on B, then we get a map r from $A \oplus B$ to itself. Let's verify $rd + dr = 1$ on $A \oplus B$.

First let's prove $(rd + dr)(a) = a$ for any $a \in A$. Take any admissible word $\lambda_I \in \Lambda(m+1)$ with $I(1) = m$ and $x \in M_{2m+2}$. We need to prove $(rd+dr)(\lambda_I \otimes x\mathrm{Sq}_0) =$

$\lambda_I \otimes x\mathrm{Sq}_0$. The $dr(\lambda_I \otimes x\mathrm{Sq}_0)$ is equal to

$$\begin{aligned} & deh(\lambda_I) \otimes x + \sum_{0 \leq 2m+2-2i < k+1} \lambda_{i-1} eh(\lambda_I) \otimes x\mathrm{Sq}^i \\ =& ehd(\lambda_I) \otimes x + \sum_{1 \leq 2m+2-2i < k+1} e(\lambda_{i-1} h(\lambda_I)) \otimes x\mathrm{Sq}^i + \lambda_m eh(\lambda_I) \otimes x\mathrm{Sq}_0 \\ =& ehd(\lambda_I) \otimes x + \sum_{0 \leq m-2i < (k-1)/2} e(\lambda_{2i-1} h(\lambda_I)) \otimes x\mathrm{Sq}^{2i} + \lambda_I \otimes x\mathrm{Sq}_0 \end{aligned}$$

because $deh = ehd$ and Remark 7.4.4. When i is odd, the term $e(\lambda_{i-1}h(\lambda_I)) \otimes x\mathrm{Sq}^i$ is equal to zero in B because $x\mathrm{Sq}^i$ lives in the image of $i : (\Omega M)_{2m+1-i} \to M_{2m+2-i}$. The $rd(\lambda_I \otimes x\mathrm{Sq}_0)$ is equal to

$$\begin{aligned} & ehd(\lambda_I) \otimes x + \sum_{-1 \leq m-2i < k} eh(\lambda_{i-1}\lambda_I) \otimes x\mathrm{Sq}^{2i} \\ =& ehd(\lambda_I) \otimes x + \sum_{0 \leq m-2i < (k-1)/2} e(\lambda_{2i-1} h(\lambda_I)) \otimes x\mathrm{Sq}^{2i} \end{aligned}$$

because $\mathrm{Sq}_0\mathrm{Sq}^i = \mathrm{Sq}^{2i}\mathrm{Sq}_0$ and Lemma 7.4.5. We also used those two facts: when $m + 1 = 2i$, $eh(\lambda_{i-1}\lambda_I) \otimes x\mathrm{Sq}_0 = 0 \in A$; $\lambda_{2i-1}h(\lambda_I) = 0 \in \Lambda_k(2m - 2i + 1)$ when $m - 2i \geq (k - 1)/2$. Then everything cancels out except for $\lambda_I \otimes x\mathrm{Sq}_0$.

Now let's prove $(rd + dr)(b) = b$ for any $b \in B$. Take any admissible word $\lambda_I \in \Lambda(m)$ and $x \in M_{m+1}$. We need prove $rd(\lambda_I \otimes x) = \lambda_I \otimes x$. The $rd(\lambda_I \otimes x)$ is equal to

$$r(d(\lambda_I) \otimes x) + \sum_{0 \leq m+1-2i < k+1} r(\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^i)$$

Since $d(\lambda_I) \in \Lambda(m)$, $d(\lambda_I) \otimes x$ lives in B and r sends it to zero. When $m - 2i \geq 0$, $\lambda_{i-1}\lambda_I$ lives in $\Lambda(m - i)$, so $\lambda_{i-1}\lambda_I \otimes x\mathrm{Sq}^i$ lives in B and r sends it to zero. When m is odd and $m + 1 = 2i$, $r(\lambda_{(m-1)/2}\lambda_I \otimes x\mathrm{Sq}_0) = eh(\lambda_{(m-1)/2}\lambda_I) \otimes x = \lambda_I \otimes x$. When m is even, $x = 0$ in $M_{m+1}/\mathrm{image}(i)$. □

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