Weakly Enriched Higher Categories

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2013

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Abstract

The goal of this thesis is to begin to lay the foundations for a theory of enriched ∞-categories. We introduce a definition of such objects, based on a non-symmetric version of Lurie's theory of ∞-operads. Our first main result is a construction of the correct homotopy theory of enriched ∞-categories as a localization of an "algebraic" homotopy theory defined using ∞-operads; this is joint work with David Gepner.

We then prove some comparison results: When a monoidal ∞-category arises from a nice monoidal model category we show that the associated homotopy theory of enriched ∞-categories is equivalent to the homotopy theory induced by the model category of enriched categories; when the monoidal structure is the Cartesian product we also show that this is equivalent to the homotopy theory of enriched Segal categories. Moreover, we prove that the homotopy theory of (∞, n)-categories enriched in spaces, obtained by iterating our enrichment procedure, is equivalent to that of n-fold complete Segal spaces.

We also introduce notions of natural transformations and correspondences in the setting of enriched ∞-categories, and use these to construct $(\infty, 2)$-categories of enriched ∞-categories, functors, and natural transformations, and double ∞-categories of enriched ∞-categories, functors, and correspondences.

Finally, we briefly discuss a non-iterative definition of enriched (∞, n)-categories, based on a version of ∞-operads over Joyal's categories Θ_n, and define what should be the correct ∞-category of these.

Thesis Supervisor: Haynes Miller
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Acknowledgments

First of all I thank Haynes Miller, for being a great advisor (even when his students display an unhealthy interest in higher category theory), and for teaching me a small part of what he knows about algebraic topology. I am indebted to Clark Barwick for inspiring this project and for teaching me an enormous amount about higher categories, and most importantly for always having a useful suggestion on the far too many occasions when I felt completely stuck; I am sure I would have made far more progress if I had listened to him more. Collaborating with David Gepner not only led to one of the main results of this thesis, but also deepened my understanding of ∞-operads and ∞-categories; I especially thank him for steering me away from a truly horrendous approach to functor categories. This thesis builds on the vast body of work produced by Jacob Lurie, and I am grateful to him for patiently answering several technical questions about his work. For the past three years I have been partially funded by the American-Scandinavian Foundation and the Norway-America Association, and I thank them for their support. I also thank everyone who has taken an interest in this project and with whom I have had interesting discussions about it, and everyone in the topology group at MIT.

On a personal note, I am grateful to my flatmates (especially Jethro and Gun) and office mates (particularly Olga, Høskuldur, Niels, and Ailsa, and our honorary office mate Eric) for putting up with me when math was not going well, and to all my friends in Boston for all the good times over the past five years. Of course I also thank my family and, last but not least, I thank Christy for making it so much harder to leave Boston.

Contents

Chapter 1

Introduction

The language of category theory has played an important role in many areas of mathematics for the past half-century. In recent years, however, taking seriously the higher-categorical nature of many structures has turned out to be a very fruitful idea. In particular, the theory of *∞-categories* has had many applications in algebraic topology and other areas of mathematics. Roughly speaking, the notion of *∞-category* (or *$(\infty,1)$-category*) is a generalization of the notion of category where in addition to objects and morphisms we also have homotopies between morphisms, homotopies between homotopies, and so on. One way to think of an ∞-category is as a category where the morphisms between two objects form a space rather than just a set — such *topological categories*, or equivalently *simplicial* categories (where the morphisms form a simplicial set), give the simplest model of ∞-categories. However, topological and simplicial categories are very rigid, which makes it hard to understand the homotopically correct functors between them, and in general make homotopy-invariant constructions (such as homotopy limits and colimits); moreover, many naturally occurring composition laws are not strictly associative, but only associative up to coherent homotopy. It is therefore usually more convenient to work with a notion of ∞-category where composition of morphisms is associative up to coherent homotopy. There are several ways to make this idea precise, including *Segal categories*, *complete Segal spaces*, and *quasicategories*.

In some cases, the morphisms between objects in an ∞-category have more structure than just forming a space; in algebraic topology, for example, we often come across ∞-categories where the morphisms naturally form a spectrum. It is possible to think of these objects as spectral categories, i.e. categories enriched in a model category of spectra (such as symmetric spectra), and more generally we can consider categories enriched in nice monoidal model categories. However, these suffer from the same problems as simplicial categories do when considered as a model for ∞-categories. This suggests that a weaker notion of enrichment, where composition is only associative up to coherent homotopy, should be useful. The goal of this thesis is to begin to lay the foundations for a theory of such *enriched ∞-categories*; specifically, we will define and study ∞-categories enriched in *monoidal ∞-categories*, which are ∞-categories equipped with a tensor product that is associative and unital up to coherent homotopy.

From an algebro-topological perspective, the most interesting monoidal ∞-category[1] is the ∞-category of spectra equipped with the smash product, and I expect that the the-

[1] Apart from the ∞-category of spaces, with the Cartesian product, but enriching in this just gives ordinary ∞-categories.

ory developed in this thesis will have interesting applications in the context of *spectral ∞-categories*, i.e. ∞-categories enriched in spectra. For instance, many naturally occurring structures that "ought to be" spectral categories can be very difficult to define, because the natural composition maps are only associative up to homotopy; I hope that in many cases these structures can be more easily described as spectral ∞-categories.

As a specific example, it has long been expected that the spectral category of *genuine G-spectra* for a finite group G "ought to be" the spectral category of spectral presheaves on a small spectral category $\mathcal{B}^G_+$; if $\mathcal{B}^G$ is the 2-category of finite G-sets, spans of finite G-sets, and isomorphisms of spans, then $\mathcal{B}^G_+$ ought to be constructed by applying group completion to the mapping groupoids of $\mathcal{B}^G$. However, in the setting of ordinary categories group completion is only a multiplicative functor when restricted to *permutative* categories (i.e monoidal categories where the tensor product is strictly associative); Guillou and May have recently constructed a version of $\mathcal{B}^G_+$ by replacing $\mathcal{B}^G$ by a category enriched in permutative categories, and using this they can show that spectral presheaves on this spectral category does indeed give genuine G-spectra [GM11b, GM11a, GM12]. However, their construction is quite complicated — by contrast, in the setting of ∞-categories it is straightforward to see that group completion is a lax monoidal functor, and since our theory of enriched ∞-categories is functorial with respect to lax monoidal functors it is trivial to construct $\mathcal{B}^G_+$ as a spectral ∞-category. Moreover, it is equally easy to construct spectral ∞-categories by applying other lax monoidal functors, such as topological Hochschild homology or topological cyclic homology; for example, this gives a rigorous construction of Morava's category of TC-*motives* [Mor11].

We will set up our theory of enriched ∞-categories entirely within the context of ∞-categories (rather than working with model categories, say); apart from greater generality, working in this setting has several advantages:

- Weak or homotopy-coherent enrichment is the only natural notion of enrichment, which allows us to define our enriched ∞-categories as certain "algebraic" objects in the ∞-categorical sense.

- It is easy to consider enriched categories with *spaces* of objects rather than just sets, which turns out to make the resulting homotopy theory nicer and easier to set up, analogously to the way complete Segal spaces are better-behaved than Segal categories or simplicial categories.

- We automatically get naturality properties that would be difficult even to define in a model-categorical framework — for example, our ∞-categories are natural in functors between monoidal ∞-categories that are lax monoidal in the appropriate ∞-categorical sense.

- Beyond just constructing a homotopy theory, our theory gives a good setting to develop ∞-categorical analogues of many concepts from enriched category theory. In this thesis we will discuss analogues of natural transformations and correspondences, and we hope to study analogues of other concepts, such as weighted colimits, in future work.

Part of this thesis is joint work with David Gepner — specifically, the results of §3.3 and §4.1–4.3 are taken from our article [GH], as are many of the results scattered in §2.1.

1.1 From Enriched Categories to Enriched ∞-Categories

To orient the reader, we now attempt to motivate our approach to enriched ∞-categories by describing how it relates to ordinary enriched categories.

1.1.1 Multicategories and Enrichment

Let's begin with the usual definition of an enriched category: if $\mathbf{V}$ is a monoidal category, a $\mathbf{V}$-*enriched category* (or $\mathbf{V}$-*category*) $\mathbf{C}$ consists of:

- a set $\operatorname{ob}\mathbf{C}$ of objects,
- for all pairs $x, y \in \operatorname{ob}\mathbf{C}$ an object $\mathbf{C}(x,y)$ in $\mathbf{V}$,
- composition maps $\mathbf{C}(x,y) \otimes \mathbf{C}(y,z) \to \mathbf{C}(x,z)$,
- units $\mathrm{id}_x \colon I \to \mathbf{C}(x,x)$.

The composition must be associative (this involves the associator isomorphism for $\mathbf{V}$) and unital. When formulated in this way, it is very hard to see how this notion ought to be generalized in the setting of ∞-categories. We should therefore look for alternative ways of defining enriched categories that have more obvious generalization; we first consider a definition in terms of multicategories.

A *multicategory* (or *non-symmetric coloured operad*) is roughly speaking a category where a morphism has a *list* of objects as its source. More precisely, a *multicategory* $\mathbf{M}$ consists of a set of objects and for objects $x_1, \ldots, x_n, y$ a set $\mathbf{M}((x_1, \ldots, x_n), y)$ of "multimorphisms" from $(x_1, \ldots, x_n)$ to y; these have an associative composition, in the sense that we can compose multimorphisms

$$(z_1, \ldots, z_{i_1}) \to y_1, \quad (z_{i_1+1}, \ldots, z_{i_2}) \to y_2, \quad \ldots, \quad (z_{i_{n-1}+1}, \ldots, z_{i_n}) \to y_n$$

with a multimorphism $(y_1, \ldots, y_n) \to x$ to get a multimorphism $(z_1, \ldots, z_{i_n}) \to x$. A multicategory with a single object is precisely a non-symmetric operad.[2]

If $\mathbf{V}$ is a monoidal category, we can view it as a multicategory by defining

$$\mathbf{V}((x_1, \ldots, x_n), y) := \mathbf{V}(x_1 \otimes \cdots \otimes x_n, y).$$

An *algebra* for a multicategory $\mathbf{M}$ in a monoidal category $\mathbf{V}$ is then just a functor of multicategories from $\mathbf{M}$ to $\mathbf{V}$ viewed as a multicategory.

Given a set X, there is a simple multicategory $\mathbf{O}_X$ such that $\mathbf{O}_X$-algebras in a monoidal category $\mathbf{V}$ are precisely $\mathbf{V}$-categories: the objects of $\mathbf{O}_X$ are $X \times X$, and the multimorphism sets are defined by

$$\mathbf{O}_X(((x_0, y_1), (x_1, y_2), \ldots, (x_{n-1}, y_n)), (y_0, x_n)) := \begin{cases} *, & \text{if } y_i = x_i,\ i = 0, \ldots, n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus an $\mathbf{O}_X$-algebra $\mathbf{C}$ in $\mathbf{V}$ assigns an object $\mathbf{C}(x,y)$ to each pair (x,y) of elements of X, with a unit $I \to \mathbf{C}(x,x)$ from the unique map $() \to ((x,x))$, and a composition map

[2] Below, we will refer to (non-symmetric) coloured operads as just (non-symmetric) operads, for consistency with the terminology used by Lurie [Lur11] and Barwick [Bar13]; here we stick to the more common terminology.

$\mathbf{C}(x,y) \otimes \mathbf{C}(y,z) \to \mathbf{C}(x,z)$ from the unique multimorphism $((x,y),(y,z)) \to (x,z)$. Looking at triples of pairs we see that this composition is associative, and it is also clearly unital, so $\mathbf{C}$ is precisely a $\mathbf{V}$-category.

If we had an ∞-categorical generalization of the theory of multicategories (which included a theory of *monoidal* ∞-categories as a special case), it would therefore make sense to define an ∞-category enriched in a monoidal ∞-category $\mathcal{V}$ with set of objects X to be an $\mathbf{O}_X$-algebra in $\mathcal{V}$. To generalize multicategories to the ∞-categorical setting we could use *simplicial multicategories*, i.e. multicategories enriched in simplicial sets. However, these suffer from the same technical problems as simplicial categories considered as a model for ∞-categories. Just as for ∞-categories, there are several better-behaved models for ∞-categorical (symmetric) multicategories, namely the *dendroidal sets* (and related constructions such as *Segal operads* and *dendroidal Segal spaces*) studied by Moerdijk together with Berger, Cisinski, and Weiss, and the ∞-*operads* of Lurie. In this thesis we will primarily use a non-symmetric variant of Lurie's theory.

We can regard the multicategories $\mathbf{O}_X$ as non-symmetric ∞-operads, and considering algebras for these in a monoidal ∞-category $\mathcal{V}$ does indeed give the right objects — for example, if X is a one-element set then $\mathbf{O}_X$-algebras are precisely A_∞-algebras in $\mathcal{V}$, which is what we expect. Moreover, the machinery of ∞-operads gives ∞-categories $\mathrm{Alg}_{\mathbf{O}_X}(\mathcal{V})$ of $\mathbf{O}_X$-algebras in a fixed monoidal ∞-category $\mathcal{V}$, and we can combine these to form an ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, which has objects $\mathcal{V}$-∞-categories and 1-morphisms $\mathcal{V}$-functors in the appropriate sense. However, a morphism $f\colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is an equivalence if and only if it is *fully faithful*, i.e. $\mathcal{C}(x,y) \to \mathcal{D}(fx,fy)$ is an equivalence for all objects x,y of $\mathcal{C}$, and a *bijection* on objects. These are clearly not the correct equivalences of $\mathcal{V}$-∞-categories — these ought to be the fully faithful and *essentially surjective* functors. To get the right ∞-category of $\mathcal{V}$-∞-categories we must therefore localize $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ to invert these.

For the localized ∞-category to be well-behaved we need this to be an *accessible* localization (the ∞-categorical analogue of left Bousfield localization of model categories) — this means that the localized ∞-category is a full subcategory of the original ∞-category consisting of *local* objects. However, this is not possible for $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ as we've defined it here: For example, if $\mathcal{V}$ is the category of sets, then $\mathrm{Alg}_{\mathrm{cat}}(\mathrm{Set})$ is just the ordinary category of categories and functors; the correct localization, on the other hand, is the $(2,1)$-category of categories, functors, and natural isomorphisms, which clearly can't be a full subcategory of the 1-category $\mathrm{Alg}_{\mathrm{cat}}(\mathrm{Set})$.

It turns out that we can avoid this problem if we allow $\mathcal{V}$-∞-categories to have *spaces* of objects, rather than just sets, which is also very natural from the ∞-categorical point of view. We would thus like to define non-symmetric ∞-operads analogous to $\mathbf{O}_X$ with X a space; one way to do this is to define *simplicial* multicategories, taking as input simplicial categories whose nerves are Kan complexes[3], but this generalization is much easier and more natural if we start from a slightly different approach to enriched categories.

1.1.2 Virtual Double Categories and Enrichment

Virtual double categories[4] are a common generalization of double categories and multicategories. Roughly speaking, a virtual double category has objects and vertical and horizontal

[3]We will in fact define and make brief but crucial use of these in §4.1.2 below.

[4]Also known as **fc**-multicategories; we will later call them *generalized* (*non-symmetric*) *operads*, for consistency with Lurie's terminology.

morphisms between them, but in addition to a collection of "squares" there are cells with a list of vertical arrows as source; we refer the reader to [CS10] or [Lei04] for more details.

We will instead consider virtual double categories from another perspective, by generalizing the *category of operators* of a multicategory: if $\mathbf{M}$ is a multicategory, its category of operators $\mathbf{M}^{\otimes}$ is a category with objects lists $(x_0, \ldots, x_n)$ of objects $x_i \in \mathbf{M}$, and a morphism $(x_0, \ldots, x_n) \to (y_0, \ldots, y_m)$ given by a morphism $\phi\colon [m] \to [n]$ in Δ and, for each $i = 0, \ldots, m$, a multimorphism $(x_{\phi(i)}, x_{\phi(i)+1}, \ldots, x_{\phi(i+1)-1}) \to y_i$ in $\mathbf{M}$. We can characterize the categories $\mathbf{E}$ over Δ^{op} that are categories of operators for multicategories — in particular, $\mathbf{E}_{[n]}$ must be equivalent to $\mathbf{E}_{[1]}^{\times n}$ via the maps $\{i, i+1\} \hookrightarrow [n]$. If we relax this to a more general "Segal condition", $\mathbf{E}_{[n]} \simeq \mathbf{E}_{[1]} \times_{\mathbf{E}_{[0]}} \cdots \times_{\mathbf{E}_{[0]}} \mathbf{E}_{[1]}$, we obtain precisely the analogous "categories of operators" for virtual double categories.

Given a set X, we can define a virtual double category $\mathbf{D}_X$ with objects X where the vertical morphisms are trivial, and there is a unique horizontal morphism between any two elements of X. Then a functor of virtual double categories from $\mathbf{D}_X$ to a monoidal category $\mathbf{V}$ is precisely a $\mathbf{V}$-category with objects X. In terms of categories of operators, this virtual double category corresponds to the category Δ_X^{op} whose objects are non-empty sequences $(x_0, \ldots, x_n)$ of elements $x_i \in X$, and a unique morphism $(x_0, \ldots, x_n) \to (x_{\phi(0)}, \ldots, x_{\phi(m)})$ for each $\phi\colon [m] \to [n]$ in Δ. If $\mathbf{V}$ is a monoidal category, and $\mathbf{V}^{\otimes}$ is its category of operators, a functor $\mathbf{D}_X \to \mathbf{V}$ corresponds to a functor $\mathbf{C}\colon \Delta_X^{\mathrm{op}} \to \mathbf{V}^{\otimes}$ over Δ^{op} such that $\mathbf{C}(x_0, \ldots, x_n) = (\mathbf{C}(x_0, x_1), \ldots, \mathbf{C}(x_{n-1}, x_n))$; it is easy to see that this is precisely a $\mathbf{V}$-category.

An ∞-categorical version of the theory of virtual double categories is provided by Lurie's *generalized ∞-operads*. This is the setting in which we will mainly develop our theory of enriched ∞-categories; the advantage of working with these rather than only with ∞-operads is that there is an easy and natural ∞-categorical definition of ∞-categories Δ_X^{op} where X is a *space*. If $\mathcal{V}$ is a monoidal ∞-category we will define an ∞-category enriched in $\mathcal{V}$ with space of objects X to be a map of generalized ∞-operads from Δ_X^{op} to $\mathcal{V}$.

1.1.3 Lax Functors and Enrichment

A third approach to enriched categories is to consider them as certain *lax functors*. Recall that if $\mathbf{C}$ and $\mathbf{D}$ are 2-categories, a lax functor F from $\mathbf{C}$ to $\mathbf{D}$ assigns

- to each object $X \in \mathbf{C}$ an object $F(X)$ in $\mathbf{D}$,
- to each 1-morphism $f\colon X \to Y$ in $\mathbf{C}$ a 1-morphism $F(f)\colon F(X) \to F(Y)$ in $\mathbf{D}$,
- to each 2-morphism $\alpha\colon f \to g$ in $\mathbf{C}$ a 2-morphism $F(\alpha)\colon F(f) \to F(g)$ in $\mathbf{D}$,
- to each composable pair of 1-morphisms $f\colon X \to Y$, $g\colon Y \to Z$, a 2-morphism $F(g) \circ F(f) \to F(g \circ f)$, satisfying associativity in the obvious sense for sequences of 3 composable 1-morphisms,
- to each object $X \in \mathbf{C}$, a 2-morphism $\mathrm{id}_{F(X)} \to F(\mathrm{id}_X)$, which must be compatible with the 2-morphisms for composable pairs of 1-morphisms.

A monoidal category $\mathbf{V}$ corresponds to a 2-category $\Sigma\mathbf{V}$ with one object, and if $\mathbf{V}$ and $\mathbf{W}$ are monoidal categories, a lax functor $\Sigma\mathbf{V} \to \Sigma\mathbf{W}$ is precisely a lax monoidal functor $\mathbf{V} \to \mathbf{W}$.

If X is a set, let $\mathbf{E}X$ denote the "codiscrete" category with objects X, and a unique morphism between any two objects. Then a $\mathbf{V}$-category with objects X, for some monoidal category $\mathbf{V}$, is the same thing as a lax functor $\mathbf{E}X \to \Sigma\mathbf{V}$.

This definition is related to the definition using virtual double categories as follows: we can regard 2-categories as double categories with no non-trivial vertical morphisms, and thus as a special kind of virtual double categories. Under this identification, a lax functor between 2-categories is precisely a morphism of virtual double categories. Moreover, the virtual double category associated to $\Sigma\mathbf{V}$ is precisely the one we obtain by regarding $\mathbf{V}$ as a multicategory. Similarly, the virtual double category associated to $\mathbf{E}X$ is the one we denote $\mathbf{D}_X$ above — thus a lax functor $\mathbf{E}X \to \Sigma\mathbf{V}$ corresponds to a morphism of virtual double categories from $\mathbf{D}_X$ to $\mathbf{V}$, which was the definition of enriched category we considered above.

In the ∞-categorical context, we can similarly regard generalized (non-symmetric) ∞-operads as a natural setting for studying lax functors between $(\infty, 2)$-categories.

1.2 Overview

The next two chapters mainly comprise background material. In Chapter 2 we review some basic definitions and results on ∞-categories and other higher-categorical structures, and also prove some technical results we will need later on. Chapter 3 reviews Barwick's theory of *operator categories* and describes how to generalize Lurie's ∞-operads to this setting; we also prove some results about ∞-categories of algebras over operads.

Chapter 4 is the heart of this thesis — here we introduce and study our theory of enriched ∞-categories. In §4.1 we use the machinery of ∞-operads from Chapter 3 to set up an "algebraic" ∞-category of enriched ∞-category, then in §4.2 we construct the correct ∞-category of enriched ∞-categories by localizing this at the fully faithful and essentially surjective functors — the key result is that this localization is given by restricting to "complete" enriched ∞-categories, which is proved analogously to the main theorem of [Rez01]. After briefly describing some simple applications of this construction in §4.3, we compare our homotopy theory to homotopy theories of categories enriched in model categories, enriched Segal categories, and iterated Segal spaces in §4.4. Then we discuss natural transformations and construct the $(\infty, 2)$-category of enriched ∞-categories, functors, and natural transformations in §4.5. We extend this to a double ∞-category of enriched ∞-categories, functors, and *correspondences* (or profunctors) in §4.6.

Finally, in Chapter 5 we begin to study the generalization of our construction to a (non-iterative) theory of *enriched (∞, n)-categories*. Unfortunately we are not able to accomplish very much in this setting, primarily because we have not yet been able to prove some key results about the appropriate theory of ∞-operads. We do, however, set up the correct ∞-category of (∞, n)-categories enriched in a given $\mathbb{E}_n$-monoidal ∞-category.

1.3 Notation and Terminology

We generally recycle the notation and terminology used by Lurie in [Lur09a, Lur11]. Here are some exceptions and reminders:

- Generic categories are generally denoted by single capital bold-face letters (e.g. $\mathbf{V}$) and generic ∞-categories by single caligraphic letters (e.g. $\mathcal{V}$). Specific categories and ∞-categories both get names in the normal text font: thus the category of small $\mathbf{V}$-categories is denoted $\mathrm{Cat}^{\mathbf{V}}$ and the ∞-category of small $\mathcal{V}$-∞-categories is denoted $\mathrm{Cat}^{\mathcal{V}}_{\infty}$.

- We make use of the elegant theory of *Grothendieck universes* to avoid set-theoretical problems; specifically, we fix three nested universes, and refer to sets contained in them as *small*, *large* and *very large*. When $\mathcal{C}$ is an ∞-category of small objects of a certain type, we generally refer to the corresponding ∞-category of large objects as $\widehat{\mathcal{C}}$. For example, Cat_∞ is the (large) ∞-category of small ∞-categories, and $\widehat{\mathrm{Cat}}_\infty$ is the (very large) ∞-category of large ∞-categories.

- As far as possible we argue using the "high-level" language of ∞-categories, without referring to their specific implementation as quasicategories. Following this philosophy we have generally not distinguished notationally between categories and their nerves, since categories are a special kind of ∞-category. However, we do indicate the nerve (using N) when we think of the nerve of a category as being a specific simplicial set; by the same principle we always indicate the nerves of simplicial categories. This should hopefully not cause any confusion.

- We will refer to the notion dual to that of Grothendieck fibration as *coGrothendieck fibration*, by analogy with the terminology of Cartesian and coCartesian fibrations in the ∞-categorical case.

Chapter 2

Background on Higher Categories

This chapter contains some background material for the main part of this thesis: in §2.1 we briefly review ∞-categories and prove some technical results, and in §2.2 we review some other higher-categorical structures we will encounter.

2.1 Preliminaries on ∞-Categories

In this thesis we will work throughout in the setting of ∞-categories. Specifically, we will make use of the theory of quasicategories, as due to the work of Joyal and Lurie it is currently by far the best-developed theory of ∞-categories. In this section we briefly review some of the main definitions and results from [Lur09a, Lur11] that we will make use of. Along the way, we also prove a number of fairly technical results that we will need later on.

2.1.1 Quasicategories

Quasicategories are a class of simplicial sets. Roughly speaking, the idea is that just as a category has a nerve in simplicial sets, an ∞-category, however we define these, should also have a nerve. The definition of quasicategory then characterizes those simplicial sets that "ought to be" nerves of ∞-categories.

Definition 2.1.1.1. Let $\mathbb{\Delta}$ denote the *simplicial indexing category*, i.e. the category whose objects are the ordered sets $[n] := \{0, \ldots, n\}$ for $n = 0, 1, \ldots$, and whose morphisms are the order-preserving maps between these. Equivalently, we may also regard $\mathbb{\Delta}$ as the category of non-empty finite ordered sets. A *simplicial set* is a presheaf of sets on $\mathbb{\Delta}$, i.e. a functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. We write Set_Δ for the category $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets.

Definition 2.1.1.2. The *n-simplex* Δ^n is the simplicial set corepresented by the object $[n] \in \mathbb{\Delta}$. The ith *horn* Λ^n_i of Δ^n is the simplicial subset obtained by removing the face opposite the ith vertex from Δ^n. The horn Λ^n_i is *inner* if $0 < i < n$.

If $\mathbf{C}$ is a category, its nerve is the simplicial set $\mathrm{N}\mathbf{C}$ with $\mathrm{N}\mathbf{C}_k := \mathrm{Hom}([k], \mathbf{C})$ where $[k]$ is the category associated to the ordered set $\{0, 1, \ldots, n\}$. We can characterize those simplicial sets that are isomorphic to nerves of categories in terms of certain *horn-filling conditions*: a simplicial set X is the nerve of a category precisely when every map from an inner horn $\Lambda^n_k \to X$ extends to a *unique* n-simplex $\Delta^n \to X$. For example, in the smallest case of a map $\Lambda^2_1 \to X$ this says that any pair of composable morphisms has a *unique* composite.

For an ∞-category we do not want such composites to be unique. Instead, a 2-simplex should describe the data of two composable morphisms and a *homotopy* from their composite to a third morphism; alternatively, since there is no preferred choice of composite, we can say that a 2-simplex exhibits this third morphism as *a* composite. Generalizing this idea to higher dimensions, we get the definition of a quasicategory:

Definition 2.1.1.3. A *quasicategory* is a simplicial set that satisfies the right lifting property with respect to the inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$. In other words, a simplicial set X is a quasicategory if and only if every inner horn $\Lambda^n_i \to X$, $0 < i < n$, can be extended to an n-simplex, but the extension need not be unique.

Following Lurie, we will generally refer to quasicategories as ∞*-categories*. If X is an ∞-category, we will often refer to its vertices as *objects* and its edges as *morphisms*.

Definition 2.1.1.4. An *inner fibration* is a morphism of simplicial sets that has the right lifting property with respect to the inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, $0 < i < n$.

Definition 2.1.1.5. If X is an ∞-category, the *interior* or *underlying space* ιX of X is the largest subspace of X that is a Kan complex. A morphism of X is an *equivalence* if it is contained in ιX.

There is a left proper combinatorial model structure on Set_Δ, originally constructed by Joyal, whose cofibrations are monomorphisms and whose fibrant objects are ∞-categories (cf. [Lur09a, Theorem 2.2.5.1]). We refer to the weak equivalences in this model structure as *categorical equivalences*.

The Joyal model structure is Cartesian closed. If $\mathcal{C}$ is an ∞-category an K is any simplicial set then we will denote the usual internal hom of simplicial sets by $\mathrm{Fun}(K, \mathcal{C})$; this is an ∞-category (cf. [Lur09a, Proposition 1.2.7.3]).

There is also a related model structure on *marked* simplicial sets:

Definition 2.1.1.6. A *marked simplicial set* (X, S) consists of a simplicial set X together with a set $S \subseteq X_1$ of edges of X that includes all the degenerate edges. We write Set^+_Δ for the category of marked simplicial sets. If X is a simplicial set, we write $X^\flat$ for X equipped with the minimal marking (X, s_0X_0) and $X^\sharp$ for X equipped with the maximal marking (X, X_1). If X is an ∞-category we write $X^\natural$ for X marked by the set of equivalences.

There is a model structure on Set^+_Δ whose cofibrations are the monomorphisms and whose fibrant objects are of the form $X^\natural$ where X is an ∞-category (cf. [Lur09a, Proposition 3.1.3.7]). The forgetful functor $\mathrm{Set}^+_\Delta \to \mathrm{Set}_\Delta$ is a right Quillen equivalence (cf. [Lur09a, Theorem 3.1.5.1]).

2.1.2 Simplicial Categories and Simplicial Groupoids

A *simplicial category* is a category enriched in simplicial sets. We write Cat_Δ for the category of simplicial categories.

Definition 2.1.2.1. A functor of simplicial categories $F\colon \mathbf{C} \to \mathbf{D}$ is *weakly fully faithful* if for all $x, y \in \mathbf{C}$ the map $\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is a weak equivalence of simplicial sets.

Definition 2.1.2.2. The functor $\pi_0\colon \mathrm{Set}_\Delta \to \mathrm{Set}$ is strong monoidal, and so induces a functor $\pi_0\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}$. We say a functor $F\colon \mathbf{C} \to \mathbf{D}$ of simplicial categories is *essentially surjective up to homotopy* if the functor $\pi_0 F$ of ordinary categories is essentially surjective.

Definition 2.1.2.3. A functor of simplicial categories $F\colon \mathbf{C} \to \mathbf{D}$ is a *local fibration* if for all $x, y \in \mathbf{C}$ the map $\mathbf{C}(x,y) \to \mathbf{D}(Fx, Fy)$ is a Kan fibration of simplicial sets.

Definition 2.1.2.4. A functor $F\colon \mathbf{C} \to \mathbf{D}$ of ordinary categories is an *isofibration* if, given $c \in \mathbf{C}$ and an isomorphism $f\colon Fc \to d$ there is an isomorphism $\bar{f}\colon c \to c'$ in $\mathbf{C}$ such that $F(\bar{f}) = f$.

Theorem 2.1.2.5 (Bergner [Ber07])**.** There is a model structure on Cat_Δ such that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is

(W) a weak equivalence if and only if F is weakly fully faithful and essentially surjective up to homotopy,

(F) a fibration if and only if F is a local fibration and $\pi_0 F$ is an isofibration.

Definition 2.1.2.6. If $i \leq j$ are positive integers, let P_{ij} be the partially ordered set of subsets of $\{i, i+1, \ldots j\}$ containing i and j, regarded as a category; if $i > j$ let $P_{ij} = \emptyset$. Let $\mathfrak{C}(\Delta^n)$ denote the simplicial category with objects $0, \ldots, n$ and $\mathfrak{C}(\Delta^n)(i,j) = \mathrm{N}P_{ij}$, with composition defined by taking unions in the obvious way. Taking colimits, this extends to a functor

$$\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$$

with right adjoint $\mathrm{N}\colon \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$ given by

$$\mathrm{N}\mathbf{C}_n = \mathrm{Hom}(\mathfrak{C}(\Delta^n), \mathbf{C}).$$

Theorem 2.1.2.7 (Joyal, Lurie [Lur09a, Theorem 2.2.5.1])**.** The adjunction $\mathfrak{C} \dashv \mathrm{N}$ is a Quillen equivalence between the Joyal model structure on Set_Δ and the Bergner model structure on Cat_Δ.

Thus if $\mathbf{C}$ is a simplicial category whose mapping objects are all Kan complexes, the simplicial set $\mathrm{N}\mathbf{C}$ is an ∞-category; this is an important way of constructing ∞-categories. For example, if $\mathbf{M}$ is a simplicial model category, and $\mathbf{M}^\circ$ denotes the full simplicial subcategory of fibrant-cofibrant objects, then $\mathrm{N}\mathbf{M}^\circ$ is an ∞-category.

Example 2.1.2.8. The *∞-category of spaces* $\mathcal{S}$ can be defined as the nerve $\mathrm{N}\mathrm{Set}_\Delta^\circ$ of the full subcategory $\mathrm{Set}_\Delta^\circ$ of Set_Δ spanned by the Kan complexes. Similarly, the *∞-category of ∞-categories* Cat_∞ can be defined as $\mathrm{N}(\mathrm{Set}_\Delta^\mathrm{J})^\circ$, where $\mathrm{Set}_\Delta^\mathrm{J}$ denotes Set_Δ equipped with the Joyal model structure.

A simplicial category can be viewed as a simplicial object in categories whose simplicial set of objects is constant. This suggests the following definition of a simplicial groupoid:

Definition 2.1.2.9. A *simplicial groupoid* is a simplicial object in groupoids with constant set of objects.

There is a model structure on simplicial groupoids where the weak equivalences are the weakly fully faithful and essentially surjective functors [DK84, Theorem 2.5], and the simplicial nerve functor restricts to a right Quillen equivalence from this to the usual model structure on simplicial sets [DK84, Theorem 3.3]. In particular, it follows that every space is modelled by a fibrant object in simplicial groupoids, which is a simplicial groupoid whose mapping spaces are Kan complexes.

Remark 2.1.2.10. Since a simplicial category can be viewed as a simplicial object in categories with constant set of objects, a simplicial groupoid $\mathcal{C}$ can be regarded as a simplicial category with an involution $i \colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ such that $i^{\mathrm{op}} \circ i = \mathrm{id}_{\mathcal{C}}$, which sends a morphism to its inverse.

2.1.3 Limits and Colimits

We now recall the definition of limits and colimits in an ∞-category; this requires first reviewing some notation:

Definition 2.1.3.1. Let $* \colon \mathbb{\Delta} \times \mathbb{\Delta} \to \mathbb{\Delta}$ denote concatenation of finite ordered sets, i.e. if I and J are finite ordered sets then $I * J$ is the set $I \amalg J$ ordered so that every element of J is greater than every element of I. Thus $[n] * [m] \cong [n + m + 1]$.

Remark 2.1.3.2. This is the restriction to $\mathbb{\Delta}$ of a monoidal structure on the category $\mathbb{\Delta}_+$ of all finite ordered sets (including $\emptyset$).

Definition 2.1.3.3. Suppose K and L are simplicial sets. Their *join* $K * L$ is the left Kan extension of $K \times L \colon \mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ along $* \colon \mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$. Concretely, we have

$$(K * L)_n = K_n \amalg L_n \amalg \coprod_{i+j=n-1} K_i \times L_j.$$

Remark 2.1.3.4. This can be regarded as the Day convolution product on presheaves on $\mathbb{\Delta}_+$ with the monoidal structure given by $*$.

Definition 2.1.3.5. Let K be a simplicial set. The *left cone* $K^{\triangleleft}$ on K is the join $\Delta^0 * K$, and the *right cone* $K^{\triangleright}$ on K is the join $K * \Delta^0$. We will often denote the "cone point", i.e. the vertex coming from Δ^0, by $-\infty \in K^{\triangleleft}$ and $\infty \in K^{\triangleright}$.

Definition 2.1.3.6. Let $p \colon K \to S$ be a map of simplicial sets. The simplicial set $S_{/p}$ is defined by the universal property

$$\mathrm{Hom}(X, S_{/p}) = \mathrm{Hom}_p(X * K, S),$$

where the right-hand side denotes the set of morphisms $X * K \to S$ that restrict to p on K. Similarly, the simplicial set $S_{p/}$ is defined by the universal property

$$\mathrm{Hom}(X, S_{p/}) = \mathrm{Hom}_p(K * X, S).$$

If $\mathcal{C}$ is an ∞-category, for any map $p \colon K \to \mathcal{C}$ the simplicial sets $\mathcal{C}_{p/}$ and $\mathcal{C}_{p/}$ are also ∞-categories (cf. [Lur09a, Proposition 1.2.9.3]).

Definition 2.1.3.7. Let $\mathcal{C}$ be an ∞-category. An object $X \in \mathcal{C}$ is a *final object* if the projection $\mathcal{C}_{/X} \to \mathcal{C}$ is a categorical equivalence. Similarly, X is an *inital object* if $\mathcal{C}_{X/} \to \mathcal{C}$ is a categorical equivalence.

Equivalently, X is a final object if and only if for every object $Y \in \mathcal{C}$ the mapping space $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is contractible (cf. [Lur09a, Corollary 1.2.12.5]).

Definition 2.1.3.8. Let $\mathcal{C}$ be an ∞-category and $p\colon K \to \mathcal{C}$ a map of simplicial sets. A *colimit* of p is a final object of $\mathcal{C}_{p/}$, and a *limit* of p is an initial object of $\mathcal{C}_{/p}$.

Remark 2.1.3.9. A colimit of p can thus be regarded as a diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ that restricts to p on K. From the definition of final objects it follows immediately that an arbitrary such diagram $\bar{p}$ is a colimit precisely when

$$\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/}$$

is a categorical equivalence.

We now recall the definition of *relative* colimits, from [Lur09a, §4.3.1]:

Definition 2.1.3.10. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, and let $p\colon K \to \mathcal{C}$ be a diagram. A diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ extending p is an *f-colimit* of p if the map

$$\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$$

is a categorical equivalence.

2.1.4 Left and Right Fibrations

Here we briefly discuss *left* and *right* fibrations, which correspond to (covariant and contravariant) functors to the ∞-category $\mathcal{S}$ of spaces.

Definition 2.1.4.1. A morphism of simplicial sets is a *left fibration* if it has the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ with $0 \leq i < n$, and a *right fibration* if it has the right lifting property with respect to $\Lambda^n_i \hookrightarrow \Delta^n$ with $0 < i \leq n$.

If S is a simplicial set, there are model structures on $(\mathrm{Set}_\Delta)_{/S}$, the *covariant* and *contravariant* model structures, whose fibrant objects are, respectively, left and right fibrations with target S (cf. [Lur09a, Proposition 2.1.4.7]).

Theorem 2.1.4.2 (Lurie [Lur09a, Theorem 2.2.1.2])**.** Let S be a simplicial set. There is a Quillen equivalence

$$(\mathrm{Set}_\Delta)_{/S} \rightleftarrows \mathrm{Fun}(\mathfrak{C}(S)^{\mathrm{op}}, \mathrm{Set}_\Delta)$$

where $(\mathrm{Set}_\Delta)_{/S}$ is equipped with the contravariant model structure and $\mathrm{Fun}(\mathfrak{C}(S)^{\mathrm{op}}, \mathrm{Set}_\Delta)$ with the projective model structure for the usual model structure on Set_Δ.

Corollary 2.1.4.3. Suppose $\mathcal{C}$ is an ∞-category. Let $\mathrm{LFib}(\mathcal{C})$ denote the ∞-category of left fibrations over $\mathcal{C}$ (for example obtained from the covariant model structure on $(\mathrm{Set}_\Delta)_{/\mathcal{C}}$. There is an equivalence

$$\mathrm{LFib}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S}).$$

Proposition 2.1.4.4. Suppose given functors $f, g\colon X \to \mathcal{S}$ and a natural transformation η from f to g. Let $p\colon F \to X$ and $q\colon G \to X$ be left fibrations associated to f and g, and let $e\colon F \to G$ be a functor over X associated to η. Then e is equivalent to a left fibration.

Proof. We may regard f and g as functors $\mathfrak{C}[X] \to \mathrm{Set}_\Delta$; without loss of generality we may assume f and g correspond to fibrant objects and η to a fibration in the projective model structure on $\mathrm{Fun}(\mathfrak{C}[X], \mathrm{Set}_\Delta)$. Since unstraightening is a right Quillen functor, we obtain a commutative diagram

$$\begin{tikzcd} F \arrow[rr, "e"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

where p and q are left fibrations associated to f and g, and e is a fibration in the covariant model structure associated to η. By [Lur09a, Proposition 2.1.4.9] the map e is then a left fibration. □

2.1.5 Cartesian and coCartesian Fibrations

Definition 2.1.5.1. Suppose $p\colon X \to S$ is an inner fibration of simplicial sets. We say an edge $f\colon x \to y$ in X is *p-Cartesian* if the map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a categorical equivalence. Similarly, f is *p-coCartesian* if

$$X_{f/} \to X_{x/} \times_{S_{p(x)/}} S_{p(f)/}$$

is a categorical equivalence.

Definition 2.1.5.2. Suppose $p\colon X \to S$ is an inner fibration of simplicial sets. An edge $f\colon x \to y$ in X is a *locally p-(co)Cartesian* if it is a p'-(co)Cartesian edge of $X \times_S \Delta^1$, where p' is the pullback of p along $p(f)\colon \Delta^1 \to S$.

Proposition 2.1.5.3 (Lurie [Lur09a, Proposition 2.4.4.3])**.** Suppose $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration of ∞-categories. A morphism $f\colon y \to z$ in $\mathcal{C}$ is p-Cartesian if and only if for every $x \in \mathcal{C}$ composition with f gives a homotopy Cartesian square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(x,z) \arrow[d] \\ \mathrm{Map}_{\mathcal{D}}(p(x),p(y)) \arrow[r] & \mathrm{Map}_{\mathcal{D}}(p(x),p(z)). \end{tikzcd}$$

Definition 2.1.5.4. A map $p\colon X \to S$ of simplicial sets is a *Cartesian fibration* if p is an inner fibration and for every object $x \in X$ and every morphism $f\colon s \to p(x)$ in S there exists a p-Cartesian morphism $\bar{f}\colon f^*x \to x$ with $p(\bar{f}) = f$. Similarly, p is a *coCartesian fibration* if p^{op} is a Cartesian fibration, i.e. if p is an inner fibration and for every object $x \in X$ and every morphism $f\colon p(x) \to s$ in S there exists a p-Cartesian morphism $\bar{f}\colon x \to f_!x$ with $p(\bar{f}) = f$.

Definition 2.1.5.5. A map $p\colon X \to S$ is a *locally (co)Cartesian fibration* if p is an inner fibration and for every edge $\sigma\colon \Delta^1 \to S$ the pullback $X \times_S \Delta^1 \to \Delta^1$ is a (co)Cartesian fibration.

Corollary 2.1.5.6. Suppose given a commutative triangle

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{B} \arrow[dl, "q"] \\
& \mathcal{C} &
\end{tikzcd}
$$

where p and q are Cartesian fibrations and f is an inner fibration that takes p-Cartesian edges to q-Cartesian edges. If for each $c \in \mathcal{C}$ the pullback $f_c\colon \mathcal{A}_c \to \mathcal{B}_c$ is a Cartesian fibration, and the functor $\mathcal{A}_{c'} \to \mathcal{A}_c$ induced by a morphism $c \to c'$ in $\mathcal{C}$ takes $f_{c'}$-Cartesian edges to f_c-Cartesian edges, then f is also a Cartesian fibration.

Proof. We must show that for every $a \in \mathcal{A}$ and every morphism $\beta\colon b \to f(a)$ in $\mathcal{B}$ there exists an f-Cartesian morphism $\beta^* a \to a$ over ϕ. Write $\gamma\colon q(b) \to p(a)$ for the image of β in $\mathcal{C}$. Since p is a Cartesian fibration, there exists a p-Cartesian morphism $\alpha\colon \gamma^* a \to a$ in $\mathcal{A}$ over γ, and by assumption $f(\alpha)$ is q-Cartesian. Since q is a Cartesian fibration, it follows that β factors as

$$b \xrightarrow{\beta'} f(\gamma^* a) \xrightarrow{f(\alpha)} f(a),$$

where β' lies over $\mathrm{id}_{q(b)}$. Now as $f_{q(b)}\colon \mathcal{A}_{q(b)} \to \mathcal{B}_{q(b)}$ is a Cartesian fibration, there exists an $f_{q(b)}$-Cartesian edge $\beta'^* \gamma^* a \to \gamma^* a$. It is easy to check using the criterion of Proposition 2.1.5.3 that the composite $\beta'^* \gamma^* a \to \gamma^* a \to a$ is f-Cartesian. $\square$

If S is a simplicial set, there are model structures on $(\mathrm{Set}_\Delta^+)_{/S}$, the *Cartesian* and *coCartesian* model structures, whose fibrant objects are, respectively, Cartesian and coCartesian fibrations with target S, with their (co)Cartesian edges marked (cf. [Lur09a, Proposition 3.1.3.7]).

Theorem 2.1.5.7 (Lurie [Lur09a, Theorem 3.2.0.1])**.** Let S be a simplicial set. There is a Quillen equivalence

$$(\mathrm{Set}_\Delta^+)_{/S} \rightleftarrows \mathrm{Fun}(\mathfrak{C}(S)^{\mathrm{op}}, \mathrm{Set}_\Delta^+)$$

where $(\mathrm{Set}_\Delta^+)_{/S}$ is equipped with the Cartesian model structure and $\mathrm{Fun}(\mathfrak{C}(S)^{\mathrm{op}}, \mathrm{Set}_\Delta^+)$ with the projective model structure with respect to the model structure on Set_Δ^+ that models ∞-categories.

Corollary 2.1.5.8. Let $\mathcal{C}$ be an ∞-category, and write $\mathrm{Cart}(\mathcal{C})$ and $\mathrm{CoCart}(\mathcal{C})$ for the ∞-categories of Cartesian and coCartesian fibrations to $\mathcal{C}$, respectively, i.e. the ∞-categories associated to the Cartesian and coCartesan model structures on $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}}$. Then there are equivalences

$$\mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty), \qquad \mathrm{CoCart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty).$$

Definition 2.1.5.9. A morphism of ∞-categories $\phi\colon \mathcal{C} \to \mathcal{D}$ is an *essentially coCartesian fibration* if there exists a factorization

$$\mathcal{C} \xrightarrow{\epsilon} \mathcal{C}' \xrightarrow{\phi'} \mathcal{D}$$

such that ϵ is a categorical equivalence and ϕ' is a coCartesian fibration.

We can describe colimits in the total space of a coCartesian fibration:

Lemma 2.1.5.10. Suppose $\pi\colon \mathcal{E} \to \mathcal{B}$ is a coCartesian fibration such that both $\mathcal{B}$ and the fibres $\mathcal{E}_b$ for all $b \in \mathcal{B}$ admit small colimits, and the functors $f_!\colon \mathcal{E}_b \to \mathcal{E}_{b'}$ preserve colimits for all morphisms $f\colon b \to b'$ in $\mathcal{B}$. Then $\mathcal{E}$ admits small colimits.

Proof. The coCartesian fibration π satisfies the conditions of [Lur09a, Corollary 4.3.1.11] for all small simplicial sets K, and so in every diagram

$$\begin{tikzcd} K \arrow[r, "p"] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\ K^{\triangleright} \arrow[r, "\bar{q}"'] \arrow[ur, dashed, "\bar{p}"] & \mathcal{B} \end{tikzcd}$$

there exists a lift $\bar{p}$ that is a π-colimit of p. Given a diagram $p\colon K \to \mathcal{E}$ we can apply this with $\bar{q}$ a colimit of $\pi \circ p$ to get a colimit $\bar{p}\colon K^{\triangleright} \to \mathcal{E}$ of p. $\square$

It is easy to see that colimits of coCartesian edges are coCartesian:

Lemma 2.1.5.11. Suppose $p\colon X \to S$ is a coCartesian fibration, and let $\bar{r}\colon K^{\triangleright} \to \mathrm{Fun}(\Delta^1, X)$ be a colimit diagram such that for every $i \in K$ the edge $\bar{r}(i,0) \to \bar{r}(i,1)$ is coCartesian. Then the edge $\bar{r}(\infty,0) \to \bar{r}(\infty,1)$ is also coCartesian.

Proof. Since colimits in functor categories are pointwise, we must show that for all $x \in X$ the diagram

$$\begin{tikzcd} \mathrm{Map}_X(\mathrm{colim}_i\, \bar{r}(i,1), x) \arrow[r] \arrow[d] & \mathrm{Map}_X(\mathrm{colim}_i\, \bar{r}(i,0), x) \arrow[d] \\ \mathrm{Map}_S(\mathrm{colim}_i\, p\bar{r}(i,1), p(x)) \arrow[r] & \mathrm{Map}_S(\mathrm{colim}_i\, p\bar{r}(i,0), p(x)) \end{tikzcd}$$

is Cartesian, which is clear since limits commute. $\square$

In good cases it is also true that the colimit of Cartesian edges is Cartesian:

Proposition 2.1.5.12. Suppose $\pi\colon \mathcal{E} \to \mathcal{B}$ is a Cartesian and coCartesian fibration, where $\mathcal{B}$ is an ∞-category with all colimits. Let $\bar{p}\colon \mathcal{I}^{\triangleright} \to \mathcal{E}$ be a colimit diagram; then $\pi \circ \bar{p}$ is a colimit diagram in $\mathcal{B}$. Suppose the associated contravariant functor $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ takes $\pi \circ \bar{p}$ to a limit diagram of ∞-categories, and that $\bar{p}$ takes each edge of $\mathcal{I}$ to a π-Cartesian morphism in $\mathcal{E}$. Then $\bar{p}$ takes every edge of $\mathcal{I}^{\triangleright}$ to a π-Cartesian morphism.

Proof. Write b for $\pi(\bar{p}(\infty))$ and p for $\bar{p}|_K$. Then the fibre $\mathcal{E}_b$ is the limit of $\mathcal{E}_{\pi p(i)}$ for i in $\mathcal{I}$. This limit is given by the ∞-category of Cartesian sections $\mathcal{I} \to \mathcal{E}$ of $\pi \circ p$. Since p takes every edge of $\mathcal{I}$ to a Cartesian edge in $\mathcal{E}$, the diagram $p = \bar{p}|_K$ corresponds to an object x of $\mathcal{E}_b$.

Write ϕ_i for the canonical map $i \to \infty$. Then for y in $\mathcal{E}_b$ we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{E}_b}(x,y) &\simeq \lim \mathrm{Map}_{\mathcal{E}_{\pi p(i)}}(p(i), \phi_i^* y) \\ &\simeq \lim \mathrm{Map}_{\mathcal{E}_b}(\phi_{i,!}p(i), y) \\ &\simeq \mathrm{Map}_{\mathcal{E}_b}(\operatorname{colim} \phi_{i,!}p(i), y) \\ &\simeq \mathrm{Map}_{\mathcal{E}_b}(\bar{p}(\infty), y).\end{aligned}$$

Thus $x \simeq \bar{p}(\infty)$. In particular $\phi_i^* \bar{p}(\infty) \simeq p(i)$, or in other words the morphism $p(i) \to \bar{p}(\infty)$ is Cartesian. □

In the setting of ordinary categories, the total space of a coGrothendieck fibration is the lax colimit of the associated functor. An ∞-categorical theory of lax colimits has not yet been developed, but we will now prove that the total space of a coCartesian fibration satisfies a version of the expected universal property:

Proposition 2.1.5.13. Suppose $D\colon \mathcal{O} \to \mathrm{Cat}_\infty$ is a functor, $\mathcal{O}_D \to \mathcal{O}$ is the associated coCartesian fibration, and $\mathcal{C}$ is a locally small ∞-category. Let $\mathcal{E} \to \mathcal{O}$ be the Cartesian fibration associated to the functor $\mathrm{Fun}(D, \mathcal{C})$ that sends $x \in \mathcal{O}$ to $\mathrm{Fun}(D(x), \mathcal{C})$. Then $\mathrm{Fun}(\mathcal{O}_D, \mathcal{C})$ is equivalent to the ∞-category of sections $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{E})$.

Proof. We first consider the case where $\mathcal{C}$ is an ∞-category $\mathcal{P}(\mathcal{D})$ of presheaves on an ∞-category $\mathcal{D}$. Then we have equivalences

$$\mathrm{Fun}(\mathcal{O}_D, \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{LFib}(\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}})$$

Composition with the coCartesian fibration $\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}} \to \mathcal{O} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{O}$ gives a functor

$$\mathrm{LFib}(\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}}) \to \mathrm{CoCart}(\mathcal{O})_{/\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}}}$$

since any functor between left fibrations over $\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}}$ gives a coCartesian-morphism-preserving functor — indeed, this shows that this functor is fully faithful. We claim that under the equivalence $\mathrm{CoCart}(\mathcal{O}) \simeq \mathrm{Fun}(\mathcal{O}, \mathrm{Cat}_\infty)$ this full subcategory corresponds to the full subcategory $\mathcal{X}$ of $\mathrm{Fun}(\mathcal{O}, \mathrm{Cat}_\infty)_{/D \times \mathcal{D}^{\mathrm{op}}}$ spanned by those natural transformations that are pointwise left fibrations.

It is clear that $\mathrm{LFib}(\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}})$ lands in the full subcategory $\mathcal{X}$, since left fibrations are closed under pullback. It thus suffices to show that any object of $\mathcal{X}$ corresponds to a right fibration over $\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}}$. The functor to $\mathcal{O}_D \times \mathcal{D}^{\mathrm{op}}$ is a coCartesian fibration by (the dual of) Corollary 2.1.5.6, and its fibres are spaces since the pullbacks to $D(x) \times \mathcal{D}^{\mathrm{op}}$ are right fibrations for all $x \in \mathcal{O}$, thus this is true.

Write $\mathcal{L}$ for the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ spanned by the left fibrations. Since left fibrations are closed under pullback, the functor $\mathcal{L} \to \mathrm{Cat}_\infty$ given by evaluation at $1 \in \Delta^1$ is a Cartesian fibration. Let $\mathcal{L}' \to \mathcal{O}$ be the pullback of $\mathcal{L}$ along $D \times \mathcal{D}^{\mathrm{op}}\colon \mathcal{O} \to \mathrm{Cat}_\infty$. This is the Cartesian fibration associated to the functor $\mathcal{O}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends x to $\mathrm{LFib}(D(x) \times \mathcal{D}^{\mathrm{op}}) \simeq \mathrm{Fun}(D(x) \times \mathcal{D}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(D(x), \mathcal{C})$, i.e. $\mathcal{L}' \to \mathcal{O}$ is equivalent to the Cartesian fibration $\mathcal{E} \to \mathcal{O}$. A section of $\mathcal{L}' \to \mathcal{O}$ clearly corresponds to a functor $\phi\colon \mathcal{O} \times \Delta^1 \to \mathrm{Cat}_\infty$ such that $\phi(x,0) \to \phi(x,1)$ is a left fibration for all $x \in \mathcal{O}$ and $\phi|_{\mathcal{O} \times \{1\}}$ is $D \times \mathcal{D}^{\mathrm{op}}$. In other words, $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{L}') \simeq \mathcal{X}$. This completes the proof when $\mathcal{C} \simeq \mathcal{P}(\mathcal{D})$.

Now suppose $\mathcal{C}$ is a full subcategory of $\mathcal{P}(\mathcal{D})$ for some ∞-category $\mathcal{D}$. Then we can identify $\mathrm{Fun}(\mathcal{O}_D, \mathcal{C})$ with a full subcategory of $\mathrm{Fun}(\mathcal{O}_D, \mathcal{P}(\mathcal{D}))$, and $\mathcal{E}$ with a full subcategory of $\mathcal{L}'$, and it is clear that under these equivalences $\mathrm{Fun}(\mathcal{O}_D, \mathcal{C})$ corresponds to $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{E})$ under the equivalence $\mathrm{Fun}(\mathcal{O}_D, \mathcal{P}(\mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{L}')$ constructed above. Since every locally small ∞-category $\mathcal{C}$ can be identified with a full subcategory of $\mathcal{P}(\mathcal{C})$ via the Yoneda embedding, this completes the proof. □

2.1.6 Adjunctions

Definition 2.1.6.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a map $p\colon \mathcal{M} \to \Delta^1$ that is both a Cartesian and a coCartesian fibration, together with equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{M}_0$ and $\mathcal{D} \xrightarrow{\sim} \mathcal{M}_1$. If $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ are functors associated to the adjunction $\mathcal{M}$ we say that f is *left adjoint* to g and g is *right adjoint* to f.

Definition 2.1.6.2. Suppose given a pair of functors

$$f\colon \mathcal{C} \rightleftarrows \mathcal{D} : g$$

between ∞-categories. A *unit transformation* for f, g is a natural transformation $u\colon \mathrm{id}_{\mathcal{C}} \to g \circ f$ such that for all $c \in \mathcal{C}$, $d \in \mathcal{D}$, the composite

$$\mathrm{Map}_{\mathcal{D}}(f(c), d) \to \mathrm{Map}_{\mathcal{C}}(gf(c), g(d)) \to \mathrm{Map}_{\mathcal{C}}(c, g(d))$$

is an equivalence of spaces.

Proposition 2.1.6.3 ([Lur09a, Proposition 5.2.2.8])**.** Suppose given a pair of functors

$$f\colon \mathcal{C} \rightleftarrows \mathcal{D} : g$$

between ∞-categories. Then f is left adjoint to g if and only if there exists a unit transformation $u\colon \mathrm{id}_{\mathcal{C}} \to g \circ f$.

Lemma 2.1.6.4. Let $\mathcal{E}$ and $\mathcal{B}$ be ∞-categories and $p\colon \mathcal{E} \to \mathcal{B}$ a functor. Suppose

(1) $\mathcal{E}$ has finite limits and p preserves these,

(2) p has a right adjoint $r\colon \mathcal{B} \to \mathcal{E}$ such that $p \circ r \simeq \mathrm{id}_{\mathcal{B}}$.

Then p is a Cartesian fibration.

Proof. Given $x \in \mathcal{E}$ and a morphism $f\colon b \to p(x)$, we must show there exists a Cartesian arrow in $\mathcal{E}$ lying over f with target x. Define $\bar{f}\colon y \to x$ by the pullback diagram

$$\begin{tikzcd} y \arrow[r, "\bar{f}"] \arrow[d] & x \arrow[d] \\ r(b) \arrow[r, "r(f)"'] & rp(x) \end{tikzcd}$$

Since p preserves pullbacks, the morphism $p(\bar{f})$ is equivalent to f. Moreover, for any $z \in \mathcal{E}$

we have a pullback diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(z,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(z,x) \arrow[d] \\
\mathrm{Map}_{\mathcal{E}}(z,r(b)) \arrow[r] & \mathrm{Map}_{\mathcal{E}}(z,rp(x)).
\end{tikzcd}$$

Under the adjunction this corresponds to the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(z,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(z,x) \arrow[d] \\
\mathrm{Map}_{\mathcal{B}}(p(z),b) \arrow[r] & \mathrm{Map}_{\mathcal{E}}(p(z),p(x))
\end{tikzcd}$$

induced by the functor p. But then $\bar{f}$ is Cartesian by Proposition 2.1.5.3. □

Proposition 2.1.6.5. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a functor between ∞-categories such that $\mathcal{E}$ has pullbacks, these are preserved by p, and for all $b \in \mathcal{B}$ the ∞-category $\mathcal{E}_{/b}$ has a final object, which lies in the fibre over b of p. Then p is a Cartesian fibration.

Proof. By Lemma 2.1.6.4 it suffices to show that p has a right adjoint $r\colon \mathcal{B} \to \mathcal{E}$ that is a section of p. Let $\mathcal{Q} \to \mathcal{B}$ be a coCartesian fibration associated to the functor $b \mapsto \mathcal{E}_{/b}$; by the dual of [Lur09a, Proposition 2.4.4.9] this fibration has an (essentially unique) section $\mathcal{B} \to \mathcal{Q}$ that sends $b \in \mathcal{B}$ to a final object in $\mathcal{E}_{/b}$. Combining this with the natural map $\mathcal{Q} \to \mathcal{E}$ associated to the forgetful functors $\mathcal{E}_{/b} \to \mathcal{E}$ we get a section $r\colon \mathcal{B} \to \mathcal{E}$ that sends $b \in \mathcal{B}$ to a final object $*_b$ of $\mathcal{E}_{/b}$. Then r is a right adjoint of p: by definition all fibres of the map $\mathrm{Map}_{\mathcal{E}}(x, *_b) \to \mathrm{Map}_{\mathcal{B}}(px, b)$ are contractible, so this map is an equivalence. □

2.1.7 Accessible and Presentable ∞-Categories

Definition 2.1.7.1. Suppose κ is a regular cardinal. A simplicial set K is *κ-small* if all the sets K_n are κ-small. A *κ-small (co)limit* is a (co)limit indexed by a κ-small simplicial set.

Definition 2.1.7.2. Suppose κ is a regular cardinal. An ∞-category $\mathcal{I}$ is *κ-filtered* if the colimit functor $\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \to \mathcal{S}$ preserves κ-small limits.

Proposition 2.1.7.3 ([Lur09a, Proposition 5.3.3.3])**.** An ∞-category $\mathcal{I}$ is κ-filtered if and only if for every κ-small simplicial set K and every map $f\colon K \to \mathcal{I}$ there exists a map $\bar{f}\colon K^{\triangleright} \to \mathcal{I}$ extending f.

Definition 2.1.7.4. Suppose κ is a regular cardinal. An object c in an ∞-category $\mathcal{C}$ is *κ-compact* if the representable functor $\mathrm{Map}_{\mathcal{C}}(c, -)$ preserves κ-filtered colimits. We denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects by $\mathcal{C}^{\kappa}$.

Definition 2.1.7.5. Suppose κ is a regular cardinal. If $\mathcal{C}$ is an ∞-category, we let $\mathrm{Ind}_{\kappa}\,\mathcal{C}$ denote the full subcategory of $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by the the functors $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ that classify right fibrations $\mathcal{E} \to \mathcal{C}$ such that $\mathcal{E}$ is κ-filtered.

Definition 2.1.7.6. Suppose κ is a regular cardinal. An ∞-category $\mathcal{C}$ is *κ-accessible* if there exists a small ∞-category $\mathcal{C}_0$ and an equivalence $\mathrm{Ind}_\kappa \mathcal{C}_0 \xrightarrow{\sim} \mathcal{C}$.

Proposition 2.1.7.7 ([Lur09a, Proposition 5.4.2.2])**.** Suppose κ is a regular cardinal. An ∞-category $\mathcal{C}$ is κ-accessible if and only if $\mathcal{C}$ has κ-filtered colimits and contains an essentially small full subcategory $\mathcal{C}'$ that consists of κ-compact objects and generates $\mathcal{C}$ under κ-filtered colimits.

Definition 2.1.7.8. We say an ∞-category is *accessible* if it is κ-accessible for some κ. If $\mathcal{C}$ is an accessible ∞-category, we say a functor $f\colon \mathcal{C} \to \mathcal{D}$ is *accessible* if it preserves κ-filtered colimits for some κ.

Definition 2.1.7.9. Suppose κ is a regular cardinal. An ∞-category is *κ-presentable* if it is κ-accessible and admits small colimits. We say an ∞-category is *presentable* if it is κ-presentable for some κ.

Theorem 2.1.7.10 (Adjoint Functor Theorem, [Lur09a, Corollary 5.5.2.9])**.** Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between presentable ∞-categories. Then F has a right adjoint if and only if it preserves small colimits, and a left adjoint if and only if it is accessible and preserves small limits.

Lemma 2.1.7.11. Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is an adjunction such that the right adjoint U preserves κ-filtered colimits. Then F preserves κ-compact objects.

Proof. Suppose $X \in \mathcal{C}$ is a κ-compact object, and $p\colon K^{\triangleright} \to \mathcal{D}$ is a κ-filtered colimit diagram. Then we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{D}}(F(X), \operatorname{colim} p) &\simeq \mathrm{Map}_{\mathcal{C}}(X, G(\operatorname{colim} p)) \simeq \mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} G \circ p) \\ &\simeq \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, G \circ p) \simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}(F(X), p).\end{aligned}$$

Thus $\mathrm{Map}_{\mathcal{D}}(F(X), -)$ preserves κ-filtered colimits, i.e. the object $F(X)$ is κ-compact. □

Definition 2.1.7.12. Let Pr^{L} be the ∞-category of presentable ∞-categories and colimit-preserving functors.

2.1.8 Localizations

Definition 2.1.8.1. Suppose $\mathcal{C}$ is an ∞-category and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ that contains all the equivalences. The *localization* $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ with respect to $\mathcal{W}$ is the ∞-category with the universal property that for any ∞-category $\mathcal{E}$, a functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{E}$ is the same thing as a functor $\mathcal{C} \to \mathcal{E}$ that sends morphisms in $\mathcal{W}$ to equivalences in $\mathcal{E}$. More precisely, we have for every $\mathcal{E}$ a pullback square

$$\begin{tikzcd} \mathrm{Map}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{W}, \iota\mathcal{E}) \arrow[d] \\ \mathrm{Map}(\mathcal{C}, \mathcal{E}) \arrow[r] & \mathrm{Map}(\mathcal{W}, \mathcal{E}). \end{tikzcd}$$

Definition 2.1.8.2. The inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ has left and right adjoints. The right adjoint, $\iota\colon \mathrm{Cat}_\infty \to \mathcal{S}$, sends an ∞-category $\mathcal{C}$ to its maximal Kan complex, i.e. its subcategory of equivalences. The left adjoint $\kappa\colon \mathrm{Cat}_\infty \to \mathcal{S}$ sends an ∞-category $\mathcal{C}$ to a Kan complex $\kappa\mathcal{C}$ such that $\mathcal{C} \to \kappa\mathcal{C}$ is a weak equivalence of spaces.

Remark 2.1.8.3. It follows that, in the situation above, the ∞-category $\mathcal{C}[\mathcal{W}^{-1}]$ is given by the pushout square in Cat_∞

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \kappa\mathcal{W} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{C}[\mathcal{W}^{-1}]. \end{tikzcd}$$

Using this we can prove the following basic fact about localizations of ∞-categories (generalizing [DK80b, Corollary 3.6]):

Lemma 2.1.8.4. Suppose $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories and $\mathcal{V} \subseteq \mathcal{C}$ and $\mathcal{W} \subseteq \mathcal{D}$ are subcategories containing all the equivalences. Let $\mathcal{C}[\mathcal{V}^{-1}]$ and $\mathcal{D}[\mathcal{W}^{-1}]$ be localizations with respect to $\mathcal{V}$ and $\mathcal{W}$. Suppose

$$F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$$

is an adjunction such that

(1) $F(\mathcal{V}) \subseteq \mathcal{W}$,

(2) $G(\mathcal{W}) \subseteq \mathcal{V}$,

(3) the unit morphism $\eta_c\colon c \to GFc$ is in $\mathcal{V}$ for all $c \in \mathcal{C}$,

(4) the counit morphism $\gamma_d\colon FGd \to d$ is in $\mathcal{W}$ for all $d \in \mathcal{D}$.

Then F and G induce an equivalence $\mathcal{C}[\mathcal{V}^{-1}] \simeq \mathcal{D}[\mathcal{W}^{-1}]$.

Proof. Let $\kappa\mathcal{V}$ and $\kappa\mathcal{W}$ be Kan complexes that are fibrant replacements for $\mathcal{V}$ and $\mathcal{W}$ in the usual model structure on simplicial sets. Then the ∞-categories $\mathcal{C}[\mathcal{V}^{-1}]$ and $\mathcal{D}[\mathcal{W}^{-1}]$ can be described as the homotopy pushouts

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \kappa\mathcal{V} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{C}[\mathcal{V}^{-1}], \end{tikzcd} \qquad \begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \kappa\mathcal{W} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{D}[\mathcal{W}^{-1}] \end{tikzcd}$$

in the Joyal model structure. Then from (1) and (2) it is clear that F and G induce functors $F'\colon \mathcal{C}[\mathcal{V}^{-1}] \to \mathcal{D}[\mathcal{W}^{-1}]$ and $G'\colon \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{V}^{-1}]$, and the natural transformations η and γ induce natural transformations $\eta'\colon \mathrm{id} \to G'F'$ and $\gamma'\colon F'G' \to \mathrm{id}$. The objects of $\mathcal{C}[\mathcal{V}^{-1}]$ and $\mathcal{D}[\mathcal{W}^{-1}]$ are the same as those of $\mathcal{C}$ and $\mathcal{D}$, so by (3) and (4) the morphisms η'_c and γ'_d are equivalences for all $c \in \mathcal{C}[\mathcal{V}^{-1}]$ and $d \in \mathcal{D}[\mathcal{W}^{-1}]$. Thus η' and γ' are natural equivalences and F' and G' are hence equivalences of ∞-categories. $\square$

Unfortunately, pushouts in Cat_∞ are in general difficult to describe. However, in good cases the functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ has a fully faithful right adjoint, i.e. we can find the localized ∞-category as a full subcategory of $\mathcal{C}$. In fact, all functors of this kind are localizations:

Definition 2.1.8.5. A functor $f\colon \mathcal{C} \to \mathcal{D}$ is a *localization* if f has a fully faithful right adjoint.

Proposition 2.1.8.6 ([Lur09a, Proposition 5.2.7.12])**.** Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a localization functor, and let $\mathcal{W}$ be the subcategory of $\mathcal{C}$ with morphisms the morphisms $f\colon c \to c'$ in $\mathcal{C}$ such that $F(f)$ is an equivalence. Then the induced functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence.

We now recall how to describe localizations in the presentable case:

Definition 2.1.8.7. Let $\mathcal{C}$ be an ∞-category and suppose S is a collection of morphisms in $\mathcal{C}$. An object $z \in \mathcal{C}$ is *S-local* if for every $s\colon x \to y$ in S, composition with S induces an equivalence

$$\mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z).$$

A morphism $f\colon x \to y$ is an *S-equivalence* if for every S-local object z, composition with f induces an equivalence

$$\mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z).$$

Definition 2.1.8.8. We say a class of morphisms in an ∞-category satisfies the *2-out-of-3 property* if for any 2-simplex

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\ y \arrow[rr, "h"'] & & z \end{tikzcd}$$

in $\mathcal{C}$, if any two out of f, g, h is in the class, so is the third.

Definition 2.1.8.9. Let $\mathcal{C}$ be an ∞-category with small colimits and let S be a collection of morphisms in $\mathcal{C}$. We say S is *strongly saturated* if it satisfies the following conditions:

(1) S is closed under pushouts along arbitrary morphisms in $\mathcal{C}$.

(2) The full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by S is stable under small colimits.

(3) S satisfies the 2-out-of-3 property.

Proposition 2.1.8.10 ([Lur09a, Proposition 5.5.4.15])**.** Let $\mathcal{C}$ be a presentable ∞-category and suppose S is a set of morphisms of $\mathcal{C}$. Let $\overline{S}$ denote the strongly saturated class of morphisms generated by S, and let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}$ spanned by the S-local objects. Then

(i) The inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint L.

(ii) The ∞-category $\mathcal{D}$ is presentable.

(iii) For every morphism f in $\mathcal{C}$, the following are equivalent:

  (a) f is an S-equivalence.
  (b) f belongs to $\overline{S}$.
  (c) Lf is an equivalence.

We end with a few results about fibrewise localizations of coCartesian fibrations:

Lemma 2.1.8.11. Suppose $\mathcal{E} \to \Delta^1$ is a coCartesian fibration, and $\mathcal{E}'$ is a full subcategory of $\mathcal{E}$ such that the inclusion $\mathcal{E}'_1 \hookrightarrow \mathcal{E}_1$ admits a left adjoint $L\colon \mathcal{E}_1 \to \mathcal{E}'_1$. Then the restriction $\mathcal{E}' \to \Delta^1$ is also a coCartesian fibration.

Proof. We must show that for each $x \in \mathcal{E}'_0$ there exists a coCartesian arrow with source x over $0 \to 1$ in Δ^1. Suppose $\phi\colon x \to y$ is such a coCartesian arrow in $\mathcal{E}$, and let $y \to Ly$ be the unit of the adjunction. Then any composite $x \xrightarrow{\phi} y \to Ly$ is a coCartesian arrow in $\mathcal{E}'$: by Proposition 2.1.5.3 it suffices to show that for all $z \in \mathcal{E}'_1$ the map $\mathrm{Map}_{\mathcal{E}'}(Ly, z) \to \mathrm{Map}_{\mathcal{E}'}(x, z)$ is an equivalence, which is clear since $\mathrm{Map}_{\mathcal{E}'}(Ly, z) \simeq \mathrm{Map}_{\mathcal{E}}(y, z)$ as $z \in \mathcal{E}'$, $\mathrm{Map}_{\mathcal{E}'}(x, z) \simeq \mathrm{Map}_{\mathcal{E}}(x, z)$ as $\mathcal{E}'$ is a full subcategory of $\mathcal{E}$, and $x \to y$ is a coCartesian morphism in $\mathcal{E}$. □

Lemma 2.1.8.12. Let $\mathcal{E} \to \mathcal{B}$ be a locally coCartesian fibration and $\mathcal{E}^0$ a full subcategory of $\mathcal{E}$ such that for each $b \in \mathcal{B}$ the induced map on fibres $\mathcal{E}^0_b \hookrightarrow \mathcal{E}_b$ admits a left adjoint $L_b\colon \mathcal{E}_b \to \mathcal{E}^0_b$. Assume these localization functors are compatible in the sense that the following condition is satisfied:

(∗) Suppose $f\colon b \to b'$ is a morphism in $\mathcal{B}$ and e is an object of $\mathcal{E}_b$. Let $e \to e'$ and $L_b e \to e''$ be locally coCartesian arrows lying over f, and let $L_{b'}e' \to L_{b'}e''$ be the unique morphism such that the diagram

$$\begin{tikzcd} e \arrow[r] \arrow[d] & e' \arrow[r] \arrow[d] & L_{b'}e' \arrow[d] \\ L_b e \arrow[r] & e'' \arrow[r] & L_{b'}e'' \end{tikzcd}$$

commutes. Then the morphism $L_{b'}e' \to L_{b'}e''$ is an equivalence.

Then

(i) the composite map $\mathcal{E}^0 \to \mathcal{B}$ is also a locally coCartesian fibration,

(ii) the inclusion $\mathcal{E}^0 \hookrightarrow \mathcal{E}$ admits a left adjoint $L\colon \mathcal{E} \to \mathcal{E}^0$ relative to $\mathcal{B}$.

Proof. (i) is immediate from the previous lemma, and then (ii) follows from [Lur11, Proposition 7.3.2.11] — condition (2) of this result is satisfied since, in the notation of condition (∗), a locally coCartesian arrow in $\mathcal{E}^0$ over f with source $L_b e$ is given by the composite $L_b e \to e'' \to L_{b'}e''$. □

Proposition 2.1.8.13. Let $\mathcal{E} \to \mathcal{B}$ be a coCartesian fibration and $\mathcal{E}^0$ a full subcategory of $\mathcal{E}$. Suppose that for each $b \in \mathcal{B}$ the induced map on fibres $\mathcal{E}^0_b \hookrightarrow \mathcal{E}_b$ admits a left adjoint $L_b\colon \mathcal{E}_b \to \mathcal{E}^0_b$ and that the functors $\phi_!\colon \mathcal{E}_b \to \mathcal{E}_{b'}$ corresponding to edges $\phi\colon b \to b'$ in $\mathcal{B}$ preserve the fibrewise local equivalences. Then

(i) the composite map $\mathcal{E}^0 \to \mathcal{B}$ is a coCartesian fibration,

(ii) the inclusion $\mathcal{E}^0 \hookrightarrow \mathcal{E}$ admits a left adjoint $L\colon \mathcal{E} \to \mathcal{E}^0$ over $\mathcal{B}$, and L preserves coCartesian arrows.

Proof. Lemma 2.1.8.12 implies (ii) and also that $\mathcal{E}^0 \to \mathcal{E} \to \mathcal{B}$ is a locally coCartesian fibration, since for a coCartesian fibration condition $(*)$ says precisely that fibrewise local equivalences are preserved by the functors $\phi_!$. It remains to show that locally coCartesian morphisms are closed under composition. Suppose $f\colon b \to b'$ and $g\colon b' \to b''$ are morphisms in $\mathcal{B}$, and that $e \in \mathcal{E}^0_b$. Let $e \to e'$ be a coCartesian arrow in $\mathcal{E}$ over f, and let $e' \to e''_1$ and $L_{b'}e' \to e''_2$ be coCartesian arrows in $\mathcal{E}$ over g. Then a locally coCartesian arrow over f in $\mathcal{E}^0$ is given by $e \to e' \to L_{b'}e'$ and a locally coCartesian arrow over g is given by $L_{b'}e' \to e''_2 \to L_{b''}e''_2$. We have a commutative diagram

$$\begin{tikzcd}
e \arrow[r] \arrow[dr] & e' \arrow[r] \arrow[d] & e''_1 \arrow[r] \arrow[d] & L_{b''}e''_1 \arrow[d] \\
& L_{b'}e' \arrow[r] & e''_2 \arrow[r] & L_{b''}e''_2
\end{tikzcd}$$

Here the composite along the top row is a locally coCartesian arrow for gf, and the composite along the bottom is the composite of locally coCartesian arrows for g and f. By condition $(*)$ the rightmost edge is an equivalence, hence the composite map $e \to L_{b''}e''_2$ is locally coCartesian. □

2.1.9 Monads

Here we briefly review the theory of *monads* in the ∞-categorical setting; for this we assume the reader is familiar with the notions of monoidal ∞-categories, associative algebra objects, and modules from [Lur11].

Definition 2.1.9.1. Suppose $\mathcal{C}$ is an ∞-category. The ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ has a monoidal structure given by composition. A *monad* in $\mathcal{C}$ is an algebra object in this monoidal ∞-category. The monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})^\circ$ acts on the ∞-category $\mathcal{C}$; if T is a monad on $\mathcal{C}$, a *T-algebra* is a left module object for T in $\mathcal{C}$. We write $\mathrm{Alg}_T(\mathcal{C})$ for the ∞-category of T-algebras.

Proposition 2.1.9.2 ([Lur11, Proposition 6.2.2.3])**.** Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories that has a right adjoint G. Then $G \circ F$ extends canonically to a monad on $\mathcal{C}$ such that G is a left module for this monad in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 2.1.9.3. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories that has a right adjoint G. Let T be the monad associated to $G \circ F$. Then G factors canonically as

$$\mathcal{D} \xrightarrow{G'} \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C},$$

through the forgetful functor $\mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$. We say the adjunction $F \dashv G$ is *monadic* if the functor G' is an equivalence.

Definition 2.1.9.4. Let $\mathbb{\Delta}_{-\infty}$ be the category with objects $[n]$, $n \geq -1$, and with morphisms $[m] \to [n]$ given by non-decreasing maps $\alpha\colon [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ such that $\alpha(-\infty) = -\infty$ (and $-\infty$ is regarded as less than the other elements of $[m], [n]$). If $\mathcal{C}$ is an ∞-category, we say an augmented simplicial object $U_\bullet\colon \mathbb{\Delta}_+^{\mathrm{op}} \to \mathcal{C}$ is *split* if it extends to a functor $\mathbb{\Delta}_{-\infty}^{\mathrm{op}} \to \mathcal{C}$, and we say a simplicial object is *split* if it extends to a split augmented simplicial object. Given a functor $G\colon \mathcal{D} \to \mathcal{C}$ we say a simplicial object $U_\bullet$ of $\mathcal{D}$ is *G-split* if $G(U_\bullet)$ is a split simplicial object of $\mathcal{C}$.

Theorem 2.1.9.5 (Barr-Beck Theorem for ∞-Categories, [Lur11, Theorem 6.2.2.5])**.** Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories that has a right adjoint G. The adjunction $F \dashv G$ is monadic if and only if G satisfies the following conditions:

(1) G is *conservative*, i.e. a morphism f in $\mathcal{D}$ is an equivalence if and only if $G(f)$ is an equivalence in $\mathcal{C}$.

(2) G preserves colimits of G-split simplicial objects in $\mathcal{D}$, and all G-split simplicial objects have colimits.

Now we make some simple observations about monadic adjunctions:

Lemma 2.1.9.6. Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a monadic adjunction such that $\mathcal{C}$ has all small colimits, $\mathcal{D}$ has sifted colimits, and U preserves sifted colimits. Then $\mathcal{D}$ has all small colimits.

Proof. Since $\mathcal{D}$ by assumption has all sifted colimits, it suffices to prove that $\mathcal{D}$ has finite coproducts. Since $\mathcal{C}$ has coproducts and F preserves colimits, the ∞-category $\mathcal{D}$ has coproducts for objects in the essential image of F.

Let $A^1, \ldots, A^n$ be a finite collection of objects in $\mathcal{D}$. By [Lur11, Proposition 6.2.2.12], there exist simplicial objects $A^i_\bullet$ in $\mathcal{D}$ such that each A^i_k is in the essential image of F and $|A^i_\bullet| \simeq A^i$. Since coproducts of elements in the essential image of F exist, we can form a simplicial diagram $\coprod_i A^i_\bullet$. By [Lur09a, Lemma 5.5.2.3], the geometric realization $|\coprod_i A^i_\bullet|$ is a coproduct of the A^i's. $\square$

Proposition 2.1.9.7. Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a monadic adjunction such that $\mathcal{C}$ is κ-presentable, $\mathcal{D}$ has small colimits, and the right adjoint U preserves κ-filtered colimits. Then $\mathcal{D}$ is κ-presentable.

Proof. Since $\mathcal{C}$ is κ-presentable, every object of $\mathcal{C}$ is a colimit of κ-compact objects. Since U preserves κ-filtered colimits, F preserves κ-compact objects by Lemma 2.1.7.11. Therefore every object in the essential image of F is a colimit of κ-compact objects. But by [Lur11, Proposition 6.2.2.12], every object of $\mathcal{D}$ is a colimit of objects in the essential image of F, so every object of $\mathcal{D}$ is a colimit of κ-compact objects. Since by assumption $\mathcal{D}$ has all small colimits, this implies that $\mathcal{D}$ is κ-presentable. $\square$

2.1.10 Groupoid Objects

Definition 2.1.10.1. Suppose $\mathcal{C}$ is an ∞-category. A *groupoid object* U of $\mathcal{C}$ is a simplicial object $U\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that for every $n \geq 0$ and every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element s, the diagram

$$\begin{tikzcd}
U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\
U(S') \arrow[r] & U(\{s\})
\end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

Definition 2.1.10.2. Suppose $\mathcal{C}$ is an ∞-category. An augmented simplicial object

$$U\colon \Delta_+^{\mathrm{op}} \to \mathcal{C}$$

is a *Čech nerve* if $U|_{\Delta^{\mathrm{op}}}$ is a groupoid object, and the diagram

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd}$$

is a pullback square. In this case, the augmented simplicial object U is determined up to equivalence by the map $u\colon U_0 \to U_{-1}$, and we say that U is the *Čech nerve* of u.

Definition 2.1.10.3. Suppose $\mathcal{C}$ is an ∞-category and U is a groupoid object in $\mathcal{C}$. We say that U is *effective* if it can be extended to a colimit diagram $\Delta_+^{\mathrm{op}} \to \mathcal{C}$ and this is a Čech nerve, i.e. if U is the restriction to Δ^{op} of the Čech nerve of $U_0 \to |U_\bullet|$.

Lemma 2.1.10.4. Suppose $U_\bullet$ is an effective groupoid object in an ∞-category $\mathcal{C}$. The following are equivalent:

(i) The map $U_0 \to |U_\bullet|$ is an equivalence.

(ii) The map $s_0\colon U_0 \to U_1$ is an equivalence.

(iii) The simplicial object $U_\bullet$ is constant, i.e. for every map $\phi\colon [n] \to [m]$ in Δ^{op} the induced map $\iota_n\mathcal{C} \to \iota_m\mathcal{C}$ is an equivalence.

Proof. We first show that (i) implies (ii): Since $U_\bullet$ is effective, it is equivalent to the Čech nerve of the map $U_0 \to |U_\bullet|$. Thus we have a pullback diagram

$$\begin{tikzcd} U_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & U_0 \arrow[d] \\ U_0 \arrow[r] & {|U_\bullet|,} \end{tikzcd}$$

so the maps d_0, d_1 are equivalences. From the 2-out-of-3 property it follows that s_0 is also an equivalence.

To show that (ii) implies (iii) first observe that if $s_0\colon U_0 \to U_1$ is an equivalence, then by the 2-out-of-3 property $d_0, d_1\colon U_1 \to U_0$ are also equivalences. Since $U_\bullet$ is a groupoid object we have pullback diagrams

$$\begin{tikzcd} U_n \arrow[r, "d_i"] \arrow[d] & U_{n-1} \arrow[d] \\ U_1 \arrow[r, "d_1"'] & U_0 \end{tikzcd}$$

(corresponding to the decomposition $\{0, \ldots, n\} = \{0, \ldots, i-1, i+1, \ldots, n\} \cup \{i-1, i\}$), and so the face maps $d_i \colon U_n \to U_{n-1}$ are equivalences for all i and n. By the 2-out-of-3 property the degeneracies $s_i \colon U_{n-1} \to U_n$ are also equivalences, hence $\phi \colon U_n \to U_m$ must be an equivalence for all $\phi \colon [n] \to [m]$ in Δ^{op}.

Finally (iii) implies (i) since the simplicial set Δ^{op} is weakly contractible. $\square$

Dually, we have the notion of a *cogroupoid object*:

Definition 2.1.10.5. A cosimplicial object $X \colon \Delta \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is a *cogroupoid object* if for every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element, the diagram

$$\begin{tikzcd} X(S \cap S') \arrow[r] \arrow[d] & X(S) \arrow[d] \\ X(S') \arrow[r] & X([n]) \end{tikzcd}$$

is a pushout square.

Remark 2.1.10.6. We could of course have used the dual version of any of the conditions of [Lur09a, Proposition 6.1.2.6] to define a cogroupoid object.

Lemma 2.1.10.7. If $X \colon \Delta \to \mathcal{C}$ is a cogroupoid object in an ∞-category $\mathcal{C}$, then for every object $Y \in \mathcal{C}$ the simplicial space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a groupoid object in spaces.

Proof. Given a partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element, the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X([n]), Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(X(S), Y) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(X(S'), Y) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(X(S \cap S'), Y) \end{tikzcd}$$

is a pullback square, by the definition of a cogroupoid object. Thus $\mathrm{Map}_{\mathcal{C}}(X, Y)$ satisfies condition (4″) of [Lur09a, Proposition 6.1.2.6]. $\square$

2.1.11 The Makkai-Paré Accessibility Theorem

An *accessible fibration* is a Cartesian fibration $\mathcal{E} \to \mathcal{B}$ such that $\mathcal{B}$ is accessible and the associated functor from $\mathcal{B}^{\mathrm{op}}$ to $\widehat{\mathrm{Cat}}_\infty$ factors through the ∞-category of accessible ∞-categories, and preserves κ-filtered limits for κ sufficiently large. In [MP89, Theorem 5.3.4] Makkai and Paré prove that the total space of an accessible fibration of ordinary categories is accessible. The ∞-categorical analogue of this result is surely also true; however, the proof of Makkai and Paré unfortunately does not seem to have a direct analogue for ∞-categories using current technology. In this subsection we will instead prove the easiest special case of this theorem, which luckily will suffice for our needs:

Theorem 2.1.11.1. Let $\mathcal{B}$ be a presentable ∞-category, and let $p \colon \mathcal{E} \to \mathcal{S}$ be a Cartesian fibration associated to the functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending X to $\mathrm{Fun}(X, \mathcal{B})$. Then $\mathcal{E}$ is accessible, and p is an accessible functor.

The key step in the proof is identifying the total space $\mathcal{E}$, which we do in the following preliminary result:

Proposition 2.1.11.2. Suppose $\mathcal{C}$ is a small ∞-category. Let $s\colon * \to \mathcal{C}^{\triangleleft}$ be the inclusion of the cone point, and let $s^*\colon \mathcal{P}(\mathcal{C}^{\triangleleft}) \to \mathcal{S}$ be the functor induced by composition with s. Then:

(i) s^* is a Cartesian fibration.

(ii) The fibre of s^* at $X \in \mathcal{S}$ is naturally equivalent to $\mathrm{Fun}(X, \mathcal{P}(\mathcal{C}))$, and under these equivalences the contravariant functor associated to s^* is $\mathrm{Fun}(–, \mathcal{P}(\mathcal{C}))$.

(iii) Suppose $\mathcal{C}$ admits κ-small colimits, and let $\mathcal{E}$ be the full subcategory of $\mathcal{P}(\mathcal{C}^{\triangleleft})$ spanned by functors $F\colon (\mathcal{C}^{\mathrm{op}})^{\triangleright} \to \mathcal{S}$ that take diagrams $\bar{q}^{\mathrm{op},\triangleleft}\colon K^{\triangleleft\triangleright} \to (\mathcal{C}^{\mathrm{op}})^{\triangleright}$ to limit diagrams in $\mathcal{S}$, where $\bar{q}\colon (K^{\mathrm{op}})^{\triangleright} \to \mathcal{C}$ is a colimit diagram in $\mathcal{C}$ and K is a κ-small simplicial set. Then the restricted functor $p\colon \mathcal{E} \to \mathcal{S}$ is a Cartesian fibration.

(iv) The fibre $\mathcal{E}_X$ is naturally equivalent to $\mathrm{Fun}(X, \mathrm{Ind}_\kappa \mathcal{C})$, and the contravariant functor associated to p preserves limits.

(v) The contravariant functor associated to p is the unique limit-preserving functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending $*$ to $\mathrm{Ind}_\kappa(\mathcal{C})$.

Proof. (i) follows from Proposition 2.1.6.5. The fibre $\mathcal{P}(\mathcal{C}^{\triangleleft})_X$ is the full subcategory of presheaves $(\mathcal{C}^{\triangleleft})^{\mathrm{op}} \to \mathcal{S}$ that send $-\infty$ to X. By the definition of overcategories, this is naturally equivalent to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{/X})$. It is also clear that the functor $f^*\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{/X}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{/Y})$ induced by a map $f\colon Y \to X$ corresponds to composition with pullback along f. Now there are natural isomorphisms $\mathcal{S}_{/X} \simeq \mathrm{Fun}(X, \mathcal{S})$, under which pullback along f correspond to composition with f, and this induces natural equivalences

$$\mathcal{P}(\mathcal{C}^{\triangleleft})_X \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{/X}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times X, \mathcal{S}) \simeq \mathrm{Fun}(X, \mathcal{P}(\mathcal{C})).$$

This gives a natural equivalence between the functor associated to s^* and $\mathrm{Fun}(–, \mathcal{P}(\mathcal{C}))$, which proves (ii).

To prove (iii) it suffices to show that if $F \in \mathcal{E}$ and $g\colon X \to s^*F$ is a morphism in $\mathcal{S}$, then g^*F is also in $\mathcal{E}$. Identify F with a functor $F'\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}_{/X}$ and suppose $\bar{q}\colon K^{\triangleleft} \to \mathcal{C}^{\mathrm{op}}$ is a κ-small limit diagram. Then it is clear that the composite $F' \circ \bar{q}$ is a limit diagram in $\mathcal{S}_{/X}$ if and only if the composite

$$K^{\triangleleft\triangleright} \xrightarrow{\bar{q}^{\triangleright}} (\mathcal{C}^{\mathrm{op}})^{\triangleright} \xrightarrow{F} \mathcal{S}$$

is a limit diagram in $\mathcal{S}$. The former condition is clearly preserved under composition with $g^*\colon \mathcal{S}_{/s^*F} \to \mathcal{S}_{/X}$, and so g^*F is also in $\mathcal{E}$.

Since $\mathcal{C}$ has κ-small colimits, by [Lur09a, Corollary 5.3.5.4] we can identify $\mathrm{Ind}_\kappa \mathcal{C}$ with the full subcategory of $\mathcal{P}(\mathcal{C})$ consisting of presheaves that preserve κ-small limits. As we observed above, the fibre $\mathcal{E}_X$ can be identified with the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{/X})$ spanned by functors that preserve κ-small limits. Since limits in functor categories are computed pointwise, under the equivalence

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{/X}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(X, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times X, \mathcal{S})$$

this corresponds to the full subcategory spanned by functors $\mathcal{C}^{\mathrm{op}} \times X \to \mathcal{S}$ such that for each $x \in X$ the restriction to $\mathcal{C}^{\mathrm{op}} \times \{x\}$ preserves κ-small limits. Under the equivalence

$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times X, \mathcal{S}) \simeq \mathrm{Fun}(X, \mathcal{P}(\mathcal{C}))$ this clearly corresponds to the full subcategory of functors $X \to \mathcal{P}(\mathcal{C})$ such that the value at each $x \in X$ is a presheaf on $\mathcal{C}$ that preserves κ-small limits, i.e. a functor from X to the full subcategory $\mathrm{Ind}_\kappa \mathcal{C}$ of these presheaves. (iv) now follows in the same way as (ii), and (v) is immediate from (iv). □

Lemma 2.1.11.3. Suppose $\mathcal{C}$ is a small ∞-category, and let $S = \{\bar{p}_\alpha \colon K_\alpha^\triangleright \to \mathcal{C}\}$ be a small set of diagrams in $\mathcal{C}$. Then the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by presheaves that take the diagrams in S to limit diagrams in $\mathcal{S}$ is accessible.

Proof. Let $j \colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denote the Yoneda embedding. A presheaf $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ takes $\bar{p}_\alpha^{\mathrm{op}}$ to a limit diagram if and only if it is local with respect to the map of presheaves

$$\mathrm{colim}(j \circ \bar{p}|_{K_\alpha}) \to j(\infty),$$

where ∞ denotes the cone point. Thus if S' is the set of these morphisms for $\bar{p}_\alpha \in S$, the subcategory in question is precisely the full subcategory of S'-local objects. Since S, and hence S', is by assumption a small set, it follows that this subcategory is an accessible localization of $\mathcal{P}(\mathcal{C})$, so in particular it is itself accessible. □

Proof of Theorem 2.1.11.1. Choose κ such that $\mathcal{B}$ is κ-presentable; then $\mathcal{B} \simeq \mathrm{Ind}_\kappa(\mathcal{B}^\kappa)$. By Proposition 2.1.11.2, the ∞-category $\mathcal{E}$ is equivalent to a full subcategory of $\mathcal{P}(\mathcal{B}^{\kappa,\triangleleft})$ spanned by presheaves F that preserve certain limit diagrams. It suffices to take a set of such diagrams (for example, we can restrict ourselves to κ-small coproduct diagrams and pushout diagrams in $\mathcal{B}^\kappa$), and thus $\mathcal{E}$ is accessible by Lemma 2.1.11.3. □

Remark 2.1.11.4. It is not necessary to assume that $\mathcal{C}$ admits κ-small colimits in Proposition 2.1.11.2 (cf. [MP89, §5.3.2] for the 1-categorical version), but this is the only case we're interested in and making this assumption considerably simplifies the proof. Thus Theorem 2.1.11.1 remains true if $\mathcal{B}$ is merely accessible instead of presentable.

2.1.12 Categorical Patterns

In this section we review Lurie's *categorical patterns* and the associated model structures.

Definition 2.1.12.1. A *categorical pattern* $\mathfrak{P} = (X, M, S, \{p_\alpha \colon K_\alpha^\triangleleft \to X)$ consists of a simplicial set X equipped with a marking M (i.e. a subset $M \subseteq X_1$ containing all the degenerate 1-simplices), a scaling T (i.e. a subset $S \subseteq X_2$ containing all the degenerate 2-simplices) and a collection of diagrams $p_\alpha \colon K_\alpha^\triangleleft \to X$ such that p_α takes every edge in $K_\alpha^\triangleleft$ to an element of M and every 2-simplex of $K_\alpha^\triangleleft$ to an element of S.

Definition 2.1.12.2. Let $\mathfrak{P} = (X, M, S, \{p_\alpha \colon K_\alpha^\triangleleft \to X\})$ be a categorical pattern. A marked simplicial set (Y, T) over (X, M) is $\mathfrak{P}$-*fibrant* if the following conditions are satisfied:

(1) The underlying map of simplicial sets $f \colon Y \to X$ is an inner fibration.

(2) For each edge $\Delta^1 \to X$ in M, the pullback $Y \times_X \Delta^1 \to \Delta^1$ is a coCartesian fibration.

(3) An edge e of Y belongs to M if and only if $f(e)$ is in M and e is locally f-coCartesian.

(4) Given a commutative diagram

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, "e"] \arrow[d] & Y \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"'] & X \end{tikzcd}$$

with $e \in M$ and $\sigma \in S$, then e determines a coCartesian edge of the pullback $Y \times_X \Delta^2 \to \Delta^2$.

(5) For every α, the coCartesian fibration $f_\alpha \colon Y \times_X K_\alpha^{\triangleleft} \to K_\alpha^{\triangleleft}$ is classified by a limit diagram $K_\alpha^{\triangleleft} \to \mathrm{Cat}_\infty$.

(6) For every α, every coCartesian section s of f_α is an f-limit diagram in Y.

Remark 2.1.12.3. In all the examples of categorical patterns we will consider in this thesis, the scaling S will simply consist of all 2-simplices in X whose edges are in M. However, to avoid confusion with Lurie's terminology we have chosen to describe the more general case in this review.

Examples 2.1.12.4.

(i) If $\mathcal{C}$ is an ∞-category, let $\mathfrak{P}_{\mathcal{C}}^{\mathrm{coCart}}$ be the categorical pattern $(\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \emptyset)$. Then $(\mathcal{E}, T) \to \mathcal{C}^\sharp$ is $\mathfrak{P}_{\mathcal{C}}^{\mathrm{coCart}}$-fibrant if and only if $\pi \colon \mathcal{E} \to \mathcal{C}$ is a coCartesian fibration, and T is the set of π-coCartesian edges in $\mathcal{E}$.

(ii) If $\mathcal{C}$ is an ∞-category, let $\mathfrak{P}_{\mathcal{C}}^{\mathrm{eq}}$ be the categorical pattern $(\mathcal{C}, \iota\mathcal{C}_1, \iota\mathcal{C}_2, \emptyset)$. Then $(\mathcal{E}, T) \to \mathcal{C}^\natural$ is $\mathfrak{P}_{\mathcal{C}}^{\mathrm{eq}}$-fibrant if and only if $\mathcal{E}$ is an ∞-category, the map $\pi \colon \mathcal{E} \to \mathcal{C}$ is a categorical fibration, and T is the set of equivalences in $\mathcal{E}$. (This follows from the description of categorical fibrations to ∞-categories in [Lur09a, Corollary 2.4.6.5].)

(iii) If $\mathcal{C}$ is an ∞-category and $\mathcal{D}$ is a subcategory if $\mathcal{C}$, let $\mathfrak{P}_{\mathcal{C},\mathcal{D}}^{\mathrm{coCart}}$ be the categorical pattern $(\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2, \emptyset)$. Then $(\mathcal{E}, T) \to (\mathcal{C}, \mathcal{D}_1)$ is $\mathfrak{P}_{\mathcal{C},\mathcal{D}}^{\mathrm{coCart}}$-fibrant if and only if $\mathcal{E}$ is an ∞-category, the map $\pi \colon \mathcal{E} \to \mathcal{C}$ is an inner fibration, and $\mathcal{E}$ has a π-coCartesian edge over every morphism in $\mathcal{D}$.

We will see more examples of categorical patterns in the next chapter.

Definition 2.1.12.5. Let $\mathbf{C}$ be a category with small colimits. A class S of morphisms in $\mathbf{C}$ is *weakly saturated* if it has the following properties:

(1) S is closed under pushouts along arbitrary morphisms in $\mathbf{C}$.

(2) S is closed under transfinite composition. More precisely, suppose α is an ordinal and $\{D_\beta\}_{\beta<\alpha}$ is a system of objects of $\mathbf{C}_{C/}$ indexed by α. For $\beta \le \alpha$ we let $D_{<\beta}$ be a colimit of $\{D_\gamma\}_{\gamma<\beta}$ in $\mathcal{C}_{C/}$. If for all $\beta < \alpha$ the map $D_{<\beta} \to D_\beta$ belongs to S, then the induced map $C \to D_{<\alpha}$ belongs to S.

(3) S is closed under retracts.

Definition 2.1.12.6. Let $\mathfrak{P} = (X, M, S, \{p_\alpha\colon K_\alpha^\triangleleft \to X\})$ be a categorical pattern. A morphism of marked simplicial sets over (X, M) is $\mathfrak{P}$-*anodyne* if it is contained in the smallest weakly saturated class of morphisms containing all morphisms of the following types:

(1) $(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \hookrightarrow (\Delta^2)^\sharp$, for every map $\Delta^2 \to X$ in S that takes every edge into M,

(2) $Q^\flat \hookrightarrow Q^\sharp$ where $Q = \Delta^0 \amalg_{\Delta^{\{0,2\}}} \Delta^3 \amalg_{\Delta^{\{1,3\}}} \Delta^0$ for any map $Q \to X$ that carries every edge of Q into M and every 2-simplex of Q into S,

(3) $\{0\}^\sharp \hookrightarrow (\Delta^1)^\sharp$ for every edge in M,

(4) $K_\alpha^\sharp \hookrightarrow (K_\alpha^\triangleleft)^\sharp$ for every α, where $K_\alpha^\triangleleft$ maps to X via p_α,

(5) $(\Lambda^n_0)^\flat \amalg_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \hookrightarrow (\Delta^n)^\flat \amalg_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp$ for every $n > 1$ and every map $\Delta^n \to X$ such that $\Delta^{\{0,1,n\}}$ belongs to S,

(6) $(\Lambda^n_i)^\flat \hookrightarrow (\Delta^n)^\flat$, for all $0 < i < n$ and all maps $\Delta^n \to X$,

(7) for every map $f\colon \Delta^n * K_\alpha \to X$ extending $p_\alpha\colon \{n\} * K_\alpha \to X$, the inclusion

$$(\partial\Delta^n * K_\alpha)^\flat \amalg_{(\{n\}*K_\alpha)^\flat} (\{n\} * K_\alpha)^\sharp \hookrightarrow (\Delta^n * K_\alpha)^\flat \amalg_{(\{n\}*K_\alpha)^\flat} (\{n\} * K_\alpha)^\sharp.$$

Proposition 2.1.12.7 ([Lur11, Proposition B.1.6])**.** Let $\mathfrak{P} = (X, M, S, \{p_\alpha\})$ be a categorical pattern. Then a marked simplicial set (Y, T) over (X, M) is $\mathfrak{P}$-fibrant if and only if it has the right lifting property with respect to $\mathfrak{P}$-anodyne maps.

Definition 2.1.12.8. Let $\mathfrak{P} = (X, M, S, \{p_\alpha\})$ be a categorical pattern. A morphism $f\colon \bar{Y} \to \bar{Z}$ of marked simplicial sets is a $\mathfrak{P}$-*equivalence* if for every $\mathfrak{P}$-fibrant object $\bar{W}$ over $\bar{X} = (X, M)$ the induced map

$$\mathrm{Map}^\sharp_{\bar{X}}(\bar{Z}, \bar{W}) \to \mathrm{Map}^\sharp_{\bar{X}}(\bar{Y}, \bar{W})$$

is a weak equivalence of Kan complexes.

Theorem 2.1.12.9 ([Lur11, Theorem B.0.19])**.** Let $\mathfrak{P} = (X, M, S, \{p_\alpha\})$ be a categorical pattern. Then there exists a left proper combinatorial simplicial model structure on the category $(\mathrm{Set}^+_\Delta)_{/(X,M)}$ such that:

(C) The cofibrations are the morphisms whose underlying morphisms of simplicial sets are monomorphisms.

(W) The weak equivalences are the $\mathfrak{P}$-equivalences.

(F) The fibrant objects are the $\mathfrak{P}$-fibrant objects.

We write $(\mathrm{Set}^+_\Delta)_{\mathfrak{P}}$ for $(\mathrm{Set}^+_\Delta)_{/(X,M)}$ equipped with this model structure.

Remark 2.1.12.10. Moreover, this model structure is enriched in the model category of marked simplicial sets — this follows from [Lur11, Remark B.2.5] (taking $\mathfrak{P}'$ to be the trivial categorical pattern on Δ^0).

Remark 2.1.12.11. Suppose $\mathfrak{P} = (X, M, S, \{p_\alpha\})$ is a categorical pattern, and let $\mathfrak{P}^-$ be the categorical pattern $(X, M, S, \emptyset)$. It follows from the proof of [Lur11, Theorem B.0.19] that the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{P}}$ is the left Bousfield localization of the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{P}^-}$ with respect to the generating $\mathfrak{P}$-anodyne maps of type (4) and (7).

Examples 2.1.12.12.

(i) If $\mathcal{C}$ is an ∞-category, the model category $(\mathrm{Set}_\Delta)^+_{\mathfrak{P}^{\mathrm{coCart}}_{\mathcal{C}}}$ is the coCartesian model structure on $(\mathrm{Set}^+_\Delta)_{/\mathcal{C}^\sharp}$. Thus the associated ∞-category is the ∞-category $\mathrm{CoCart}(\mathcal{C})$ of coCartesian fibration over $\mathcal{C}$, which is equivalent to $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$.

(ii) If $\mathcal{C}$ is an ∞-category, the model category $(\mathrm{Set}_\Delta)^+_{\mathfrak{P}^{\mathrm{eq}}_{\mathcal{C}}}$ is the over-category model structure on $(\mathrm{Set}^+_\Delta)_{/\mathcal{C}^\natural}$ from the model structure on Set^+_Δ. The associated ∞-category is thus the over-category $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$.

(iii) If $\mathcal{C}$ is an ∞-category and $\mathcal{D}$ is a subcategory of $\mathcal{C}$, the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{P}^{\mathrm{coCart}}_{\mathcal{C},\mathcal{D}}}$ gives an ∞-category of functors $\mathcal{E} \to \mathcal{C}$ that have coCartesian morphisms over the morphisms in $\mathcal{D}$; we write $\mathrm{CoCart}(\mathcal{C}, \mathcal{D})$ for this ∞-category.

Definition 2.1.12.13. Let $\mathfrak{P} = (X, M, S, \{p_\alpha\})$ and $\mathfrak{Q} = (Y, N, T, \{q_\beta\})$ be categorical patterns. A *morphism of categorical patterns* $f\colon \mathfrak{P} \to \mathfrak{Q}$ is a morphism of simplicial sets $f\colon X \to Y$ such that $f(M) \subseteq N$, $f(S) \subseteq T$, and for every α the composite

$$K_\alpha^\triangleleft \xrightarrow{p_\alpha} X \xrightarrow{f} Y$$

is in $\{q_\beta\}$.

Proposition 2.1.12.14 ([Lur11, Proposition B.2.9])**.** Let $f\colon \mathfrak{P} \to \mathfrak{Q}$ be a map of categorical patterns. Then composition with f induces a left Quillen functor

$$f_!\colon (\mathrm{Set}^+_\Delta)_{\mathfrak{P}} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{Q}}.$$

The right adjoint f^* is given by pullback along f.

Example 2.1.12.15. If $\mathfrak{P} = (X, M, S, \varnothing)$ is any categorical pattern with no limit diagrams, the map $X \to \Delta^0$ gives a map of categorical patterns $\mathfrak{P} \to \mathfrak{P}_0 := \mathfrak{P}^{\mathrm{eq}}_{\Delta^0}$ and so a colimit-preserving forgetful functor from the ∞-category associated to $(\mathrm{Set}^+_\Delta)_{\mathfrak{P}}$ to Cat_∞.

Remark 2.1.12.16. Under certain rather complicated conditions, the functor f^* is also a left Quillen functor, i.e. it has a right adjoint f_* that is a right Quillen functor — see [Lur11, Proposition B.4.1].

2.1.13 Some Technical Results

Here we collect a small number of results that do not fit anywhere else in our discussion. First we prove a characterization of certain colimits in relative functor categories; I thank Jacob Lurie for explaining the proof of this result to me.

Theorem 2.1.13.1. Let K be a weakly contractible simplicial set. Suppose $p\colon X \to S$ is a coCartesian fibration such that for all $s \in S$ the fibre X_s admits K-indexed colimits, and for all edges $f\colon s \to t$ in S the functor $f_!\colon X_s \to X_t$ preserves K-indexed colimits. Then for any map $g\colon T \to S$,

(i) the ∞-category $\mathrm{Fun}_S(T, X)$ admits K-indexed colimits,

(ii) a map $K^{\triangleright} \to \mathrm{Fun}_S(T, X)$ is a colimit diagram if and only if for all $t \in T$ the composite

$$K^{\triangleright} \to \mathrm{Fun}_S(T, X) \to X_{g(t)}$$

is a colimit diagram,

(iii) if E is a set of edges of T, the full subcategory of $\mathrm{Fun}_S(T, X)$ spanned by functors that take the edges in E to coCartesian edges of X is closed under K-indexed colimits in $\mathrm{Fun}_S(T, X)$.

Proof. The ∞-category $\mathrm{Fun}_S(T, X)$ is a fibre of the functor $p_*\colon \mathrm{Fun}(T, X) \to \mathrm{Fun}(T, S)$ induced by composition with p. The functor p_* is a coCartesian fibration by [Lur09a, Proposition 3.1.2.1]. Since the functors $f_!$ preserve K-indexed colimits, by [Lur09a, Proposition 4.3.1.10] a diagram $\bar{q}\colon K^{\triangleright} \to \mathrm{Fun}_S(T, X)$ is a colimit diagram if and only if the composite $\bar{q}'\colon K^{\triangleright} \to \mathrm{Fun}_S(T, X) \to \mathrm{Fun}(T, X)$ is a p_*-colimit diagram. By [Lur09a, Corollary 4.3.1.11], K-indexed p_*-colimits exist in $\mathrm{Fun}(T, X)$, which proves (i).

Moreover, a diagram in $\mathrm{Fun}(T, X)$ is a colimit diagram if and only if it is a p_*-colimit diagram and its image in $\mathrm{Fun}(T, S)$ is a colimit diagram. Since $\bar{q}'$ lands in one of the fibres of p_*, the projection to $\mathrm{Fun}(T, S)$ is constant, which means it is a colimit as K is weakly contractible. Thus $\bar{q}'$ is a p_*-colimit diagram if and only if it is a colimit diagram in $\mathrm{Fun}(T, X)$. By [Lur09a, Corollary 5.1.2.3] this means that $\bar{q}'$ is a colimit diagram if and only if for all $t \in T$ the induced maps $K^{\triangleright} \to X$ are colimit diagrams. A diagram in X is a colimit if and only if it is a p-colimit and the projection to S is a colimit. Since K is weakly contractible, applying [Lur09a, Proposition 4.3.1.10] we see that this is true if and only if the induced map $K^{\triangleright} \to X_{g(t)}$ is a colimit diagram in $X_{g(t)}$. This proves (ii).

Suppose $e\colon t \to t'$ is an edge of T and $q\colon K \to \mathrm{Fun}_S(T, X)$ is a diagram such that for all vertices $k \in K$ the functor $q(k)\colon T \to X$ takes e to a p-coCartesian edge of X. Let $\bar{q}\colon K^{\triangleright} \to \mathrm{Fun}_S(T, X)$ be a colimit diagram extending q. To prove (iii) we must show that the functor $\bar{q}(\infty)$ also takes e to a coCartesian edge of X. From our description of colimits in $\mathrm{Fun}_S(T, X)$ it follows that this is equivalent to showing that coCartesian edges of X are closed under colimits, which is true by Lemma 2.1.5.11. $\square$

Proposition 2.1.13.2. Let $\mathcal{I}$ be a category and $p\colon \mathcal{I} \to \mathrm{Cat}_\infty$ a functor. Let $\mathcal{D}$ be an ∞-category and $\eta\colon \mathcal{I} \times \Delta^1 \to \mathrm{Cat}_\infty$ a natural transformation from p to the constant functor at $\mathcal{D}$. Let $\mathcal{K} \to \mathcal{I}$ be a coCartesian fibration associated to p; the natural transformation η induces a map $q\colon \mathcal{K} \to \mathcal{D} \times \mathcal{I} \to \mathcal{D}$. Suppose each of the diagrams $\eta_\alpha\colon p(\alpha) \to \mathcal{D}$ has a colimit; by [Lur09a] there exists an (essentially unique) map $q_+\colon \mathcal{K}_+ \to \mathcal{D}$, where

$$\mathcal{K}_+ := \mathcal{K} \times \Delta^1 \amalg_{\mathcal{K} \times \{1\}} \mathcal{I},$$

that restricts to q on $\mathcal{K}$ and to a colimit of η_α on $p(\alpha)^{\triangleright} \simeq \mathcal{K}_+ \times_{\mathcal{I}} \{\alpha\}$. Then the maps $\mathcal{D}_{q/} \leftarrow \mathcal{D}_{q_+/} \to \mathcal{D}_{q_+|_{\mathcal{I}}/}$ are trivial fibrations.

In particular, we have equivalences $\operatorname{colim} q \simeq \operatorname{colim} q_+|_{\mathcal{I}} \simeq \operatorname{colim}_{\alpha \in \mathcal{I}} \operatorname{colim}_{p(\alpha)} \eta_\alpha$.

Proof. It follows from [Lur09a, Lemma 4.2.3.5] that the inclusion $\mathcal{I} \hookrightarrow \mathcal{K}_+$ is right anodyne, hence $\mathcal{D}_{q_+/} \to \mathcal{D}_{q_+|_{\mathcal{I}}/}$ is a trivial fibration. On the other hand, $\mathcal{D}_{q_+/} \to \mathcal{D}_{q/}$ is a trivial fibration since q_+ is clearly a left Kan extension of q along $\mathcal{K} \hookrightarrow \mathcal{K}_+$. $\square$

2.2 Other Higher-Categorical Structures

In this section we review some other higher-categorical structures that we will encounter, namely Segal spaces, double ∞-categories, $(\infty, 2)$-categories, and Θ_n-spaces. We will think of all of these as being constructed within an ambient theory of ∞-categories (rather than describing them as model categories, say).

2.2.1 Segal Spaces

Segal spaces are an alternative definition of $(\infty, 1)$-categories, introduced by Rezk [Rez01].

Definition 2.2.1.1. Suppose $\mathcal{C}$ is an ∞-category with finite limits. A *category object* in $\mathcal{C}$ is a simplicial object $F\colon \triangle^{\mathrm{op}} \to \mathcal{C}$ such that for each n the map

$$F_n \to F_1 \times_{F_0} \cdots \times_{F_0} F_1$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$ and $\{i\} \hookrightarrow [n]$ is an equivalence. A *Segal space* is a category object in the ∞-category $\mathcal{S}$ of spaces.

Let δ_n denote the simplicial space obtained from the simplicial set Δ^n by composing with the inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$. A simplicial space is then a Segal space if and only if it is local with respect to the map

$$\mathrm{seg}_n\colon \delta_n \to \delta_1 \amalg_{\delta_0} \cdots \amalg_{\delta_0} \delta_1.$$

Definition 2.2.1.2. Let $\mathrm{Seg}(\mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\triangle^{\mathrm{op}}, \mathcal{S})$ spanned by the Segal spaces; this is the localization of $\mathrm{Fun}(\triangle^{\mathrm{op}}, \mathcal{S})$ with respect to the maps seg_*.

Remark 2.2.1.3. Similarly, if $\mathcal{C}$ is a κ-presentable ∞-category, the ∞-category $\mathrm{Cat}(\mathcal{C})$ of category objects is the localization of $\mathrm{Fun}(\triangle^{\mathrm{op}}, \mathcal{C})$ with respect to the morphisms $\mathrm{seg}_n \otimes c$, where c is a κ-compact object of $\mathcal{C}$.

Definition 2.2.1.4. The inclusion $\mathrm{Gpd}(\mathcal{S}) \hookrightarrow \mathrm{Seg}(\mathcal{S})$ admits a right adjoint $\iota\colon \mathrm{Seg}(\mathcal{S}) \to \mathrm{Gpd}(\mathcal{S})$. We say a Segal space F is *complete* if the groupoid object ιF is constant.

Remark 2.2.1.5. By Lemma 2.1.10.4, a Segal space F is complete if and only if the map $\iota F(s^0)\colon \iota F[0] \to \iota F[1]$ is an equivalence.

Definition 2.2.1.6. Let j denote the inclusion $\{[0]\} \to \triangle^{\mathrm{op}}$. Composition with j gives a functor $\mathrm{Fun}(\triangle^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$, which has a right adjoint j_*, given by right Kan extension. It is clear that j_*X is a Segal space for all $X \in \mathcal{S}$. We write E^n for the Segal space $j_*\{0, \ldots, n\}$.

Proposition 2.2.1.7 (Rezk [Rez01, Proposition 6.4])**.** A Segal space is complete if and only if it is local with respect to the morphism $E^1 \to E^0$.

Definition 2.2.1.8. Let $\mathrm{CSS}(\mathcal{S})$ denote the full subcategory of $\mathrm{Seg}(\mathcal{S})$ spanned by the complete Segal spaces; by Proposition 2.2.1.7 this is the localization of $\mathrm{Seg}(\mathcal{S})$ with respect to the morphism $E^1 \to E^0$.

Theorem 2.2.1.9 (Joyal-Tierney [JT07])**.** The ∞-category $\mathrm{CSS}(\mathcal{S})$ is equivalent to Cat_∞.

Lemma 2.2.1.10. Suppose $X_\bullet$ is a Segal space. Then the following are equivalent:

(i) The functor $X_\bullet$ is constant.

(ii) The map $s_0\colon X_0 \to X_1$ is an equivalence.

Proof. This follows by induction using the Segal condition and the simplicial identitites. □

2.2.2 Double ∞-Categories and $(\infty, 2)$-Categories

Just as a double category is an internal category in the category of categories, so a double ∞-category should be an internal category in the ∞-category of ∞-categories:

Definition 2.2.2.1. A *double ∞-category* is a category object in Cat_∞. We write $\mathrm{Cat}(\mathrm{Cat}_\infty)$ for the full subcategory of $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ spanned by the double ∞-categories.

Remark 2.2.2.2. Using the equivalence $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{CoCart}(\mathbb{\Delta}^{\mathrm{op}})$, we can equivalently define a double ∞-category to be a coCartesian fibration $\mathcal{E} \to \mathbb{\Delta}^{\mathrm{op}}$, such that the functors

$$\mathcal{E}_{[n]} \to \mathcal{E}_{[1]} \times_{\mathcal{E}_{[0]}} \cdots \times_{\mathcal{E}_{[0]}} \mathcal{E}_{[1]}$$

induced by the morphisms $\{i, i+1\} \hookrightarrow [n]$ and $\{i\} \hookrightarrow [n]$ are equivalences.

Definition 2.2.2.3. A *double Segal space* is a category object in $\mathrm{Seg}(\mathcal{S})$, i.e. a bisimplicial space $\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ all of whose rows and columns are Segal spaces. We write $\mathrm{Cat}^2(\mathcal{S})$ for the ∞-category of double Segal spaces.

Using the equivalence $\mathrm{Cat}_\infty \simeq \mathrm{CSS}(\mathcal{S})$, we can also regard a double ∞-category as a category object in complete Segal spaces, i.e. a double Segal space all of whose rows are complete Segal spaces.

Definition 2.2.2.4. A double Segal space is *complete* if all its rows and columns are complete Segal spaces.

Lemma 2.2.2.5. The following are equivalent for a double ∞-category $\mathcal{C}_\bullet$:

(i) $\mathcal{C}_\bullet$ corresponds to a complete double Segal space under the equivalence $\mathrm{Cat}(\mathrm{Cat}_\infty) \simeq \mathrm{Cat}(\mathrm{CSS})$.

(ii) $\mathrm{Map}(\Delta^n, \mathcal{C}_\bullet)$ is a complete Segal space for all n.

(iii) $\mathcal{C}_\bullet$ is local with respect to $E^1 \times \Delta^n \to \Delta^n$ for all n.

If $\mathcal{C}_\bullet$ satisfies these equivalent conditions, we say that $\mathcal{C}_\bullet$ is a *complete* double ∞-category. Write CDbl_∞ for the full subcategory of $\mathrm{Cat}(\mathrm{Cat}_\infty)$ spanned by the complete double ∞-categories; this is an accessible localization of $\mathrm{Cat}(\mathrm{Cat}_\infty)$. We claim that CDbl_∞ is the "correct" ∞-category of double ∞-categories, but will not justify this further here.

Lemma 2.2.2.6. Suppose $\mathcal{C}_\bullet$ is a double ∞-category. Then $\mathcal{C}_\bullet$ is complete if and only if $\mathrm{Map}(\Delta^n, \mathcal{C}_\bullet)$ is a complete Segal space for $n = 0, 1$.

Proof. Write $\mathcal{C}_\bullet^{\Delta^n}$ for the Segal space $\mathrm{Map}(\Delta^n, \mathcal{C}_\bullet)$. Suppose we know $\mathcal{C}_\bullet^{\Delta^n}$ is a complete Segal space, where $n \geq 1$. Then the pushout diagram of ∞-categories

$$\begin{tikzcd}
\Delta^{\{n\}} \arrow[r] \arrow[d] & \Delta^{\{n,n+1\}} \arrow[d] \\
\Delta^n \arrow[r] & \Delta^{n+1}
\end{tikzcd}$$

induces a pullback diagram of Segal spaces

$$\begin{tikzcd}
\mathcal{C}_\bullet^{\Delta^{n+1}} \arrow[r] \arrow[d] & \mathcal{C}_\bullet^{\Delta^n} \arrow[d] \\
\mathcal{C}_\bullet^{\Delta^1} \arrow[r] & \mathcal{C}_\bullet^{\Delta^0}.
\end{tikzcd}$$

This means that in the diagram

$$\begin{tikzcd}
\mathcal{C}_0^{\Delta^{n+1}} \arrow[r] \arrow[d] & \mathcal{C}_0^{\Delta^n} \times_{\mathcal{C}_0^{\Delta^0}} \mathcal{C}_0^{\Delta^1} \arrow[d] \\
\mathrm{Map}(E^1, \mathcal{C}_\bullet^{\Delta^{n+1}}) \arrow[r] & \mathrm{Map}(E^1, \mathcal{C}_\bullet^{\Delta^n}) \times_{\mathrm{Map}(E^1, \mathcal{C}_\bullet^{\Delta^0})} \mathrm{Map}(E^1, \mathcal{C}_\bullet^{\Delta^1})
\end{tikzcd}$$

the horizontal maps are equivalences. But the right vertical map is also an equivalence, since $\mathcal{C}_\bullet^{\Delta^k}$ is by assumption complete for $k = 0, 1, n$. Thus $\mathcal{C}_\bullet^{\Delta^{n+1}}$ is also complete. By induction, $\mathcal{C}_\bullet^{\Delta^k}$ is therefore complete for all k, i.e. $\mathcal{C}_\bullet$ is a complete double ∞-category. $\square$

Just as we can think of 2-categories as a special kind of double category, we can think of $(\infty, 2)$-categories as a special kind of double ∞-category — this gives Barwick's notion of a *2-fold Segal space*:

Definition 2.2.2.7. A double Segal space X is a *2-fold Segal space* if the 0th row X_0 is constant. Write $\mathrm{Seg}^2(\mathcal{S})$ for the full subcategory of $\mathrm{Cat}^2(\mathcal{S})$ spanned by the 2-fold Segal spaces.

Definition 2.2.2.8. A 2-fold Segal space X is *complete* if all its rows X_i are complete Segal spaces and the 0th column $X_{\bullet,0}$ is a complete Segal space. We write $\mathrm{CSS}^2(\mathcal{S})$ for the full subcategory of $\mathrm{Seg}^2(\mathcal{S})$ spanned by the complete 2-fold Segal spaces.

The ∞-category $\mathrm{CSS}^2(\mathcal{S})$ is the "correct" ∞-category of $(\infty, 2)$-categories. Under the equivalence $\mathrm{Cat}(\mathrm{Cat}_\infty) \simeq \mathrm{Cat}(\mathrm{CSS}(\mathcal{S}))$ it is clear that complete 2-fold Segal spaces correspond to double ∞-categories $\mathcal{C}_\bullet$ such that $\mathcal{C}_0$ is a space and $\mathcal{C}_\bullet^{\Delta^0}$ is a complete Segal space. We write $\mathrm{Cat}_{(\infty,2)}$ for the full subcategory of $\mathrm{Cat}(\mathrm{Cat}_\infty)$ spanned by these objects.

Lemma 2.2.2.9. Let $X_{\bullet\bullet}$ be a double Segal space. Suppose $x, y \in X_{00}$ and $\phi \in X_{11}$ satisfies $d_0^h \phi \simeq s_0^v x \in X_{10}$ and $d_1^h \phi \simeq s_0^v y$ (where the superscripts h and v refer to the horizontal and vertical simplicial structure maps, respectively). Then ϕ is an equivalence in the Segal space $X_{1\bullet}$ if and only if ϕ is an equivalence in $X_{\bullet 1}$ and $d_i^v \phi$ is an equivalence in $X_{0,\bullet}$ for $i = 0, 1$.

Proof. Write $f = d_1 \phi$ and $g = d_0 \phi$. Suppose ϕ is an equivalence in $X_{1,\bullet}$ and let ψ be an inverse. It is clear that $f^{-1} = d_1 \psi$ and $g^{-1} = d_0 \psi$ are inverses of f and g, respectively. Consider the object of X_{23} represented by the diagram

$$\begin{array}{|c|c|c|} \hline \mathrm{id}_f & \mathrm{id}_{f^{-1}} & \phi \\ \hline \mathrm{id}_f & \psi & \mathrm{id}_g \\ \hline \end{array}.$$

Let α be a composite of the bottom row. If we compose the diagram horizontally and then vertically it is clear that we get the vertical composite of ϕ and α. On the other hand, if we compose first vertically and then horizontally we get the horizontal composite of ψ and ϕ, which is the identity. Since these give equivalent objects of X_{11} we see that the vertical composite of ϕ and α is id_f. Similarly, considering the object represented by

id_f	ψ	id_g
ϕ	$\mathrm{id}_{g^{-1}}$	id_g

we see that the vertical composite of α and ϕ is id_g. Thus α is an inverse for ϕ with respect to vertical composition, i.e. ϕ is an equivalence in $X_{1\bullet}$.

Now suppose ϕ is an equivalence in $X_{\bullet 1}$ such that f and g are equivalences. Let ψ be a (vertical) inverse for ϕ and choose inverses f^{-1} and g^{-1} of f and g. Considering the objects

ϕ	$\mathrm{id}_{g^{-1}}$	id_g	$\mathrm{id}_{f^{-1}}$
id_g	$\mathrm{id}_{g^{-1}}$	ψ	$\mathrm{id}_{f^{-1}}$

$\mathrm{id}_{g^{-1}}$	ψ	$\mathrm{id}_{f^{-1}}$	id_f
$\mathrm{id}_{g^{-1}}$	id_f	$\mathrm{id}_{f^{-1}}$	ϕ

of X_{24} we see that the horizontal composite $\mathrm{id}_{f^{-1}} \circ \psi \circ \mathrm{id}_{g^{-1}}$ is a horizontal inverse of ϕ. □

Lemma 2.2.2.10. Suppose $\mathcal{C}_\bullet$ is a double ∞-category such that $\mathcal{C}_0$ is a space and $\mathcal{C}_\bullet^{\Delta^0}$ is complete. Then a morphism ϕ in $\mathcal{C}_1$ is an equivalence in $\mathcal{C}_\bullet^{\Delta^1}$ if and only if $d_i\phi$, $i = 0, 1$, are equivalences in $\mathcal{C}_\bullet^{\Delta^0}$ and ϕ is an equivalence in $\mathcal{C}_1$.

Proposition 2.2.2.11. $\mathrm{Cat}_{(\infty,2)}$ is a full subcategory of CDbl_∞, i.e. if $\mathcal{C}_\bullet$ is a double ∞-category such that $\mathcal{C}_0$ is a space and $\mathcal{C}_\bullet^{\Delta^0}$ is complete, then $\mathcal{C}_\bullet$ is a complete double ∞-category.

Proof. By Lemma 2.2.2.6 it suffices to prove that $\mathcal{C}_\bullet^{\Delta^1}$ is a complete Segal space. Thus we need to show that the morphism $(\mathcal{C}_1^{\Delta^1})^{\mathrm{eq}} \to \mathcal{C}_0^{\Delta^1}$, where $(\mathcal{C}_1^{\Delta^1})^{\mathrm{eq}}$ denotes the subspace of $\mathcal{C}_1^{\Delta^1}$ consisting of the components corresponding to equivalences in the Segal space $\mathcal{C}_\bullet^{\Delta^1}$, is an equivalence of spaces. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Map}(\Delta^1, \mathcal{C}_0) & \longrightarrow & \mathrm{Map}(\Delta^1, \mathcal{C}_1)^{\mathrm{eq}} \\ \downarrow & & \downarrow \\ \mathrm{Map}(\Delta^0, \mathcal{C}_0) & \longrightarrow & \mathrm{Map}(\Delta^0, \mathcal{C}_1)^{\mathrm{eq}}. \end{array}$$

Here the left vertical map is an equivalence since $\mathcal{C}_0$ is a space and the bottom horizontal map is an equivalence since $\mathcal{C}_\bullet^{\Delta^0}$ is complete. To prove that the top horizontal map is an equivalence it therefore suffices to show that the right vertical map is an equivalence.

Observe that $\mathrm{Map}(\Delta^1, (\mathcal{C}_1^{\Delta^0})^{\mathrm{eq}})$ is a full subcategory of $\mathrm{Map}(\Delta^1, \mathcal{C}_1)$, as is $\mathrm{Map}(\Delta^1, \mathcal{C}_1)^{\mathrm{eq}}$. By Lemma 2.2.2.10 these subcategories have the same objects, so we get an equivalence $\mathrm{Map}(\Delta^1, \mathcal{C}_1)^{\mathrm{eq}} \xrightarrow{\sim} \mathrm{Map}(\Delta^1, (\mathcal{C}_1^{\Delta^0})^{\mathrm{eq}})$. Since $(\mathcal{C}_1^{\Delta^0})^{\mathrm{eq}}$ is a space, it follows that $\mathrm{Map}(\Delta^1, \mathcal{C}_1^{\mathrm{eq}}) \simeq (\mathcal{C}_1^{\Delta^0})^{\mathrm{eq}}$, which completes the proof. □

The automorphism of double Segal spaces that swaps the two simplicial directions corresponds to an automorphism of CDbl_∞. Under this automorphism, the double ∞-

categories corresponding to complete 2-fold Segal spaces get taken to complete double ∞-categories $\mathcal{C}_\bullet$ such that $\mathcal{C}_\bullet^{\Delta^0}$ is constant. Thus, these give yet another model for $(\infty,2)$-categories. We will now prove a criterion for a double ∞-category to be an $(\infty,2)$-category of this kind:

Proposition 2.2.2.12. Suppose $\mathcal{C}_\bullet$ is a double ∞-category such that $s_0\colon \mathcal{C}_0 \to \mathcal{C}_1$ is essentially surjective (hence an isomorphism on π_0 by the simplicial identities) and $\mathcal{C}^{\Delta^1}$ is complete. Let $\mathcal{C}_1^{\mathrm{eq}}$ be the subcategory of $\mathcal{C}_1$ whose morphisms are those that are equivalences in $\mathrm{Map}(\Delta^1,\mathcal{C}_\bullet)$. Then $s_0\colon \mathcal{C}_0 \to \mathcal{C}_1^{\mathrm{eq}}$ is an equivalence of ∞-categories.

Proof. Since s_0 is essentially surjective by assumption, it suffices to prove that it is fully faithful, i.e. that for all $x,y \in \mathcal{C}_0$ the induced map $\mathcal{C}_0(x,y) \to \mathcal{C}_1^{\mathrm{eq}}(x,y)$ is an equivalence. We can identify $\mathcal{C}_1^{\mathrm{eq}}(x,y)$ with the fibre $\mathrm{Map}(\Delta^1,\mathcal{C}_1)^{\mathrm{eq}}_{x,y}$ of the projection

$$(\mathcal{C}_1^{\Delta^1})^{\mathrm{eq}} \to (\mathcal{C}_1^{\Delta^0})^{\times 2}$$

at (s_0x, s_0y), where $(\mathcal{C}_1^{\Delta^1})^{\mathrm{eq}}$ denotes the subspace of $\mathcal{C}_1^{\Delta^1}$ whose components correspond to the equivalences in $\mathcal{C}_\bullet^{\Delta^1}$. Now since $\mathcal{C}_\bullet^{\Delta^1}$ is a complete Segal space, the map s_0 induces an equivalence $\mathcal{C}_0^{\Delta^1} \xrightarrow{\sim} (\mathcal{C}_1^{\Delta^1})^{\mathrm{eq}}$. Passing to fibres over $(x,y) \in (\mathcal{C}_0^{\Delta^0})^{\times 2}$ this shows that s_0 indeed induces an equivalence $\mathcal{C}_0(x,y) \xrightarrow{\sim} \mathcal{C}_1^{\mathrm{eq}}(x,y)$. □

Corollary 2.2.2.13. Suppose $\mathcal{C}_\bullet$ is a double ∞-category such that $s_0\colon \mathcal{C}_0 \to \mathcal{C}_1$ is essentially surjective and $\mathcal{C}^{\Delta^1}$ is complete. Then the Segal space $\mathcal{C}_\bullet^{\Delta^0}$ is constant.

Proof. By Lemma 2.2.1.10 it suffices to show that $\mathcal{C}_0^{\Delta^0} \to \mathcal{C}_1^{\Delta^0}$ is an equivalence of spaces. By Lemma 2.2.2.9 the inclusion $\mathcal{C}_1^{\Delta^0} \to \mathcal{C}_1$ factors through $\mathcal{C}_1^{\mathrm{eq}}$, hence we have an equivalence $\mathcal{C}_1^{\Delta^0} \xrightarrow{\sim} (\mathcal{C}_1^{\Delta^0})^{\mathrm{eq}}$. But the composite $\mathcal{C}_0^{\Delta^0} \to (\mathcal{C}_1^{\Delta^0})^{\mathrm{eq}}$ is an equivalence by Proposition 2.2.2.12, hence by the 2-out-of-3 property so is the map $\mathcal{C}_0^{\Delta^0} \to \mathcal{C}_1^{\Delta^0}$. □

Suppose X is a complete double Segal space. Then we can extract two complete 2-fold Segal spaces from X, by restricting to the subobject lying over the constant part of the 0th row or column. In other words, we can extract a vertical and a horizontal $(\infty,2)$-category from a double ∞-category.

Definition 2.2.2.14. Let $\mathrm{Vert},\mathrm{Hor}\colon \mathrm{CDbl}_\infty \to \mathrm{Cat}_{(\infty,2)}$ be the corresponding functors on complete double ∞-categories.

There are many other models for $(\infty,2)$-categories in the literature. In this thesis we will also make use of *marked simplicial categories*, i.e. categories enriched in the model category Set_Δ^+ of marked simplicial sets; see [Lur09b] for a comparison of these with complete 2-fold Segal spaces and other models, and [BSP11] for axioms characterizing the ∞-category of $(\infty,2)$-categories.

2.2.3 Θ_n-Spaces

We now briefly review the theory of Θ_n*-spaces*, which give a model for (∞,n)-categories. These were introduced by Rezk [Rez10] (but our discussion is also based on the summary given in [BSP11]). We first review the definition of the categories Θ_n — these were originally introduced by Joyal, but we use the inductive definition due to Berger [Ber07]:

Definition 2.2.3.1. Let $\Theta_0 = *$, and for $n > 0$ define the category Θ_n inductively as follows:

- objects of Θ_n are of the form $[n](X_1, \ldots, X_n)$, where $[n] \in \Delta$ and $X_i \in \Theta_{n-1}$;
- a morphism $[n](X_1, \ldots, X_n) \to [m](Y_1, \ldots, Y_m)$ consists of a morphism $\phi \colon [n] \to [m]$ in Δ and morphisms $\psi_{ij} \colon X_i \to Y_j$ where $0 < i \leq m$ and $\phi(i-1) < j \leq \phi(i)$.

Composition is defined in the obvious way.

The category Θ_n can also be regarded as a full subcategory of the category of strict n-categories spanned by certain free strict n-categories (cf. [Ber07, §3]).

Definition 2.2.3.2. We define the following important functors between Θ_n's:

(i) The functor $j_n \colon \Theta_{n-1} \to \Theta_n$ corresponds to the inclusion of $(n-1)$-categories into n-categories; we define it inductively by letting $j_1 \colon * = \Theta_0 \to \Theta_1 = \Delta$ be the inclusion of the object $[0]$ and defining

$$j_n([m](X_1, \ldots, X_m)) = [m](j_{n-1}X_1, \ldots, j_{n-1}X_m)$$

for $n > 1$.

(ii) The functor $\sigma_n \colon \Theta_{n-1} \to \Theta_n$ corresponds to "suspending" an $(n-1)$-category to an n-category with two objects and that $(n-1)$-category as the morphisms between them. More precisely,

$$\sigma_n(X) = [1](X).$$

(Notice that $\sigma_n j_{n-1} = j_n \sigma_{n-1}$.)

(iii) The functor $p_n \colon \Theta_n \to \Theta_{n-1}$ corresponds to "collapsing" the n-morphisms in an n-category to produce an $(n-1)$-category. More precisely, we define $p_1 \colon \Delta^{\mathrm{op}} = \Theta_1 \to \Theta_0 = *$ to be the unique functor to the final object, and set

$$p_n([m](X_1, \ldots, X_m)) = [m](p_{n-1}X_1, \ldots, p_{n-1}X_m)$$

for $n > 1$.

Definition 2.2.3.3. The k-cell C_k^n (or just C_k) in Θ_n ($k = 0, \ldots, n$) is defined by $C_k = j_n C_k$ for $k < n$ and $C_n = \sigma_n C_{n-1}$ (with C_0^0 being the unique object of Θ_0). Equivalently we have $C_k^n = j^{n-k} \sigma^k C_0^0$. The k-cell C_k corresponds to the "free k-morphism".

Definition 2.2.3.4. Recall that a morphism $\phi \colon [n] \to [m]$ in Δ is *inert* if it is the inclusion of a subinterval of $[m]$, i.e. if $\phi(i) = \phi(0) + i$ for all i. By induction, we define a morphism $(\phi, \psi_{ij}) \colon [n](X_1, \ldots, X_n) \to [m](Y_1, \ldots, Y_m)$ in Θ_k to be *inert* if $\phi \colon [n] \to [m]$ is inert, and $\psi_{i\phi(i)} \colon X_i \to Y_{\phi(i)}$ is inert for each $i = 1, \ldots, n$. Let $\mathbb{G}_n$ denote the subcategory of Θ_n with objects the cells C_0, …, C_n and morphisms the inert morphisms between these. For $X \in \Theta_n$, we write $(\mathbb{G}_n)_{/X}$ for the full subcategory of $\mathbb{G}_n \times_{\Theta_n} (\Theta_n)_{X/}$ spanned by the inert morphisms $C_k \to X$.

Definition 2.2.3.5. Let $y \colon \Theta_n \to \mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$ denote the Yoneda embedding. For X in Θ_n, the *Segal morphism* seg_X in $\mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$ is the obvious morphism

$$\operatorname*{colim}_{C_k \to X \in (\mathbb{G}_n)_{/X}} y(C_k) \to y(X).$$

We say a presheaf $\mathcal{F} \in \mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$ is a *Θ_n-space* if it is local with respect to the Segal morphisms seg_X for all $X \in \Theta_n$, i.e. if the natural map

$$\mathcal{F}(X) \to \lim_{C_k \to X \in (\mathbb{G}_n)_{/X}} \mathcal{F}(C_k)$$

is an equivalence for all $X \in \Theta_n$. We write $\mathrm{Seg}_{\Theta_n}(\mathcal{S})$ for the full subcategory of $\mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$ spanned by the Θ_n-spaces; this is an accessible localization of $\mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$ and so is a presentable ∞-category.

Remark 2.2.3.6. If $n = 1$, a Θ_1-space is precisely a Segal space.

Definition 2.2.3.7. Composition with $j \colon \Theta_{n-1} \to \Theta_n$ gives a functor

$$j^* \colon \mathrm{Seg}_{\Theta_n}(\mathcal{S}) \to \mathrm{Seg}_{\Theta_{n-1}}(\mathcal{S});$$

this corresponds to taking the underlying $(\infty, n-1)$-category of an (∞, n)-category. The functor j^* has left and right adjoints $j_!$ and j_*, given by left and right Kan extension, respectively (it is easy to see that this preserves the Segal conditions). The functor j_* freely adds n-morphisms between all parallel $(n-1)$-morphisms, while the functor $j_!$ gives the inclusion of $(\infty, n-1)$-categories into (∞, n)-categories. Similarly, the functor $p \colon \Theta_n \to \Theta_{n-1}$ induces $p^* \colon \mathrm{Seg}_{\Theta_{n-1}}(\mathcal{S}) \to \mathrm{Seg}_{\Theta_n}(\mathcal{S})$ with left adjoint $p_!$. Since $p \circ j = \mathrm{id}_{\Theta_{n-1}}$ we have $j^* p^* \simeq \mathrm{id}$.

Definition 2.2.3.8. If $X \in \Theta_n$, let X^* denote the Θ_{n-1}-space defined by

$$X^*(Y) = \mathrm{Hom}_{\Theta_n}(jY, X).$$

We write E^X for the Θ_n-space $j_* X^*$, and if $\mathcal{F}$ is a Θ_n-space we write $\iota_\bullet \mathcal{F} \colon \Theta_n^{\mathrm{op}} \to \mathcal{S}$ for the functor $\iota_X \mathcal{F} := \mathrm{Map}(E^X, \mathcal{F})$, and we define $\iota \mathcal{F} := p_! \iota_\bullet \mathcal{F}$.

Lemma 2.2.3.9. $E^{jX} \simeq yjX$ for $X \in \Theta_{n-1}$. Thus $j^* \iota_\bullet \mathcal{F} \simeq j^* \mathcal{F}$ for any Θ_n-space $\mathcal{F}$.

Conjecture 2.2.3.10. The functor $E^{(-)} \colon \Theta_n \to \mathrm{Seg}_{\Theta_n}(\mathcal{S})$ is a co-Θ_n-Segal object, i.e.

$$\operatorname*{colim}_{C_k \to X \in (\mathbb{G}_n)_{/X}} E^{C_k} \to E^X$$

is an equivalence for all X. Moreover, the cosimplicial object $E^{\sigma^{n-1}(-)}$ is a cogroupoid object.

Corollary 2.2.3.11. If $\mathcal{F}$ is a Θ_n-space, then so is $\iota_\bullet \mathcal{F}$.

Definition 2.2.3.12. A Θ_n-space X is *complete* if the natural map

$$j^* X \simeq j^* \iota_\bullet X \to j^* p^* p_! \iota_\bullet X \simeq p_! \iota_\bullet X = \iota X$$

is an equivalence, and the Θ_{n-1}-space $j^* X$ is complete. (We define all Θ_0-spaces to be complete.) We write $\mathrm{CSS}_{\Theta_n}(\mathcal{S})$ for the full subcategory of $\mathrm{Seg}_{\Theta_n}(\mathcal{S})$ spanned by the complete Θ_n-spaces.

Definition 2.2.3.13. The *free k-equivalence* $\mathrm{Eq}_k \in \mathrm{Seg}_{\Theta_n}(\mathcal{S})$ ($k = 1, \ldots, n$) is defined by $\mathrm{Eq}_n := j_*(C_n)^*$ and $\mathrm{Eq}_k := j_! \mathrm{Eq}_k$ for $k < n$.

Lemma 2.2.3.14. $\mathrm{Eq}_k \simeq \sigma_! \mathrm{Eq}_{k-1}$ for $k > 1$. In particular, $\mathrm{Eq}_k \simeq j_!^{n-k} \sigma_!^k \mathrm{Eq}_1$.

Definition 2.2.3.15. Define morphisms $\epsilon_k \colon \mathrm{Eq}_k \to yC_{k-1}$ by letting

$$\epsilon_n \colon j_*(C_n)^* \to j_*(C_{n-1})^* = yC_{n-1}$$

be induced by the unique map $C_n \to C_{n-1}$, and $\epsilon_k := j_! \epsilon_{k-1}$ for $k < n$. Equivalently, let ϵ_1 be the unique morphism from Eq_1 to the final object C_0 and let $\epsilon_k := \sigma_! \epsilon_{k-1} = \sigma_!^{k-1} \epsilon_1$ for $k > 1$.

Proposition 2.2.3.16. A Θ_n-space is complete if and only if it is local with respect to the morphisms ϵ_k, $k = 1, \ldots, n$.

Sketch Proof. Let us first show that a Θ_n-space $\mathcal{F}$ is local with respect to ϵ_n if and only if the morphism $j^*\mathcal{F} \to \iota\mathcal{F}$ is an equivalence. From the Segal conditions it is easy to see that a morphism of Θ_{n-1}-spaces $\phi \colon \mathcal{G} \to \mathcal{H}$ is an equivalence if and only if $\phi(C_k) \colon \mathcal{G}(C_k) \to \mathcal{H}(C_k)$ is an equivalence of spaces for $k = 0, \ldots, n-1$. Observe that for any Θ_n-space $\mathcal{F}$, for $k < n-1$ the space $\iota\mathcal{F}(C_k) = p_! \iota_\bullet \mathcal{F}(C_k)$ is equivalent to $\iota_\bullet \mathcal{F}(C_k) \simeq \iota_{C_k}\mathcal{F} \simeq \mathcal{F}(C_k)$ since C_k is a final object of $\Theta_n \times_{\Theta_{n-1}} (\Theta_{n-1})_{/C_k}$ (for $k = n-1$ this is not the case, since e.g. C_n gives two non-equivalent objects of this category). Thus the map $j^*\mathcal{F} \to \iota\mathcal{F}$ is an equivalence if and only if $\mathcal{F}(C_{n-1}) \to (\iota\mathcal{F})(C_{n-1}) = \mathrm{colim}_{(X, pX \to C_{n-1})} \mathcal{F}(X)$ is an equivalence.

Now consider $\sigma^{n-1} \colon \mathbb{\Delta} = \Theta_1 \to \Theta_n$; this gives a cofinal map $\mathbb{\Delta} \to \Theta_n \times_{\Theta_{n-1}} (\Theta_{n-1})_{/C_k}$. Thus it suffices to show that

$$\mathcal{F}(C_{n-1}) \to |\iota_\bullet \mathcal{F}(\sigma^{n-1}[\bullet])| \simeq |\mathrm{Map}(E^{\sigma^{n-1}[\bullet]}, \mathcal{F})|$$

is an equivalence if and only if $\mathcal{F}$ is local with respect to ϵ_n. But $E^{\sigma^{n-1}[\bullet]}$ is a cogroupoid object, so by Lemma 2.1.10.4 this morphism is an equivalence if and only if $\mathcal{F}(C_{n-1}) \simeq \mathrm{Map}(yC_{n-1}, \mathcal{F}) \to \mathrm{Map}(E^{\sigma^{n-1}[1]}, \mathcal{F}) \simeq \mathrm{Map}(\mathrm{Eq}_n, \mathcal{F})$ is an equivalence, i.e. if and only if $\mathcal{F}$ is local with respect to ϵ_n.

For $k < n$ observe that $j^*\mathcal{F}$ is local with respect to ϵ_k if and only if $\mathcal{F}$ is local with respect to $j_! \epsilon_k = \epsilon_k$, so by induction we conclude the $\mathcal{F}$ is complete if and only if it is local with respect to ϵ_k for $k = 1, \ldots, n$. □

Definition 2.2.3.17. If $\mathcal{F}$ is a Θ_n-space and $x, y \in \mathcal{F}(C_{n-1})$, write $\mathcal{F}(x, y)$ for the fibre of $\mathcal{F}(C_n)$ at $(x, y) \in \mathcal{F}(C_{n-1}) \times \mathcal{F}(C_{n-1})$. A morphism of Θ_n-spaces $\phi \colon \mathcal{F} \to \mathcal{G}$ is *fully faithful* if for all $x, y \in \mathcal{F}(C_{n-1})$ the morphism $\mathcal{F}(x, y) \to \mathcal{G}(\phi(x), \phi(y))$ is an equivalence. We say that ϕ is *fully faithful and essentially surjective* if ϕ is fully faithful and the morphism of Θ_{n-1}-spaces $\iota\phi \colon \iota\mathcal{F} \to \iota\mathcal{G}$ is fully faithful and essentially surjective. (We say a morphism of Θ_0-spaces is fully faithful and essentially surjective if and only if it is an equivalence.)

Lemma 2.2.3.18. A morphism of complete Θ_n-spaces is fully faithful and essentially surjective if and only if it is an equivalence.

Proof. Observe that a morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ of Θ_n-spaces is an equivalence if and only if it is fully faithful and $j^*\phi \colon j^*\mathcal{F} \to j^*\mathcal{G}$ is an equivalence of Θ_{n-1}-spaces. If $\mathcal{F}$ and $\mathcal{G}$ are complete it follows that ϕ is an equivalence if and only if ϕ is fully faithful and $\iota\phi$ is an equivalence. Since $\iota\mathcal{F}$ and $\iota\mathcal{G}$ are by assumption also complete, by induction we conclude that ϕ is an equivalence if and only if it is fully faithful and essentially surjective. □

Conjecture 2.2.3.19. The fully faithful and essentially surjective morphisms between Θ_n-spaces are precisely the morphisms in the saturated class generated by $\epsilon_1, \ldots, \epsilon_n$.

Chapter 3

∞-Operads over Operator Categories

In this chapter we indicate how to generalize Lurie's theory of ∞-operads to the setting of Barwick's operator categories. These are categories that can be used to parametrize multiplicative structures; apart from symmetric and non-symmetric ∞-operads, we are particularly interested in the interpolating ∞-operads with $\mathbb{E}_n$-symmetry, which we will use to define enriched (∞, n)-categories in Chapter 5.

In §3.1 we review the theory of operator categories and the 1-categorical notions of operads, monoids, and monoidal categories over an operator category. Then in §3.2 we define ∞-operads and monoidal ∞-categories over operator categories, and in §3.3 we describe some results for non-symmetric ∞-operads that we unfortunately do not yet know how to generalize to more general ∞-operads.

3.1 Review of Operator Categories

In this section we summarize part of Clark Barwick's theory of *operator categories*. Much of this material has now appeared in [Bar13]; the remainder (possibly excepting the rather trivial material in §3.1.7 and §3.1.8) is based either on earlier preprints or on conversations with Barwick and Chris Schommer-Pries. Since much of this section is only intended to motivate our definitions in §3.2 of ∞-categorical generalizations of the concepts we discuss here, we have often omitted proofs and even details of some definitions.

3.1.1 Basic Definitions and Examples

An *operator category* is a category that can be thought of as parametrizing a type of multiplicative structure. The definition is simple:

Definition 3.1.1.1. An *operator category* is a small category Φ that

(i) is *locally finite*, i.e. for all $I, J \in \Phi$ the set $\mathrm{Hom}_\Phi(I, J)$ is finite,

(ii) has a terminal object $*$,

(iii) has *fibres* J_i for every morphism $J \to I$ at all points $i\colon * \to I$, i.e. the pullbacks

$$\begin{tikzcd} J_i \arrow[r] \arrow[d] & J \arrow[d] \\ * \arrow[r, "i"'] & I \end{tikzcd}$$

exist.

Examples 3.1.1.2. The following are all operator categories:

(1) The trivial one-object category $\mathbf{0}$; this parametrizes trivial multiplicative structures.

(2) The category $\mathbb{O}$ of finite ordered sets (possibly empty). This parametrizes associative monoids, monoidal categories, and non-symmetric operads.

(3) The category $\mathbb{F}$ of finite sets. This parametrizes commutative monoids, symmetric monoidal categories, and symmetric operads.

The basic notion of a morphism between operator categories is an *admissible functor*:

Definition 3.1.1.3. If Φ and Ψ are operator categories, an *admissible functor* $F\colon \Phi \to \Psi$ is a functor that preserves the terminal object and all fibres.

However, for many purposes it is better to consider a more restricted class of morphisms, the *operator morphisms*:

Definition 3.1.1.4. Suppose Φ is an operator category. If I is an object of Φ, we write $|I|$ for the set $\mathrm{Hom}_\Phi(*, I)$. We say an admissible functor $F\colon \Phi \to \Psi$ is an *operator morphism* if the map $|I| \to |F(I)|$ is a bijection for all $I \in \Phi$.

Remark 3.1.1.5. Any admissible functor $F\colon \Phi \to \Psi$ such that $|I| \to |F(I)|$ is *surjective* for all $I \in \Phi$ is necessarily an operator morphism (cf. [Bar13, Proposition 1.8]).

Example 3.1.1.6. For any operator category Φ, the functor $|{-}|$ gives an operator morphism $\Phi \to \mathbb{F}$. This is the unique operator morphism from Φ to $\mathbb{F}$, and we will also denote it by $u^\Phi\colon \Phi \to \mathbb{F}$.

Remark 3.1.1.7. Below, in §3.1.8, we will see that certain *subcategories* of operator categories also play an interesting role, despite the inclusions not being admissible functors.

3.1.2 Wreath Products

The *wreath product* of operator categories gives a monoidal structure on the category of operator categories and operator morphisms.

Definition 3.1.2.1. Let Ψ be an operator category, and let $n_\Psi\colon \mathbb{F}^{\mathrm{op}} \to \mathrm{Cat}$ be the functor that sends a finite set S to $\mathrm{Fun}(S, \Psi) \cong \Psi^{\times |S|}$. If Φ is another operator category, composing with the unique operator morphism $u^\Phi = |{-}|\colon \Phi \to \mathbb{F}$ gives a functor $\Phi^{\mathrm{op}} \to \mathrm{Cat}$. We define the *wreath product* $\Psi \wr \Phi \to \Phi$ to be a Grothendieck fibration associated to this functor.

Remark 3.1.2.2. Thus, an object of $\Psi \wr \Phi$ is determined by an object $I \in \Phi$ and, for each $i \in |I|$, an object $J_i \in \Psi$. We write $I(J_i)_{i\in|I|}$ for this object. A morphism $I(J_i)_{i\in|I|} \to I'(J'_k)_{k\in|I'|}$ consists of a morphism $f\colon I \to I'$ in Φ and, for each $i \in |I|$, a morphism $J_i \to J'_{f\circ i}$ in Ψ.

Remark 3.1.2.3. The wreath product of operator categories has a universal property: a $\Psi \wr \Phi$-algebra is, roughly speaking, a Ψ-algebra in Φ-algebras; we will give several more precise statements along these lines below.

Remark 3.1.2.4. The wreath product $\Psi \wr \Phi$ is functorial with respect to all admissible functors in the first variable, but only with respect to operator morphisms in the second variable.

Proposition 3.1.2.5 ([Bar13, Proposition 3.9])**.** The operation $\wr$ gives a monoidal structure on the category of operator categories and operator morphisms.

Remark 3.1.2.6. The unit for the wreath product is the trivial operator category $\mathbf{0}$. This is also the initial operator category (and the zero object with respect to admissible functors) so given operator categories Φ and Ψ there are canonical maps

$$i_\Phi\colon \Phi \simeq \mathbf{0} \wr \Phi \to \Psi \wr \Phi, \quad I \mapsto I(*)_{i\in|I|},$$

$$j_\Psi\colon \Psi \simeq \Psi \wr \mathbf{0} \to \Psi \wr \Phi, \quad J \mapsto *(J),$$

$$p_\Phi\colon \Psi \wr \Phi \to \mathbf{0} \wr \Phi \simeq \Phi, \quad I(J_i)_{i\in|I|} \mapsto I.$$

The functors i_Φ and j_Ψ are operator morphisms, whereas p_Φ is merely an admissible functor.

The wreath product allows us to define the key examples of operator categories we will be interested in in this thesis:

Example 3.1.2.7. We write $\mathbb{O}(n)$ for the n-fold wreath power $\mathbb{O}^{\wr n}$ of the operator category $\mathbb{O}$ of finite ordered sets. In the setting of ordinary categories $\mathbb{O}(2)$ parametrizes braided monoidal categories and braided operads, while $\mathbb{O}(n)$ parametrizes symmetric monoidal categories and symmetric operads for $n > 2$. When working with ∞-categories, however, $\mathbb{O}(n)$ gives $\mathbb{E}_n$-monoidal ∞-categories, as we will see below.

3.1.3 Monoidal Categories and Operads

We will now justify the claim that operator categories parameterize multiplicative structures by defining Φ-monoidal categories and Φ-operads, where Φ is an operator category.

Definition 3.1.3.1. A Φ-*monoidal category* is a category $\mathbf{C}$ equipped with:

(i) For each $I \in \Phi$ a functor $\bigotimes_I\colon \mathbf{C}^{\times|I|} \to \mathbf{C}$, such that $\bigotimes_* = \mathrm{id}$.

(ii) For each morphism $f\colon J \to I$ in Φ, a natural isomorphism

$$\alpha_f\colon \bigotimes_J \xrightarrow{\sim} \bigotimes_I \circ \left(\bigotimes_{J_i}\right)_{i\in|I|}.$$

These isomorphisms must be functorial, i.e. $\alpha_{f\circ g} = \alpha_f \circ \alpha_g$ and $\alpha_{\mathrm{id}_I} = \mathrm{id}_{\otimes_I}$. If $\mathbf{C}$ and $\mathbf{D}$ are Φ-monoidal categories, a *lax Φ-monoidal functor* from $\mathbf{C}$ to $\mathbf{D}$ is a functor $F\colon \mathbf{C} \to \mathbf{D}$ together with natural transformations $\bigotimes_I \circ F \to F \circ \bigotimes_I$, compatible with the natural isomorphisms α_f for f in Φ. A *strong Φ-monoidal functor* is a lax Φ-monoidal functor such that these natural transformations are natural isomorphisms. We write Mon^{Φ} for the category of Φ-monoidal categories and strong Φ-monoidal functors, and $\mathrm{Mon}^{\Phi,\mathrm{lax}}$ for the category of Φ-monoidal categories and lax Φ-monoidal functors.

Examples 3.1.3.2.

(1) A $\mathbf{0}$-monoidal category is a category.

(2) An $\mathbb{O}$-monoidal category is a monoidal category.

(3) An $\mathbb{O}(2)$-monoidal category is a braided monoidal category.

(4) An $\mathbb{F}$-monoidal category is a symmetric monoidal category, as is an $\mathbb{O}(n)$-monoidal category for $n > 2$.

More generally, we can consider Φ-operads; here we will restrict ourselves to Φ-operads in sets:

Definition 3.1.3.3. A *Φ-operad* $\mathbf{M}$ consists of a set $\operatorname{ob}\mathbf{M}$ of objects and, given $I \in \Phi$, a collection $(x_i)_{i\in|I|}$ of objects indexed by the points of I, and an object y, a set

$$\mathbf{M}_I((x_i), y)$$

of multimorphisms from (x_i) to y. Given a morphism $J \to I$ in Φ we have a composition operation

$$\prod_{i\in|I|} \mathbf{M}_{J_i}((x_j)_{j\in|J_i|}, y_i) \times \mathbf{M}_I((y_i)_{i\in|I|}, z) \to \mathbf{M}_J((x_j)_{j\in|J|}, z).$$

This must be associative in the obvious sense, and there is also an identity morphism $\mathrm{id}_x \in \mathbf{M}_*(x, x)$ for all objects x.

Remark 3.1.3.4. For consistency with Lurie's terminology for ∞-categories we have chosen to use the term Φ-operad instead of Φ-multicategory or coloured Φ-operad for this concept.

Remark 3.1.3.5. An obvious variant of this definition gives a notion of Φ-operads *enriched* in, for example, a symmetric monoidal category. In the next section we will make use of *simplicial Φ-operads*, which are Φ-operads enriched in the category of simplicial sets.

Definition 3.1.3.6. A *functor of Φ-operads* $F\colon \mathbf{M} \to \mathbf{N}$ consists of a function $\operatorname{ob}\mathbf{M} \to \operatorname{ob}\mathbf{N}$ and a function $\mathbf{M}_I((x_i), y) \to \mathbf{N}_I((F(x_i))_i, F(y)$ for each $I \in \Phi$ and $x_i, y \in \operatorname{ob}\mathbf{M}$; these must preserve identities and be compatible with composition in the obvious sense. We write Opd^{Φ} for the category of Φ-operads and functors.

Definition 3.1.3.7. If $F, G\colon \mathbf{M} \to \mathbf{N}$ are functors of Φ-operads, a *natural transformation* $\eta\colon F \to G$ consists of, for each $x \in \mathbf{M}$ a morphism $\eta_x \in \mathbf{N}_*(Fx, Gx)$, compatible with compositions in the obvious way. We write OPD^{Φ} for the 2-category of Φ-operads, functors, and natural transformations.

Example 3.1.3.8. We can consider a Φ-monoidal category $\mathbf{C}$ as a Φ-operad by defining $\mathbf{C}_I((x_i), y)$ to be the set of morphisms $\mathbf{C}(\bigotimes_I(x_i), y)$, with composition defined using the natural isomorphisms α_f and ordinary composition in $\mathbf{C}$. Then $\mathrm{Mon}^{\Phi,\mathrm{lax}}$ is the full subcategory of Opd^{Φ} spanned by Φ-operads of this form.

Examples 3.1.3.9.

(1) A $\mathbf{0}$-operad is a category.

(2) An $\mathbb{O}$-operad is a multicategory or (coloured) non-symmetric operad.

(3) An $\mathbb{O}(2)$-operad is a braided multicategory or (coloured) braided operad.

(4) An $\mathbb{F}$-operad is a symmetric multicategory or (coloured) symmetric operad, as is an $\mathbb{O}(n)$-operad for $n > 2$.

Remark 3.1.3.10. A Φ-operad $\mathbf{O}$ with a single object can equivalently be described by sets $\mathbf{O}(I)$ for $I \in \Phi$ and composition morphisms

$$\mathbf{O}(I) \times \prod_{i \in |I|} \mathbf{O}(J_i) \to \mathbf{O}(J)$$

for each morphism $J \to I$ in Φ.

Definition 3.1.3.11. If $\mathbf{C}$ is a Φ-monoidal category (or more generally a Φ-operad), and $\mathbf{O}$ is a Φ-operad, an $\mathbf{O}$-*algebra* in $\mathbf{C}$ is a functor of Φ-operads $A \colon \mathbf{O} \to \mathbf{C}$. We write $\mathrm{Alg}^{\Phi}_{\mathbf{O}}(\mathbf{C})$ for the category of $\mathbf{O}$-algebras in $\mathbf{C}$, i.e. the mapping category $\mathrm{OPD}^{\Phi}(\mathbf{O}, \mathbf{C})$.

Remark 3.1.3.12. Suppose $f \colon \Phi \to \Psi$ is an operator morphism. Then f allows us to regard a Ψ-operad as a Φ operad, giving a functor $f^* \colon \mathrm{Opd}^{\Psi} \to \mathrm{Opd}^{\Phi}$: if $\mathbf{M}$ is a Ψ-operad, then $f^*\mathbf{M}$ has the same objects as $\mathbf{M}$, and

$$f^*\mathbf{M}_I((x_i), y) := \mathbf{M}_{f(I)}((x_i), y).$$

This functor has a left adjoint $f_! \colon \mathrm{Opd}^{\Phi} \to \mathrm{Opd}^{\Psi}$, which forms the "free" Ψ-operad on a Φ-operad. For example, $u^{\mathbb{O}}_! \colon \mathrm{Opd}^{\mathbb{O}} \to \mathrm{Opd}^{\mathbb{F}}$ gives the usual way of regarding a non-symmetric operad as a symmetric operad.

Remark 3.1.3.13. If Φ is an operator category, we let $\mathbf{U}(\Phi)$ denote the $\mathbb{F}$-operad $u^{\Phi}_! *$. For example, $\mathbf{U}(\mathbb{O})$ is the usual associative (symmetric) operad. The functor $u^{\Phi}_! \colon \mathrm{Opd}^{\Phi} \to \mathrm{Opd}^{\mathbb{F}}_{/\mathbf{U}(\Phi)}$ is often an equivalence, for example if Φ is $\mathbb{O}$.

The wreath product of operator categories can be extended to a wreath product of operads: If $\mathbf{O}$ is a Φ-operad and $\mathbf{Q}$ is a Ψ-operad, both with a single object, then $\mathbf{Q} \wr \mathbf{O}$ is a $\Psi \wr \Phi$-operad, also with a single object, with

$$(\mathbf{Q} \wr \mathbf{O})(I(J_i)_{i \in |I|}) = \mathbf{O}(I) \times \prod_{i \in |I|} \mathbf{Q}(J_i).$$

The general definition is somewhat more complicated:

Definition 3.1.3.14. Suppose $\mathbf{M}$ is a Φ-operad an $\mathbf{N}$ is a Ψ-operad. Then $\mathbf{N} \wr \mathbf{M}$ is a $\Psi \wr \Phi$-operad with objects $\operatorname{ob} \mathbf{M} \times \operatorname{ob} \mathbf{N}$ and multimorphism sets defined by

$$(\mathbf{M} \wr \mathbf{N})_{I(J_i)}((m_{(i,j)}, n_{(i,j)}), (m', n')) := \mathbf{M}_I((m_i), m') \times \prod_{i \in |I|} \mathbf{N}_{J_i}((n_{i,j}), n'),$$

if $m_{(i,j)}$ is equal to m_i for all $j \in J_I$, and $\emptyset$ otherwise, with composition defined in the obvious way.

Remark 3.1.3.15. The wreath product of operads has a universal property, which we can roughly describe as follows: suppose $\mathbf{M}$ is a Φ-operad, $\mathbf{N}$ is a Ψ-operad, and $\mathbf{X}$ is a $\Psi \wr \Phi$-operad. Then the category $\mathrm{Alg}^{\Phi}_{\mathbf{M}}(j^*_{\Phi}\mathbf{X})$ has a natural Ψ-operad structure, and there is an equivalence

$$\mathrm{Alg}^{\Psi \wr \Phi}_{\mathbf{N} \wr \mathbf{M}}(\mathbf{X}) \simeq \mathrm{Alg}^{\Psi}_{\mathbf{N}}(\mathrm{Alg}^{\Phi}_{\mathbf{M}}(j^*_{\Phi}\mathbf{X})).$$

3.1.4 Perfect Operator Categories and Monoids

We will now introduce the most important class of operator categories, namely the so-called *perfect* operator categories, which includes all the examples we are interested in here.

Definition 3.1.4.1. A *point classifier* for an operator category Φ is an object $(T, t \colon * \to T) \in \Phi_{*/}$ such that for any object $(V, v \colon * \to V) \in \Phi_*$ there exists a unique morphism $V \to T$ in Φ such that

$$\begin{array}{ccc} * & \xrightarrow{v} & V \\ \downarrow & & \downarrow \\ * & \xrightarrow[t]{} & T \end{array}$$

is a pullback square.

Definition 3.1.4.2. An operator category Φ is *perfect* if it has a point classifier (T, t) and the functor $(-)_t \colon \Phi_{/T} \to \Phi$ that takes the fibre at t has a right adjoint $T_* \colon \Phi \to \Phi_{/T}$. We refer to t as the *special point* of T and its other points as *generic points*. We write $T \colon \Phi \to \Phi$ for the composite of T_* with the forgetful functor to Φ.

Example 3.1.4.3. The operator categories $\mathbb{O}$ and $\mathbb{F}$ are perfect, with point classifiers $\{1\} \to \{0, 1, 2\}$ and $\{1\} \to \{0, 1\}$, respectively.

Proposition 3.1.4.4 ([Bar13, Proposition 5.11])**.** If Φ and Ψ are perfect operator categories, with point classifiers $t \to T$ and $t' \to T'$, respectively, then $\Psi \wr \Phi$ is also perfect, with point classifier $T(I_i)$ where $I_t = T'$ and $I_i = *$ for $i \neq t$.

Example 3.1.4.5. The operator categories $\mathbb{O}(n)$ are perfect for all n.

Theorem 3.1.4.6 ([Bar13, Theorem 6.9])**.** If Φ is a perfect operator category, then the functor T is a monad on Φ.

Definition 3.1.4.7. Suppose Φ is a perfect operator category. The *Leinster category* $\mathcal{L}^{\Phi}$ of Φ is the Kleisli category of the monad T. In other words, the objects of $\mathcal{L}^{\Phi}$ are the same as those of Φ, but morphisms are given by

$$\mathrm{Hom}_{\mathcal{L}^{\Phi}}(I, J) = \mathrm{Hom}_{\Phi}(I, TJ),$$

with composition defined using the monadic structure of T. We will generally denote a map from I to J in the Leinster category with a barred arrow, $I \nrightarrow J$.

Example 3.1.4.8. Suppose Φ is a perfect operator category. A point $i\colon * \to I$ corresponds to a unique morphism $I \to T$ such that i is the pullback of $t\colon * \to T$, and so to a morphism $i^\vee\colon I \nrightarrow *$ in $\mathcal{L}^\Phi$.

Proposition 3.1.4.9. Let Ψ be an operator category, and let $n_\Psi\colon \Gamma = (\mathcal{L}^{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}$ be the functor sending a pointed finite set S to the category $\mathrm{Fun}_*(S, \mathcal{L}^\Psi)$ of functors that take the base point of S to $* \in \mathcal{L}^\Psi$, i.e. $(\mathcal{L}^\Psi)^{\times|S|-1}$. If Φ is another operator category, composing with the functor $\mathcal{L}^\Phi \to \mathcal{L}^{\mathbb{F}}$ induced by the unique operator morphism $u^\Phi = |-|\colon \Phi \to \mathbb{F}$ gives a functor $(\mathcal{L}^\Phi)^{\mathrm{op}} \to \mathrm{Cat}$. The Leinster category $\mathcal{L}^{\Phi \wr \Psi}$ is equivalent to the total space of the Grothendieck fibration associated to this functor.

Proof. This is a special case of [Bar13, Proposition 7.7]. □

Examples 3.1.4.10.

(i) The Leinster category $\mathcal{L}^{\mathbf{0}}$ is just $\mathbf{0}$.

(ii) The Leinster category $\mathcal{L}^{\mathbb{O}}$ is the opposite category $\mathbb{\Delta}^{\mathrm{op}}$ of the simplicial indexing category $\mathbb{\Delta}$ (cf. [Bar13, Example 7.6]).

(iii) The Leinster category $\mathcal{L}^{\mathbb{F}}$ is the category Γ^{op} of finite pointed sets (cf. [Bar13, Example 7.5]).

(iv) The Leinster category $\mathcal{L}^{\mathbb{O}(n)}$ is the opposite category Θ_n^{op} of Joyal's category Θ_n (cf. [Bar13, Example 7.8]).

Using the Leinster category we can define Φ-*monoids* when Φ is a perfect operator category:

Definition 3.1.4.11. Let Φ be a perfect operator category, and suppose $\mathbf{C}$ is a category with finite products. A Φ-*monoid* M in $\mathbf{C}$ is a functor $M\colon \mathcal{L}^\Phi \to \mathbf{C}$ such that for every object I, the morphism $M(I) \to \prod_{i \in |I|} M(*)$ induced by the morphisms $i^\vee\colon I \nrightarrow *$ is an isomorphism. We write $\mathrm{Mnd}_\Phi(\mathbf{C})$ for the obvious category of Φ-monoids in $\mathbf{C}$.

Examples 3.1.4.12.

(i) A $\mathbf{0}$-monoid is just an object.

(ii) An $\mathbb{O}$-monoid is an associative monoid.

(iii) An $\mathbb{F}$-monoid is a commutative monoid, as is an $\mathbb{O}(n)$-monoid for $n > 1$.

Remark 3.1.4.13. The wreath product of operator categories also has a universal property in terms of monoids: Let Φ and Ψ be perfect operator categories, and suppose $\mathbf{C}$ is a category with finite products. Then there is an equivalence of categories

$$\mathrm{Mnd}_{\Phi \wr \Psi}(\mathbf{C}) \simeq \mathrm{Mnd}_\Phi(\mathrm{Mnd}_\Psi(\mathbf{C})),$$

i.e. a $\Phi \wr \Psi$-monoid in $\mathbf{C}$ is equivalent to a Φ-monoid in Ψ-monoids in $\mathbf{C}$.

Remark 3.1.4.14. Let $f\colon \Phi \to \Psi$ be an operator morphism. Then composition with $\mathcal{L}^f$ takes Ψ-monoids to Φ-monoids and so induces a functor $f^*\colon \mathrm{Mnd}_\Psi(\mathbf{C}) \to \mathrm{Mnd}_\Phi(\mathbf{C})$. If $\mathbf{C}$ is a presentable category where the Cartesian product preserves colimits in each variable then f^* has a left adjoint $f_!\colon \mathrm{Mnd}_\Phi(\mathbf{C}) \to \mathrm{Mnd}_\Psi(\mathbf{C})$.

3.1.5 The Inert-Active Factorization System

If Φ is a perfect operator category, there is an important factorization system on the Leinster category $\mathcal{L}^{\Phi}$:

Definition 3.1.5.1. Let Φ be a perfect operator category. If $\phi\colon I \nrightarrow J$ is a morphism in $\mathcal{L}^{\Phi}$, consider the pullback square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & J \arrow[d, "u_J"] \\ I \arrow[r, "\phi"'] & TJ. \end{tikzcd}$$

We say that ϕ is *active* if the morphism $F \to I$ is an isomorphism, and *inert* if the morphim $F \to J$ is an isomorphism.

Remark 3.1.5.2. A morphism $\phi\colon I \nrightarrow J$ is active if and only if it is in the image of Φ, i.e. it is of the form

$$I \xrightarrow{f} J \xrightarrow{u_J} TJ,$$

where f is a morphism in Φ. It is clear from the definition that every active morphism is of this form, and the converse holds because the diagram

$$\begin{tikzcd} J \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & J \arrow[d, "u_J"] \\ J \arrow[r, "u_J"'] & TJ \end{tikzcd}$$

is a pullback square in Φ for all J.

Examples 3.1.5.3.

(i) A morphism $f\colon [n] \to [m]$ in $\mathbb{\Delta}$ corresponds to an active morphism in $\mathcal{L}^{\mathbb{O}} \simeq \mathbb{\Delta}^{\mathrm{op}}$ if and only if $f(0) = 0$ and $f(n) = m$, and an inert morphism if and only if f is the inclusion of a subinterval, i.e. $f(j) = f(0) + j$ for $j = 0, \ldots, n$.

(ii) A morphism $f\colon \langle n \rangle \to \langle m \rangle$ in $\mathbb{\Gamma}^{\mathrm{op}} \simeq \mathcal{L}^{\mathbb{F}}$ is active if and only if $f^{-1}(*) = \{*\}$, and inert if and only if $|f^{-1}(i)| = 1$ for $i \neq *$.

Proposition 3.1.5.4 ([Bar13, Lemma 8.3])**.** If Φ is a perfect operator category, then the inert and active morphisms form a factorization system on $\mathcal{L}^{\Phi}$.

Proof. We first show that any morphism $\phi\colon I \nrightarrow J$ has a factorization as an inert morphism followed by an active morphism. We may regard $I \to TJ$ as a morphism in $\Phi_{/T}$ via the map $T(J \to *)$, i.e. a morphism from $\bar{I} = (I \to T)$ to $T_*(J)$. This is adjoint to a morphism $\alpha\colon \bar{I}_t \to J$, giving a factorization

$$\bar{I} \to T(\bar{I}_t) \xrightarrow{T(\alpha)} T(J).$$

The fibre $\bar{I}_t$ is the fibre product F in the pullback square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & J \arrow[d, "u_J"] \\ I \arrow[r, "\phi"'] & TJ \end{tikzcd}$$

in Φ, so we have factored ϕ as a composite

$$I \xrightarrow{\beta} TF \xrightarrow{T(\alpha)} TJ,$$

i.e. as a composite $(u_J\alpha) \circ \beta$ in $\mathcal{L}^{\Phi}$. The morphism $u_J\alpha$ is obviously active. Here F is also the fibre of the composite $I \to TJ \to T$ at the special point t, i.e. F is $(I \to T)_t$. Since the diagram

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "u_F"'] & J \arrow[d, "u_J"] \\ TF \arrow[r, "T\alpha"'] & TJ \end{tikzcd}$$

is a pullback, it is clear that β is inert.

This also shows that any inert morphism $\phi\colon I \nrightarrow J$ factors as the counit $I \to T(I_t)$ followed by an isomorphism. Since this counit does not change when we compose with an active map, it follows that the inert-active factorization is unique up to isomorphism. □

Remark 3.1.5.5. This factorization system is probably a particular case of that described by Weber [Web04] on the Kleisli categories of certain monads; this observation is due to David Gepner. However, the "generic morphisms", which are the equivalent of our inert morphisms, do not have as nice a description in this more general setting.

Definition 3.1.5.6. We write $\mathcal{L}^{\Phi}_{\text{int}}$ and $\mathcal{L}^{\Phi}_{\text{act}}$ for the subcategories of $\mathcal{L}^{\Phi}$ where the morphisms are the inert and active morphisms in $\mathcal{L}^{\Phi}$, respectively.

Definition 3.1.5.7. An object A of Φ is an *atom* if there exists an inert morphism $* \nrightarrow A$ in $\mathcal{L}^{\Phi}$, i.e. A is the fibre at $t\colon * \to T$ of some other point $* \to T$. The *globular category* $\mathcal{G}^{\Phi}$ is then the full subcategory of $\mathcal{L}^{\Phi}_{\text{int}}$ spanned by the atoms. If I is an object of Φ, we write $\mathcal{G}^{\Phi}_{I/}$ for $\mathcal{G}^{\Phi} \times_{\mathcal{L}^{\Phi}_{\text{int}}} (\mathcal{L}^{\Phi}_{\text{int}})_{I/}$.

Example 3.1.5.8. Suppose Φ and Ψ are perfect operator categories. Then the atoms of $\Psi \wr \Phi$ are $*(A)$, where A is an atom of Ψ, and $A()$ where A is an atom of Φ other than $*$ (which must necessarily have no points).

Examples 3.1.5.9.

(i) The unique object of $\mathbf{0}$ is an atom, and $\mathcal{G}^{\mathbf{0}}$ is just $\mathbf{0}$.

(ii) The atoms of $\mathbb{F}$ are $\emptyset$ and $*$, and $\mathcal{G}^{\mathbb{F}}$ is

$$* \to \emptyset.$$

(iii) The atoms of $\mathbb{O}$ are $\emptyset$ and $*$, and $\mathcal{G}^{\mathbb{O}}$ is

$$* \rightrightarrows \emptyset.$$

(iv) The atoms of $\mathbb{O}(n)$ are $C_k^{\mathbb{O}(n)}$, $k = 0, \ldots, n$, which we can inductively define by $C_0^{\mathbb{O}(1)} = \emptyset$, $C_1^{\mathbb{O}(1)} = *$, and

$$C_k^{\mathbb{O}(n)} = \begin{cases} *(C_{k-1}^{\mathbb{O}(n-1)}), & k = 1, \ldots, n, \\ \emptyset, & k = 0. \end{cases}$$

The category $\mathcal{G}^{\mathbb{O}(n)}$ is

$$C_n^{\mathbb{O}(n)} \rightrightarrows C_{n-1}^{\mathbb{O}(n)} \rightrightarrows \cdots \rightrightarrows C_0^{\mathbb{O}(n)},$$

i.e. $\mathbb{G}_n^{\mathrm{op}}$. In terms of the description of the objects of Θ_n as certain strict n-categories, the object $C_k^{\mathbb{O}(n)}$ corresponds to the k-cell or free k-morphism.

Definition 3.1.5.10. Let Φ be a perfect operator category, and let $\mathbf{C}$ be a category with finite limits. A *Φ-category object M* in $\mathbf{C}$ is a functor $M \colon \mathcal{L}^{\Phi} \to \mathbf{C}$ such that $M|_{\mathcal{L}^{\Phi}_{\mathrm{int}}}$ is a right Kan extension of $M|_{\mathcal{G}^{\Phi}}$. In particular, for $I \in \Phi$ the object $M(I)$ is the limit of $M|_{\mathcal{G}^{\Phi}_{I/}}$.

Example 3.1.5.11. An $\mathbb{O}(n)$-category object in sets is a strict n-category.

Definition 3.1.5.12. A perfect operator category Φ is *self-categorical* if the functor

$$I^* := \mathrm{Hom}_{\mathcal{L}^{\Phi}}(I, –) \colon \mathcal{L}^{\Phi} \to \mathrm{Set}$$

is a Φ-category object for all $I \in \Phi$.

Examples 3.1.5.13. The operator categories $\mathbb{O}(n)$ are self-categorical for all n, whereas $\mathbb{F}$ is not.

3.1.6 The May-Thomason Category of a Φ-Operad

For a perfect operator category Φ we can give an alternative definition of Φ-operads as certain functors to $\mathcal{L}^{\Phi}$, by considering the *May-Thomason category* of a Φ-operad. This is the definition we will generalize to define ∞-operads in the next section.

Definition 3.1.6.1. Let Φ be a perfect operator category. If $\mathbf{M}$ is a Φ-operad, the *May-Thomason category* $\mathbf{M}^{\otimes}$ of $\mathbf{M}$ has objects pairs $(I, (x_i)_{i \in |I|})$ where $I \in \Phi$ and $x_i \in \mathbf{M}$, and a morphism $(I, (x_i)) \to (J, (y_j))$ is given by a morphism $I \nrightarrow J$ in $\mathcal{L}^{\Phi}$ and for each $j \in |J|$ a morphism in $\mathbf{M}_{I_j}((x_i)_{i \in |I|}, y_j)$, where I_j is the fibre of $I \to TJ$ at

$$* \xrightarrow{j} J \xrightarrow{u_J} TJ.$$

There is an obvious projection $\mathbf{M}^{\otimes} \to \mathcal{L}^{\Phi}$.

Remark 3.1.6.2. If $\mathbf{M}$ is a Φ-operad enriched in a symmetric monoidal category $\mathbf{C}$ that has coproducts and whose tensor product commutes with these, then the same definition applied to $\mathbf{M}$ gives a $\mathbf{C}$-category $\mathbf{M}^{\otimes}$.

Proposition 3.1.6.3. A functor $\pi \colon \mathbf{C} \to \mathcal{L}^{\Phi}$ is equivalent to the May-Thomason category of a Φ-operad if and only if the following conditions hold:

(i) If $\phi\colon I \nrightarrow J$ is an inert morphism in $\mathcal{L}^{\Phi}$ and $x \in \mathbf{C}_I$ then there exists a π-coCartesian morphism $x \to \phi_! x$.

(ii) For every $I \in \Phi$ the functor $\mathbf{C}_I \to \prod_{i \in |I|} \mathbf{C}_*$ induced by the coCartesian arrows over $i^{\vee}$ for $i \in |I|$ is an equivalence of categories.

(iii) Given a morphism $\phi\colon I \nrightarrow J$ in $\mathcal{L}^{\Phi}$ and $y \in \mathbf{C}_J$, the coCartesian morphisms $y \to y_j$ induced by the inert morphisms $j^{\vee}\colon J \nrightarrow *$ give an isomorphism

$$\mathbf{C}_{\phi}(x, y) \xrightarrow{\sim} \prod_{j \in |J|} \mathbf{C}_{j \circ \phi}(x, y_j),$$

where $\mathbf{C}_{\phi}(x, y)$ denotes the subset of $\mathbf{C}(x, y)$ of morphisms that map to ϕ in $\mathcal{L}^{\Phi}$.

Moreover, a functor $F\colon \mathbf{C} \to \mathbf{D}$ over $\mathcal{L}^{\Phi}$ between such categories corresponds to a functor of Φ-operads if and only if it preserves coCartesian morphisms over inert morphisms in $\mathcal{L}^{\Phi}$. We can thus equivalently *define* a Φ-operad to be a functor $\mathbf{C} \to \mathcal{L}^{\Phi}$ satisfying (i)–(iii).

Remark 3.1.6.4. A functor $\pi\colon \mathbf{C} \to \mathcal{L}^{\Phi}$ is equivalent to the May-Thomason category of a Φ-monoidal category if and only if it satisfies conditions (i)–(iii) above, and is also a coGrothendieck fibration, i.e. for any morphism $\phi\colon I \nrightarrow J$ in $\mathcal{L}^{\Phi}$ and any $x \in \mathbf{C}_I$ there exists a π-coCartesian morphism $x \to \phi_! x$. A functor $F\colon \mathbf{C} \to \mathbf{D}$ between such categories over $\mathcal{L}^{\Phi}$ corresponds to a strong monoidal functor of Φ-monoidal categories if and only if it preserves *all* coCartesian arrows.

3.1.7 Generalized Operads and Multiple Categories

Replacing the Segal conditions for monoids with those for category objects in the characterization of May-Thomason categories above gives a generalization of the notion of Φ-operad:

Definition 3.1.7.1. Let Φ be a perfect operator category. A *generalized Φ-operad* is a functor $\pi\colon \mathbf{C} \to \mathcal{L}^{\Phi}$ such that the following conditions hold:

(i) If $\phi\colon I \nrightarrow J$ is an inert morphism in $\mathcal{L}^{\Phi}$ and $x \in \mathbf{C}_I$ then there exists a π-coCartesian morphism $x \to \phi_! x$.

(ii) For every $I \in \Phi$ the functor $\mathbf{C}_I \to \lim_{I \nrightarrow A \in \mathcal{G}^{\Phi}_{I/}} \mathbf{C}_A$ induced by the coCartesian arrows over $I \nrightarrow A$ is an equivalence of categories.

(iii) Given a morphism $\phi\colon I \nrightarrow J$ in $\mathcal{L}^{\Phi}$ and $y \in \mathbf{C}_J$, the coCartesian morphisms $y \to y_{\alpha}$ induced by the inert morphisms $\alpha\colon J \nrightarrow A$ in $\mathcal{G}^{\Phi}_{J/}$ give an isomorphism

$$\mathbf{C}_{\phi}(x, y) \xrightarrow{\sim} \lim_{\alpha} \mathbf{C}_{\alpha \circ \phi}(x, y_{\alpha}).$$

Example 3.1.7.2. A generalized $\mathbb{O}$-operad is the same as a *virtual double category* as defined by Cruttwell and Shulman [CS10], or **fc**-*multicategory* as defined by Leinster [Lei04]. Generalized $\mathbb{O}(n)$-operads for a general n may be regarded as the most general objects for which we can define a notion of "lax functor" extending that for n-categories.

Remark 3.1.7.3. A generalized Φ-operad $\mathbf{C} \to \mathcal{L}^{\Phi}$ is (equivalent to the May-Thomason category of) a Φ-operad precisely when the fibre $\mathbf{C}_A$ is equivalent to $*$ when $A \in \Phi$ is an atom other than $*$.

Definition 3.1.7.4. If $\mathbf{C}$ and $\mathbf{D}$ are generalized Φ-operads, a *functor of generalized Φ-operads* $F\colon \mathbf{C} \to \mathbf{D}$ is a functor over $\mathcal{L}^{\Phi}$ that preserves coCartesian arrows lying over inert morphisms in $\mathcal{L}^{\Phi}$. A *natural transformation* of functors between generalized Φ-operads is just an ordinary natural transformation of such functors. We write $\mathrm{Opd}^{\Phi,\mathrm{gen}}$ for the category of generalized Φ-operads and functors, and $\mathrm{OPD}^{\Phi,\mathrm{gen}}$ for the 2-category of Φ-operads, functors, and natural transformations.

Definition 3.1.7.5. A *Φ-multiple category* is a generalized Φ-operad $\mathbf{C} \to \mathcal{L}^{\Phi}$ that is also a coGrothendieck fibration.

Example 3.1.7.6. An $\mathbf{O}$-multiple category is a double category.

Definition 3.1.7.7. A *lax monoidal functor* between Φ-multiple categories is just a functor of generalized Φ-operads; we write $\mathrm{Mult}^{\Phi,\mathrm{lax}}$ for the full subcategory of $\mathrm{Opd}^{\Phi,\mathrm{gen}}$ spanned by the Φ-multiple categories. A *strong monoidal functor* between Φ-multiple categories is a functor over $\mathcal{L}^{\Phi}$ that preserves *all* coCartesian morphisms; we write Mult^{Φ} for the category of Φ-multiple categories and strong monoidal functors.

Remark 3.1.7.8. An operator morphism does *not* generally induce a pullback functor on generalized operads or multiple categories.

3.1.8 Subcategories of Operator Categories

We will now observe that subcategories of perfect operator categories determined by atoms are often themselves operator categories:

Lemma 3.1.8.1. Let Φ be a perfect operator category. If A is an atom of Φ then A is a subobject of $*$. In particular, the forgetful functor $\Phi_{/A} \to \Phi$ is fully faithful.

Proof. We have a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d, "t"] \\ * \arrow[r, "s"'] & T \end{tikzcd}$$

where t is the special point. Thus a morphism $X \to A$ corresponds to a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d, "t"] \\ * \arrow[r, "s"'] & T. \end{tikzcd}$$

Since $*$ is the final object of Φ we see that such a diagram is unique if it exists. In other words, an object of Φ admits at most one morphism to A. $\square$

Lemma 3.1.8.2. Let Φ be a perfect operator category and A an atom of Φ such that Φ has *A-fibres*, i.e. pullbacks along morphisms from A. Then $\Phi_{/A}$ is a perfect operator category with respect to the restriction of T.

Proof. It is clear that $\Phi_{/A}$ is an operator category if A-fibres exist in Φ. We have a pullback diagram

$$\begin{tikzcd} A \arrow[r, "u_A"] \arrow[d] & TA \arrow[d] \\ * \arrow[r, "t"'] & T. \end{tikzcd}$$

Given a morphism $Y \to TA$ the pullback along $A \hookrightarrow TA$ is therefore $Y_t \to A$. In the diagram

$$\begin{tikzcd} \mathrm{Hom}(Y_t, X) \arrow[r] \arrow[d] & \mathrm{Hom}_T(Y, TX) \arrow[d] \\ \mathrm{Hom}(Y_t, A) \arrow[r] & \mathrm{Hom}_T(Y, TA) \end{tikzcd}$$

the horizontal morphisms are isomorphisms since $(-)_t$ is left adjoint to T. Taking fibres at $Y_t \to A$ we get a natural isomorphism

$$\mathrm{Hom}_A(Y_t, X) \xrightarrow{\sim} \mathrm{Hom}_{TA}(Y, TX)$$

hence pullback along $A \hookrightarrow TA$ is left adjoint to $T\colon \Phi_{/A} \to \Phi_{/TA}$. In particular, TA is a point classifier for $\Phi_{/A}$. $\square$

Remark 3.1.8.3. In this case the induced functor $\mathcal{L}^{\Phi_{/A}} \to \mathcal{L}^{\Phi}$ clearly preserves the inert-active factorization system.

Restricting a Φ-operad or Φ-monoidal category to $\Phi_{/A}$ always gives a trivial $\Phi_{/A}$-operad. However, in good cases we can restrict generalized Φ-operads to $\Phi_{/A}$:

Definition 3.1.8.4. Let Φ be a perfect operator category. We say an atom A of Φ is *clean* if

(i) Φ has A-fibres.

(ii) If I is in $\Phi_{/A}$, then $\mathcal{G}^{\Phi_{/A}}_{I/} \to \mathcal{G}^{\Phi}_{I/}$ is an equivalence.

Lemma 3.1.8.5. Let Φ be a perfect operator category, and suppose A is a clean atom in Φ. Then pullback along the inclusion $\mathcal{L}^{\Phi_{/A}} \hookrightarrow \mathcal{L}^{\Phi}$ induced by the inclusion $j^A\colon \Phi_{/A} \hookrightarrow \Phi$ gives functors

$$j^{A,*}\colon \mathrm{Opd}^{\Phi,\mathrm{gen}} \to \mathrm{Opd}^{\Phi_{/A},\mathrm{gen}},$$
$$j^{A,*}\colon \mathrm{Mult}^{\Phi} \to \mathrm{Mult}^{\Phi_{/A}}.$$

Remark 3.1.8.6. These functors have left adjoints $j^A_!$ and right adjoints j^A_*.

Example 3.1.8.7. The atoms $C_k^{\mathbb{O}(n)}$ in $\mathbb{O}(n)$ are all clean. The subcategory $\mathbb{O}(n)_{/C_k}$ can be identified with $\mathbb{O}(k)$, so we have inclusions $j^n_k\colon \mathbb{O}(k) \hookrightarrow \mathbb{O}(n)$. The corresponding inclusion $\Theta_k \hookrightarrow \Theta_n$ is the obvious inclusion of the basic k-categories as n-categories with no non-trivial i-morphisms for $i > k$, i.e. j^{n-k} in the notation of §2.2.3.

3.2 ∞-Operads

In this section we indicate how to extend parts of Lurie's theory of (symmetric) ∞-operads to the setting of operator categories. Given the material in §3.1 this is mostly a straightforward generalization of definitions and results from [Lur11] — the exception is the discussion of wreath products in §3.2.6 which is based on results of Barwick from [Bar13].

3.2.1 Basic Definitions

Let Φ be a perfect operator category. We will now define the basic objects we will study in this section, as well as the appropriate morphisms between them:

Definition 3.2.1.1. A Φ-∞-operad is an inner fibration $p\colon \mathcal{O}^{\otimes} \to \mathcal{L}^{\Phi}$ such that:

(i) For each inert map $\phi\colon I \twoheadrightarrow J$ in $\mathcal{L}^{\Phi}$ and every $X \in \mathcal{O}^{\otimes}_{I}$, there exists a p-coCartesian edge $X \to \phi_! X$ over ϕ.

(ii) For every I in $\mathcal{L}^{\Phi}$, the map
$$\mathcal{O}^{\otimes}_{I} \to \prod_{i \in |I|} \mathcal{O}^{\otimes}_{*}$$
induced by the inert maps $i^{\vee}\colon I \twoheadrightarrow *$ is an equivalence.

(iii) Given $C \in \mathcal{O}^{\otimes}_{I}$ and coCartesian morphisms $i^{\vee}_{!}\colon C \to C_i$ for each inert map $i^{\vee}\colon I \twoheadrightarrow *$, the object C is a p-limit of the C_i's.

Remark 3.2.1.2. If $\mathcal{O}^{\otimes}$ is a Φ-∞-operad we will sometimes denote the fibre $\mathcal{O}^{\otimes}_{*}$ at $* \in \mathcal{L}^{\Phi}$ by $\mathcal{O}$.

Definition 3.2.1.3. A Φ-*monoidal ∞-category* is a Φ-∞-operad that is also a coCartesian fibration.

Definition 3.2.1.4. A *generalized Φ-∞-operad* is an inner fibration $p\colon \mathcal{M} \to \mathcal{L}^{\Phi}$ such that:

(i) For each inert map $\phi\colon I \twoheadrightarrow J$ in $\mathcal{L}^{\Phi}$ and every $X \in \mathcal{M}_I$, there exists a p-coCartesian edge $X \to \phi_! X$ over ϕ.

(ii) For every I in $\mathcal{L}^{\Phi}$, the map
$$\mathcal{M}_I \to \lim_{I \twoheadrightarrow A \in \mathcal{G}^{\Phi}_{I/}} \mathcal{M}_A$$
induced by the inert morphisms $I \twoheadrightarrow A$ is an equivalence.

(iii) Every coCartesian section $(\mathcal{G}^{\Phi}_{I/})^{\triangleleft} \to \mathcal{M}$ is a p-limit diagram.

Remark 3.2.1.5. Condition (iii) in the definition says, roughly speaking, that given $C \in \mathcal{M}_I$, $D \in \mathcal{M}_J$, and $\phi\colon J \twoheadrightarrow I$, the map
$$\mathrm{Map}^{\phi}_{\mathcal{M}}(D, C) \to \lim_{\xi \in \mathcal{G}^{\Phi}_{I/}} \mathrm{Map}^{\xi\phi}_{\mathcal{M}}(D, \xi_! C)$$
is an equivalence, where the superscripts denote the obvious fibres over maps in $\mathcal{L}^{\Phi}$.

Definition 3.2.1.6. A Φ-*multiple ∞-category* is a generalized Φ-∞-operad that is also a coCartesian fibration.

Definition 3.2.1.7. We refer to $\mathbb{O}$-∞-operads as *non-symmetric ∞-operads*, generalized $\mathbb{O}$-∞-operads as *generalized non-symmetric ∞-operads*, $\mathbb{O}$-monoidal ∞-categories as *monoidal ∞-categories*, and $\mathbb{O}$-multiple ∞-categories as *double ∞-categories*. Similarly, we refer to $\mathbb{F}$-∞-operads as *symmetric ∞-operads*, generalized $\mathbb{F}$-∞-operads as *generalized symmetric ∞-operads*, and $\mathbb{F}$-monoidal ∞-categories as *symmetric monoidal ∞-categories*.

Definition 3.2.1.8. Let $\pi\colon \mathcal{M} \to \mathcal{L}^{\Phi}$ be a generalized Φ-∞-operad. We say a morphism f in $\mathcal{M}$ is *inert* if it is coCartesian and $\pi(f)$ is an inert morphism in $\mathcal{L}^{\Phi}$, and *active* if $\pi(f)$ is an active morphism in $\mathcal{L}^{\Phi}$.

Lemma 3.2.1.9. The active and inert morphisms form a factorization system on any generalized Φ-∞-operad.

Proof. This is a special case of [Lur11, Proposition 2.1.2.5]. □

Definition 3.2.1.10. Let $\mathcal{M}, \mathcal{N} \to \mathcal{L}^{\Phi}$ be (generalized) Φ-∞-operads. A morphism of (generalized) Φ-∞-operads from $\mathcal{M}$ to $\mathcal{N}$ is a commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr] & & \mathcal{N} \\ & \mathcal{L}^{\Phi} \arrow[ur] & \end{tikzcd}$$

such that F takes inert morphisms in $\mathcal{M}$ to inert morphisms in $\mathcal{N}$. A morphism of (generalized) Φ-∞-operads is a *fibration of (generalized) Φ-∞-operads* if it is also a categorical fibration, and a *coCartesian fibration of (generalized) Φ-∞-operads* if it is also a coCartesian fibration.

Definition 3.2.1.11. If $\mathcal{M}$ and $\mathcal{N}$ are generalized Φ-∞-operads, then an $\mathcal{M}$-*algebra* in $\mathcal{N}$ is just a morphism of generalized Φ-∞-operads $\mathcal{M} \to \mathcal{N}$. We write $\mathrm{Alg}^{\Phi}_{\mathcal{M}}(\mathcal{N})$ for the full subcategory of $\mathrm{Fun}_{\mathcal{L}^{\Phi}}(\mathcal{M}, \mathcal{N})$ spanned by the $\mathcal{M}$-algebras. Similarly, if $\mathcal{M}$ and $\mathcal{N}$ are generalized Φ-∞-operads over a generalized Φ-∞-operad $\mathcal{Q}$, then we write $\mathrm{Alg}^{\Phi}_{\mathcal{M}/\mathcal{Q}}(\mathcal{N})$ for the full subcategory of $\mathrm{Fun}_{\mathcal{Q}}(\mathcal{M}, \mathcal{N})$ spanned by the functors that preserve inert morphisms.

Remark 3.2.1.12. We will also refer to a morphism of (generalized) Φ-∞-operads between Φ-monoidal ∞-categories, or more generally Φ-multiple ∞-categories, as a *lax monoidal functor*.

Definition 3.2.1.13. A *strong monoidal functor* between Φ-multiple ∞-categories is a lax monoidal functor that preserves *all* coCartesian morphisms. If $\mathcal{M}$ and $\mathcal{N}$ are Φ-multiple ∞-categories, we write $\mathrm{Fun}^{\otimes}(\mathcal{M}, \mathcal{N})$ for the full subcategory of $\mathrm{Fun}_{\mathcal{L}^{\Phi}}(\mathcal{M}, \mathcal{N})$ spanned by the strong monoidal functors.

Definition 3.2.1.14. If $\mathcal{M}$ is a generalized Φ-∞-operad, then an $\mathcal{M}$-*multiple ∞-category* $\mathcal{N}$ is a coCartesian fibration of generalized Φ-∞-operads $\mathcal{N} \to \mathcal{M}$. Similarly, an $\mathcal{M}$-*monoidal ∞-category* is an $\mathcal{M}$-multiple ∞-category $\mathcal{E}^{\otimes} \to \mathcal{M}$ such that $\mathcal{E}^{\otimes}$ is a Φ-∞-operad. A *strong monoidal functor* between $\mathcal{M}$-multiple ∞-categories is a morphism of generalized Φ-∞-operads over $\mathcal{M}$ that preserves all coCartesian morphisms.

One source of Φ-∞-operads is simplicial Φ-operads:

Definition 3.2.1.15. A simplicial Φ-operad $\mathcal{O}$ is *fibrant* if all the simplicial sets $\mathcal{O}_I((x_i)_{i\in|I|}, y)$ where $x_i, y \in \operatorname{ob}\mathcal{O}$ are Kan complexes.

Lemma 3.2.1.16. Suppose $\mathcal{O}$ is a fibrant simplicial Φ-operad, and let $\mathcal{O}^{\otimes}$ denote the simplicially enriched version of the May-Thomason category. Then the projection $\mathrm{N}\mathcal{O}^{\otimes} \to \mathcal{L}^{\Phi}$ is a Φ-∞-operad.

Proof. As [Lur11, Proposition 2.1.1.27]. □

3.2.2 Model Categories of ∞-Operads

Using Lurie's theory of *categorical patterns*, we now construct ∞-categories and $(\infty, 2)$-categories of the objects defined above.

Definition 3.2.2.1. Let Φ be a perfect operator category. Write I_Φ for the set of inert morphisms in $\mathcal{L}^{\Phi}$, M_Φ for the set of 2-simplices in $\mathrm{N}\mathcal{L}^{\Phi}$ all of whose edges are inert morphisms, and for $I \in \Phi$ let K_I be the set of inert morphisms $I \nrightarrow *$ in $\mathcal{L}^{\Phi}$. Then we define $\mathfrak{O}_\Phi$ to be the categorical pattern

$$(\mathrm{N}\mathcal{L}^{\Phi}, I_\Phi, M_\Phi, \{K_I^{\triangleleft} \to \mathcal{L}^{\Phi}\}).$$

Lemma 3.2.2.2. A map $(X, S) \to (\mathcal{L}^{\Phi}, I_\Phi)$ is $\mathfrak{O}_\Phi$-fibred if and only if the underlying map $X \to \mathcal{L}^{\Phi}$ is a Φ-∞-operad and S is the set of inert morphisms in X.

Definition 3.2.2.3. Let $\mathfrak{O}_\Phi^{\mathrm{gen}}$ be the categorical pattern

$$(\mathrm{N}\mathcal{L}^{\Phi}, I_\Phi, M_\Phi, \{(\mathcal{G}^{\Phi}_{I/})^{\triangleleft} \to \mathcal{L}^{\Phi}\}).$$

Lemma 3.2.2.4. A map $(X, S) \to (\mathcal{L}^{\Phi}, I_\Phi)$ is $\mathfrak{O}_\Phi^{\mathrm{gen}}$-fibred if and only if the underlying map $X \to \mathcal{L}^{\Phi}$ is a generalized Φ-∞-operad and S is the set of inert morphisms in X.

Definition 3.2.2.5. Let Φ be a perfect operator category. The categorical patterns $\mathfrak{O}_\Phi$ and $\mathfrak{O}_\Phi^{\mathrm{gen}}$ induce two model structures on $(\mathrm{Set}_\Delta^+)_{/(\mathcal{L}^\Phi, I_\Phi)}$. We call these the Φ-∞-*operad model structure* and the *generalized* Φ-∞-*operad model structure*, respectively.

Definition 3.2.2.6. The ∞-categories $\mathrm{Opd}_\infty^{\Phi}$ and $\mathrm{Opd}_\infty^{\Phi,\mathrm{gen}}$ of Φ-∞-operads and generalized Φ-∞-operads are the ∞-categories associated to the simplicial model categories $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_\Phi}$ and $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_\Phi^{\mathrm{gen}}}$, respectively. Since these model categories are enriched in marked simplicial sets by Remark 2.1.12.10, they also define $(\infty, 2)$-categories $\mathrm{OPD}_\infty^{\Phi}$ and $\mathrm{OPD}_\infty^{\Phi,\mathrm{gen}}$.

Remark 3.2.2.7. If $\mathcal{M}$ and $\mathcal{N}$ are generalized Φ-∞-operads, then the ∞-category $\mathrm{Alg}^{\Phi}_{\mathcal{M}}(\mathcal{N})$ of $\mathcal{M}$-algebras in $\mathcal{N}$ is the mapping ∞-category in the $(\infty, 2)$-category $\mathrm{OPD}_\infty^{\Phi,\mathrm{gen}}$.

Definition 3.2.2.8. Let Φ be a perfect operator category. Define $\mathfrak{M}_\Phi$ to be the categorical pattern

$$(\mathrm{N}\mathcal{L}^{\Phi}, (\mathrm{N}\mathcal{L}^{\Phi})_1, (\mathrm{N}\mathcal{L}^{\Phi})_2, \{K_I^{\triangleleft} \to \mathcal{L}^{\Phi}\}).$$

Lemma 3.2.2.9. A map $(X, S) \to \mathcal{L}^{\Phi,\sharp}$ is $\mathfrak{M}_\Phi$-fibred if and only if the underlying map $X \to \mathcal{L}^{\Phi}$ is a Φ-monoidal ∞-category and S is the set of coCartesian morphisms in X.

Definition 3.2.2.10. Let Φ be a perfect operator category. Define $\mathfrak{M}_\Phi^{\mathrm{gen}}$ to be the categorical pattern

$$(\mathrm{N}\mathcal{L}^{\Phi}, (\mathrm{N}\mathcal{L}^{\Phi})_1, (\mathrm{N}\mathcal{L}^{\Phi})_2, \{(\mathcal{G}^{\Phi}_{I/})^{\triangleleft} \to \mathcal{L}^{\Phi}\}).$$

Lemma 3.2.2.11. A map $(X, S) \to \mathcal{L}^{\Phi,\sharp}$ is $\mathfrak{M}_{\Phi}^{\mathrm{gen}}$-fibred if and only if the underlying map $X \to \mathcal{L}^{\Phi}$ is a Φ-multiple ∞-category and S is the set of coCartesian morphisms in X.

Definition 3.2.2.12. Let Φ be a perfect operator category. The categorical patterns $\mathfrak{M}_{\Phi}$ and $\mathfrak{M}_{\Phi}^{\mathrm{gen}}$ induce two model structures on $(\mathrm{Set}_{\Delta}^{+})_{/\mathcal{L}^{\Phi,\sharp}}$. We call these the *$\Phi$-monoidal ∞-category model structure* and the *Φ-multiple ∞-category model structure*, respectively.

Definition 3.2.2.13. The ∞-categories $\mathrm{Mon}_{\infty}^{\Phi}$ and $\mathrm{Mult}_{\infty}^{\Phi}$of Φ-monoidal ∞-categories and Φ-multiple ∞-categories, and strong monoidal functors, are the ∞-categories associated to the simplicial model categories $(\mathrm{Set}_{\Delta}^{+})_{\mathfrak{M}_{\Phi}}$ and $(\mathrm{Set}_{\Delta}^{+})_{\mathfrak{M}_{\Phi}^{\mathrm{gen}}}$, respectively. Since these model categories are enriched in marked simplicial sets by Remark 2.1.12.10, they also define $(\infty, 2)$-categories $\mathrm{MON}_{\infty}^{\Phi}$ and $\mathrm{MULT}_{\infty}^{\Phi}$.

Definition 3.2.2.14. If $\mathcal{M}$ is a generalized Φ-∞-operad, we write $\mathrm{Mon}_{\infty}^{\Phi,\mathcal{M}}$ for the full subcategory of $\mathrm{CoCart}(\mathcal{M})$ spanned by the $\mathcal{M}$-monoidal ∞-categories, and $\mathrm{Mult}_{\infty}^{\Phi,\mathcal{M}}$ for the full subcategory spanned by the $\mathcal{M}$-multiple ∞-categories.

Proposition 3.2.2.15. The identity is a right (marked simplicially enriched) Quillen functor $(\mathrm{Set}_{\Delta}^{+})_{\mathfrak{O}_{\Phi}} \to (\mathrm{Set}_{\Delta}^{+})_{\mathfrak{O}_{\Phi}^{\mathrm{gen}}}$ and $(\mathrm{Set}_{\Delta}^{+})_{\mathfrak{M}_{\Phi}} \to (\mathrm{Set}_{\Delta}^{+})_{\mathfrak{M}_{\Phi}^{\mathrm{gen}}}$.

Proof. As [Lur11, Corollary 2.3.2.6]. □

Corollary 3.2.2.16. The inclusions $\mathrm{Opd}_{\infty}^{\Phi} \to \mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}}$ and $\mathrm{Mon}_{\infty}^{\Phi} \to \mathrm{Mult}_{\infty}^{\Phi}$ have left adjoints $\mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}} \to \mathrm{Opd}_{\infty}^{\Phi}$ and $\mathrm{Mult}_{\infty}^{\Phi} \to \mathrm{Mon}_{\infty}^{\Phi}$.

Remark 3.2.2.17. There are obvious maps of categorical patterns $\mathfrak{O}_{\Phi} \to \mathfrak{M}_{\Phi}$ and $\mathfrak{O}_{\Phi}^{\mathrm{gen}} \to \mathfrak{M}_{\Phi}^{\mathrm{gen}}$. These induce adjunctions

$$\mathrm{Opd}_{\infty}^{\Phi} \rightleftarrows \mathrm{Mon}_{\infty}^{\Phi},$$

$$\mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}} \rightleftarrows \mathrm{Mult}_{\infty}^{\Phi}.$$

Definition 3.2.2.18. We write $\mathrm{Mon}_{\infty}^{\Phi,\mathrm{lax}}$ and $\mathrm{Mult}_{\infty}^{\Phi,\mathrm{lax}}$ for the full subcategories of $\mathrm{Opd}_{\infty}^{\Phi}$ and $\mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}}$ spanned by the Φ-monoidal ∞-categories and Φ-multiple ∞-categories, respectively.

Definition 3.2.2.19. Let Φ and Ψ be perfect operator categories, and let $f\colon \Phi \to \Psi$ be an operator morphism. Then f induces a map of categorical patterns $\mathfrak{O}_{\Phi} \to \mathfrak{O}_{\Psi}$, and so an adjunction

$$f_{!}\colon \mathrm{Opd}_{\infty}^{\Phi} \rightleftarrows \mathrm{Opd}_{\infty}^{\Psi} : f^{*}.$$

Example 3.2.2.20. If Φ is a perfect operator category, the operator morphism u^{Φ} induces a functor $u_{!}^{\Phi}\colon \mathrm{Opd}_{\infty}^{\Phi} \to \mathrm{Opd}_{\infty}^{\mathbb{F}}$. We write $\mathcal{U}_{\Phi}^{\otimes}$ for the symmetric ∞-operad $u_{!}^{\Phi}\mathcal{L}^{\Phi}$.

Remark 3.2.2.21. An operator morphism does not in general induce functors between ∞-categories of generalized ∞-operads.

Definition 3.2.2.22. Let Φ be a perfect operator category, and let A be a clean atom in Φ. Then the inclusion $j^{A}\colon \Phi_{/A} \to \Phi$ induces a map of categorical patterns $\mathfrak{O}_{\Phi_{/A}}^{\mathrm{gen}} \to \mathfrak{O}_{\Phi}^{\mathrm{gen}}$, and so an adjunction

$$j_{!}^{A}\colon \mathrm{Opd}_{\infty}^{\Phi_{/A},\mathrm{gen}} \to \mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}} \colon j^{A,*}.$$

Conjecture 3.2.2.23. The functor $j^{A,*}$ induced by a clean atom A in a perfect operator category Φ also has a right adjoint $j_{*}^{A}\colon \mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}} \colon \mathrm{Opd}_{\infty}^{\Phi_{/A},\mathrm{gen}}$.

Definition 3.2.2.24. By [Lur11, Proposition 6.3.1.14] the ∞-category Pr^{L} of presentable ∞-categories and colimit-preserving functors has a symmetric monoidal structure $(\mathrm{Pr}^{\mathrm{L}})^{\otimes}$ such that a colimit-preserving functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is equivalent to a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that is colimit-preserving in each variable. If Φ is a perfect operator category, we write $\mathrm{Mon}_{\infty}^{\Phi,\mathrm{Pr}}$ for the ∞-category $\mathrm{Alg}^{\Phi}_{\mathcal{L}^{\Phi}}(u^{\Phi,*}(\mathrm{Pr}^{\mathrm{L}})^{\otimes})$. We will refer to objects of $\mathrm{Mon}_{\infty}^{\Phi,\mathrm{Pr}}$ as *presentably Φ-monoidal ∞-categories*. These are Φ-monoidal ∞-categories $\mathcal{C}^{\otimes} \to \mathcal{L}^{\Phi}$ such that $\mathcal{C}$ is a presentable ∞-category and the operations $\phi_{!}\colon \mathcal{C}^{\times |I|} \simeq \mathcal{C}^{\otimes}_{I} \to \mathcal{C}$ induced by active morphisms $\phi\colon I \to *$ preserve colimits in each variable. Morphisms in $\mathrm{Mon}_{\infty}^{\Phi,\mathrm{Pr}}$ correspond to strong monoidal functors $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ such that $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits.

3.2.3 Trivial Generalized ∞-Operads

Definition 3.2.3.1. Let $\mathcal{M}$ be a generalized Φ-∞-operad. Define the generalized Φ-∞-operad $\mathcal{M}_{\mathrm{triv}}$ by the pullback diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{triv}} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{L}^{\Phi}_{\mathrm{int}} & \longrightarrow & \mathcal{L}^{\Phi} \end{array}$$

This is the *trivial generalized Φ-∞-operad over* $\mathcal{M}$.

Definition 3.2.3.2. Let $\mathfrak{O}^{\mathrm{triv}}_{\Phi}$ denote the categorical pattern

$$(\mathrm{N}\mathcal{L}^{\Phi}_{\mathrm{int}}, (\mathrm{N}\mathcal{L}^{\Phi}_{\mathrm{int}})_1, (\mathrm{N}\mathcal{L}^{\Phi}_{\mathrm{int}})_2, \{(\mathcal{G}^{\Phi}_{I/})^{\triangleleft} \to \mathcal{L}^{\Phi}_{\mathrm{int}}\}).$$

Remark 3.2.3.3. An object (X, S) of $(\mathrm{Set}^{+}_{\Delta})_{/\mathcal{L}^{\Phi,\sharp}_{\mathrm{int}}}$ is thus $\mathfrak{O}^{\mathrm{triv}}_{\Phi}$-fibrant if $X \to \mathcal{L}^{\Phi}_{\mathrm{int}}$ is a coCartesian fibration, S is the set of coCartesian edges, and the Segal morphisms $X_I \to \lim_{(I \twoheadrightarrow A) \in \mathcal{G}^{\Phi}_{I/}} X_A$ are equivalences.

Under the equivalence between coCartesian fibrations and functors the ∞-category associated to the model category $(\mathrm{Set}^{+}_{\Delta})_{\mathfrak{O}^{\mathrm{triv}}_{\Phi}}$ therefore corresponds to the full subcategory of $\mathrm{Fun}(\mathcal{L}^{\Phi}_{\mathrm{int}}, \mathrm{Cat}_{\infty})$ spanned by the functors that are right Kan extensions along the inclusion $\gamma^{\Phi}\colon \mathcal{G}^{\Phi} \to \mathcal{L}^{\Phi}_{\mathrm{int}}$. Thus we have proved the following:

Lemma 3.2.3.4. The ∞-category associated to the model category $(\mathrm{Set}^{+}_{\Delta})_{\mathfrak{O}^{\mathrm{triv}}_{\Phi}}$ is equivalent to $\mathrm{Fun}(\mathcal{G}^{\Phi}, \mathrm{Cat}_{\infty})$.

The obvious map of categorical patterns $\mathfrak{O}^{\mathrm{triv}}_{\Phi} \to \mathfrak{O}^{\mathrm{gen}}_{\Phi}$ then induces an adjoint pair of functors

$$\gamma^{\Phi}_{!}\colon \mathrm{Fun}(\mathcal{G}^{\Phi}, \mathrm{Cat}_{\infty}) \rightleftarrows \mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty} : \gamma^{\Phi,*}.$$

Since composition with the inclusion $\mathcal{L}^{\Phi}_{\mathrm{int}} \to \mathcal{L}^{\Phi}$ takes $\mathfrak{O}^{\mathrm{triv}}_{\Phi}$-fibrant objects to $\mathfrak{O}^{\mathrm{gen}}_{\Phi}$-fibrant objects, the left adjoint $\gamma^{\Phi}_{!}$ sends a functor $\mathcal{G}^{\Phi} \to \mathrm{Cat}_{\infty}$ to its right Kan extension to $\mathcal{L}^{\Phi}_{\mathrm{int}} \to \mathrm{Cat}_{\infty}$, then to the composite $\mathcal{E} \to \mathcal{L}^{\Phi}_{\mathrm{int}} \to \mathcal{L}^{\Phi}$, where $\mathcal{E} \to \mathcal{L}^{\Phi}_{\mathrm{int}}$ is the associated coCartesian fibration. In particular, if $\mathcal{M}$ is a generalized Φ-∞-operad, then $\mathcal{M}_{\mathrm{triv}}$ is $\gamma^{\Phi}_{!}\gamma^{*}_{\Phi}\mathcal{M}$, and the natural map $\mathcal{M}_{\mathrm{triv}} \to \mathcal{M}$ is the adjunction morphism.

Taking the $(\infty, 2)$-categories associated to the categorical patterns into account, we get the following:

Proposition 3.2.3.5. Let $F\colon \mathcal{G}^{\Phi} \to \mathrm{Cat}_{\infty}$ be a functor, and let $\mathcal{F} \to \mathcal{G}^{\Phi}$ be the associated coCartesian fibration. If $\mathcal{M}$ is a generalized Φ-∞-operad let $\mathcal{M}_{\mathrm{glob}}$ denote the pullback of $\mathcal{M}$ along $\mathcal{G}^{\Phi} \to \mathcal{L}^{\Phi}$. Then there is a natural equivalence between $\mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F}(\mathcal{M})$ and the full subcategory $\mathrm{Fun}^{\mathrm{coCart}}_{\mathcal{G}^{\Phi}}(\mathcal{F}, \mathcal{M}_{\mathrm{glob}})$ of $\mathrm{Fun}_{\mathcal{G}^{\Phi}}(\mathcal{F}, \mathcal{M}_{\mathrm{glob}})$ spanned by functors that preserve coCartesian arrows. In particular, if $\mathcal{O}^{\otimes}$ is a Φ-∞-operad, then

$$\mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F}(\mathcal{O}^{\otimes}) \simeq \mathrm{Fun}(F(*), \mathcal{O}).$$

Corollary 3.2.3.6. Let $F\colon \mathcal{G}^{\Phi} \to \mathrm{Cat}_{\infty}$ be a functor, and let $\mathcal{F} \to \mathcal{G}^{\Phi}$ be the associated coCartesian fibration. Given a morphism of generalized Φ-∞-operads $\mathcal{M} \to \mathcal{N}$ there is a natural equivalence between $\mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F/\mathcal{N}}(\mathcal{M})$ and the full subcategory $\mathrm{Fun}^{\mathrm{coCart}}_{\mathcal{N}_{\mathrm{glob}}}(\mathcal{F}, \mathcal{M}_{\mathrm{glob}})$ of $\mathrm{Fun}_{\mathcal{N}_{\mathrm{glob}}}(\mathcal{F}, \mathcal{M}_{\mathrm{glob}})$ spanned by functors that preserve coCartesian arrows. In particular, if $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a morphism of Φ-∞-operads, then

$$\mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F/\mathcal{P}^{\otimes}}(\mathcal{O}^{\otimes}) \simeq \mathrm{Fun}_{\mathcal{P}}(F(*), \mathcal{O}).$$

Proof. Apply Proposition 3.2.3.5 to identify the fibre in the pullback square

$$\begin{tikzcd}
\mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F/\mathcal{N}}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F}(\mathcal{M}) \arrow[d] \\
* \arrow[r] & \mathrm{Alg}^{\Phi}_{\gamma_!^{\Phi}F}(\mathcal{N}).
\end{tikzcd}$$

□

3.2.4 Monoids and Category Objects

Definition 3.2.4.1. Let Φ be a perfect operator category. Suppose $\mathcal{M}$ is a small generalized Φ-∞-operad and $\mathcal{C}$ an ∞-category with small limits. An $\mathcal{M}$-*monoid object* in $\mathcal{C}$ is a functor $F\colon \mathcal{M} \to \mathcal{C}$ such that its restriction $F|_{\mathcal{M}_{\mathrm{triv}}}$ is a right Kan extension of $F|_{\mathcal{M}_*}$ along the inclusion $\mathcal{M}_* \hookrightarrow \mathcal{M}_{\mathrm{triv}}$. Write $\mathrm{Mnd}^{\Phi}_{\mathcal{M}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{C})$ spanned by the $\mathcal{M}$-monoid objects. When $\mathcal{M}$ is $\mathcal{L}^{\Phi}$ we refer to $\mathcal{L}^{\Phi}$-monoids as just Φ-monoids and write $\mathrm{Mnd}^{\Phi}(\mathcal{C})$ for $\mathrm{Mnd}^{\Phi}_{\mathcal{L}^{\Phi}}(\mathcal{C})$.

Definition 3.2.4.2. Let Φ be a perfect operator category. Suppose $\mathcal{M}$ is a small generalized Φ-∞-operad and $\mathcal{C}$ is an ∞-category with small limits. An $\mathcal{M}$-*category object* in $\mathcal{C}$ is a functor $F\colon \mathcal{M} \to \mathcal{C}$ such that its restriction $F|_{\mathcal{M}_{\mathrm{triv}}}$ is a right Kan extension of $F|_{\mathcal{M}_{\mathrm{glob}}}$ along the inclusion $\mathcal{M}_{\mathrm{glob}} \hookrightarrow \mathcal{M}_{\mathrm{triv}}$. Write $\mathrm{Cat}^{\Phi}_{\mathcal{M}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{C})$ spanned by the $\mathcal{M}$-category objects. When $\mathcal{M}$ is $\mathcal{L}^{\Phi}$ we refer to $\mathcal{L}^{\Phi}$-category objects as just Φ-*category objects* and write $\mathrm{Cat}^{\Phi}(\mathcal{C})$ for $\mathrm{Cat}^{\Phi}_{\mathcal{L}^{\Phi}}(\mathcal{C})$.

Definition 3.2.4.3. Let Φ be a perfect operator category. A *Segal Φ-space* is a Φ-category object in the ∞-category $\mathcal{S}$ of spaces. We write $\mathrm{Seg}^{\Phi}_{\infty}$ for the ∞-category $\mathrm{Cat}^{\Phi}_{\mathcal{L}^{\Phi}}(\mathcal{S})$ of Segal Φ-spaces.

Example 3.2.4.4. Segal $\mathbb{O}(n)$-spaces are precisely Θ_n-spaces as defined in §2.2.3.

Proposition 3.2.4.5. Suppose $\mathcal{C}$ is an ∞-category with small limits, and denote the pullback of the Cartesian symmetric monoidal structure on $\mathcal{C}$ to a Φ-monoidal structure by $\mathcal{C}^{\times} \to \mathcal{L}^{\Phi}$. Then for any generalized Φ-∞-operad $\mathcal{M}$ we have $\mathrm{Alg}^{\Phi}_{\mathcal{M}}(\mathcal{C}^{\times}) \simeq \mathrm{Mnd}^{\Phi}_{\mathcal{M}}(\mathcal{C})$.

Proof. As [Lur11, Proposition 2.4.2.5]. □

Proposition 3.2.4.6. Let $\mathcal{M}$ be a Φ-multiple ∞-category. We have equivalences $\mathrm{Mon}^{\Phi,\mathcal{M}}_{\infty} \simeq \mathrm{Mnd}^{\Phi}_{\mathcal{M}}(\mathrm{Cat}_{\infty})$ and $\mathrm{Mult}^{\Phi,\mathcal{M}}_{\infty} \simeq \mathrm{Cat}^{\Phi}_{\mathcal{M}}(\mathrm{Cat}_{\infty})$.

Proof. We can identify $\mathrm{Mon}^{\Phi,\mathcal{M}}_{\infty}$ with the full subcategory of the ∞-category of coCartesian fibrations over $\mathcal{M}$ spanned by the $\mathcal{M}$-monoidal ∞-categories. Under the equivalence between coCartesian fibrations over $\mathcal{M}$ and functors $\mathcal{M} \to \mathrm{Cat}_{\infty}$ these correspond precisely to those functors satisfying the condition for a monoid object. Similarly, the double ∞-categories correspond to the category objects. □

Proposition 3.2.4.7. Let Φ be a perfect operator category and suppose $\mathcal{M}$ is a generalized Φ-∞-operad. Pullback along $\mathcal{M} \to u^{\Phi}_{!}\mathcal{M}$ gives an equivalence $\mathrm{Mon}^{\Phi,\mathcal{M}}_{\infty} \simeq \mathrm{Mon}^{\mathbb{F},u^{\Phi}_{!}\mathcal{M}}_{\infty}$. In particular, we have an equivalence $\mathrm{Mon}^{\Phi}_{\infty} \simeq \mathrm{Mon}^{\mathbb{F},\mathcal{U}^{\otimes}_{\Phi}}_{\infty}$.

Proof. Using Proposition 3.2.4.6 we have a sequence of equivalences

$$\begin{aligned}\mathrm{Mon}^{\Phi,\mathcal{M}}_{\infty} &\simeq \mathrm{Mnd}^{\Phi}_{\mathcal{M}}(\mathrm{Cat}_{\infty}) \simeq \mathrm{Alg}^{\Phi}_{\mathcal{M}}(\mathrm{Cat}^{\times}_{\infty}) \simeq \mathrm{Alg}^{\mathbb{F}}_{u^{\Phi}_{!}\mathcal{M}}(\mathrm{Cat}^{\times}_{\infty}) \\ &\simeq \mathrm{Mnd}^{\mathbb{F}}_{u^{\Phi}_{!}\mathcal{M}}(\mathrm{Cat}_{\infty}) \simeq \mathrm{Mon}^{\Phi,u^{\Phi}_{!}\mathcal{M}}_{\infty}.\end{aligned}$$

□

3.2.5 Filtered Colimits of ∞-Operads

Colimits of (generalized) Φ-∞-operads are in general difficult to describe explicitly. However, we will now show that filtered colimits can be computed in Cat_{∞}:

Lemma 3.2.5.1. Let q be a diagram in $\mathrm{Opd}^{\Phi}_{\infty}$ or $\mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty}$, and let $\mathcal{Q}$ be a colimit of q composed with the forgetful functor to Cat_{∞}; there is a canonical map $\mathcal{Q} \to \mathcal{L}^{\Phi}$. If $\mathcal{Q} \to \mathcal{L}^{\Phi}$ is a (generalized) Φ-∞-operad, then this is the colimit of q.

Proof. By Example 2.1.12.15, the object $\mathcal{Q} \to \mathcal{L}^{\Phi}$ is the colimit of the diagram obtained by composing q with the inclusion to $\mathrm{CoCart}(\mathcal{L}^{\Phi}, \mathcal{L}^{\Phi}_{\mathrm{int}})$. But by Remark 2.1.12.11 the ∞-categories $\mathrm{Opd}^{\Phi}_{\infty}$ and $\mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty}$ are localizations of $\mathrm{CoCart}(\mathcal{L}^{\Phi}, \mathcal{L}^{\Phi}_{\mathrm{int}})$, so the colimit of q is obtained by localizing the colimit in $\mathrm{CoCart}(\mathcal{L}^{\Phi}, \mathcal{L}^{\Phi}_{\mathrm{int}})$. Thus if this colimit is already a (generalized) Φ-∞-operad, it is also the colimit in the full subcategory of (generalized) Φ-∞-operads. □

Lemma 3.2.5.2. The forgetful functors $\mathrm{Opd}^{\Phi}_{\infty}, \mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty} \to \mathrm{Cat}_{\infty}$ preserve filtered colimits.

Proof. Let p be a filtered diagram in $\mathrm{Opd}^{\Phi}_{\infty}$ or $\mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty}$, and let $\mathcal{P}$ be a colimit of the diagram obtained by composing p with the forgetful functor to Cat_{∞}. By Lemma 3.2.5.1 to prove that $\mathcal{P} \to \mathcal{L}^{\Phi}$ is the colimit of the diagram p it suffices to show that $\mathcal{P} \to \mathcal{L}^{\Phi}$ is a (generalized) Φ-∞-operad.

In other words, we must show that $\mathcal{P}$, considered as an object of $\mathrm{CoCart}(\mathcal{L}^{\Phi}, \mathcal{L}^{\Phi}_{\mathrm{int}})$, is local with respect to the generating $\mathfrak{P}$-anodyne maps, where $\mathfrak{P}$ is the categorical pattern

$\mathfrak{O}_\Phi$ or $\mathfrak{O}_\Phi^{\mathrm{gen}}$. Compact objects in $\mathrm{CoCart}(\mathcal{L}^\Phi, \mathcal{L}^\Phi_{\mathrm{int}})$ are detected in Cat_∞ (since the right adjoint to the forgetful functor, which sends an ∞-category $\mathcal{C}$ to $\mathcal{C} \times \mathcal{L}^\Phi \to \mathcal{L}^\Phi$, clearly preserves colimits). It is therefore clear from Definition 2.1.12.6 that for each generating $\mathfrak{P}$-anodyne map $f\colon A \to B$ both A and B are compact objects in $\mathrm{CoCart}(\mathcal{L}^\Phi, \mathcal{L}^\Phi_{\mathrm{int}})$. It follows that in the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(B, \mathcal{P}) \arrow[r] \arrow[d] & \mathrm{colim}_\alpha \mathrm{Map}(B, p(\alpha)) \arrow[d] \\
\mathrm{Map}(A, \mathcal{P}) \arrow[r] & \mathrm{colim}_\alpha \mathrm{Map}(A, p(\alpha))
\end{tikzcd}$$

the horizontal maps are equivalences, since p is a filtered diagram. The right vertical map is also an equivalence, since $p(\alpha)$ is a (generalized) Φ-∞-operad for all α. Thus the left vertical morphism must also be an equivalence, and so $\mathcal{P}$ is local with respect to f. In other words, $\mathcal{P}$ is a (generalized) Φ-∞-operad, as required. □

3.2.6 Wreath Products

Definition 3.2.6.1. Suppose Φ and Ψ are perfect operator categories. Let $W\colon \mathcal{L}^\Psi \times \mathcal{L}^\Phi \to \mathcal{L}^{\Psi \wr \Phi}$ be the functor that sends (J, I) to $I((J)_{i \in |I|})$ and a morphism $(\psi\colon J \nrightarrow J', \phi\colon I \nrightarrow I')$ to the morphism $I((J)_{i\in|I|}) \nrightarrow I'((J')_{i\in|I'|})$ corresponding to the morphism $I((J)_{i\in|I|}) \to T(I'((J')_{i\in|I'|})) = TI'((K_i))$ (where $K_i = TJ'$ if $i \in |I'| \subseteq |TI'|$ and $*$ otherwise) determined by $\phi\colon I \to I'$ and $J_i \to K_{\phi(i)}$ being either ψ or the unique morphism to $*$, according to whether $\phi(i) \in |I'|$ or not.

Definition 3.2.6.2. Suppose Φ and Ψ are perfect operator categories, and suppose $X \in (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Phi}$ and $Y \in (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Psi}$. Then we define $Y \wr X \in (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{\Psi\wr\Phi}}$ to be the product $Y \times X$, regarded as a marked simplicial set over $\mathcal{L}^{\Phi\wr\Psi}$ via

$$Y \times X \to \mathcal{L}^\Psi \times \mathcal{L}^\Phi \xrightarrow{W} \mathcal{L}^{\Psi\wr\Phi}.$$

Theorem 3.2.6.3 (Barwick, [Bar13, Theorem 9.6])**.** The functor

$$\wr\colon (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Psi} \times (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_\Phi} \to (\mathrm{Set}^+_\Delta)_{\mathfrak{O}_{\Psi\wr\Phi}}$$

is a left Quillen functor in each variable.

Remark 3.2.6.4. Theorem 3.2.6.3 is proved by applying [Lur11, Proposition B.2.9], since W gives a morphism of categorical patterns $\mathfrak{O}_\Psi \times \mathfrak{O}_\Phi \to \mathfrak{O}_{\Psi\wr\Phi}$. This is not the case if we consider generalized ∞-operads however, and so this result does not obviously generalize to this setting.

Consequently, $\wr$ induces a functor of ∞-categories

$$\wr\colon \mathrm{Opd}^\Psi_\infty \times \mathrm{Opd}^\Phi_\infty \to \mathrm{Opd}^{\Psi\wr\Phi}_\infty,$$

with right adjoints

$$\mathrm{Alg}^{\Psi,\Psi\wr\Phi}\colon (\mathrm{Opd}^\Psi_\infty)^{\mathrm{op}} \times \mathrm{Opd}^{\Psi\wr\Phi}_\infty \to \mathrm{Opd}^\Phi_\infty,$$

$$\mathrm{Alg}^{\Phi,\Psi\wr\Phi}\colon (\mathrm{Opd}_\infty^{\Phi})^{\mathrm{op}} \times \mathrm{Opd}_\infty^{\Psi\wr\Phi} \to \mathrm{Opd}_\infty^{\Psi}$$

with respect to the two variables. In other words, if $\mathcal{O}^\otimes$ is a Φ-∞-operad, $\mathcal{P}^\otimes$ is a Ψ-∞-operad, and $\mathcal{Q}^\otimes$ is a $\Psi \wr \Phi$-∞-operad, then we have canonical equivalences

$$\mathrm{Alg}^{\Psi\wr\Phi}_{\mathcal{P}^\otimes\wr\mathcal{O}^\otimes}(\mathcal{Q}^\otimes) \simeq \mathrm{Alg}^{\Phi}_{\mathcal{O}^\otimes}(\mathrm{Alg}^{\Psi,\Psi\wr\Phi}_{\mathcal{P}^\otimes}(\mathcal{Q}^\otimes)),$$

$$\mathrm{Alg}^{\Psi\wr\Phi}_{\mathcal{P}^\otimes\wr\mathcal{O}^\otimes}(\mathcal{Q}^\otimes) \simeq \mathrm{Alg}^{\Psi}_{\mathcal{P}^\otimes}(\mathrm{Alg}^{\Phi,\Psi\wr\Phi}_{\mathcal{O}^\otimes}(\mathcal{Q}^\otimes)).$$

Lemma 3.2.6.5. The underlying ∞-category of the Ψ-∞-operad $\mathrm{Alg}^{\Phi,\Psi\wr\Phi}_{\mathcal{O}^\otimes}(\mathcal{Q}^\otimes)$ can be identified with $\mathrm{Alg}^{\Phi}_{\mathcal{O}^\otimes}(i_\Phi^*\mathcal{Q}^\otimes)$, and the underlying ∞-category of the Φ-∞-operad $\mathrm{Alg}^{\Psi,\Psi\wr\Phi}_{\mathcal{P}^\otimes}(\mathcal{Q}^\otimes)$ with $\mathrm{Alg}^{\Psi}_{\mathcal{P}^\otimes}(j_\Psi^*\mathcal{Q}^\otimes)$.

Proof. Considering $* = \mathcal{L}^{\mathbf{0}} \to \mathcal{L}^{\Phi}$ as an object of $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_\Phi}$ (whose fibrant replacement is given by $\mathcal{L}^{\Phi}_{\mathrm{int}}$) we see that

$$* \wr \mathcal{O}^\otimes = i_{\Phi,!}\mathcal{O}^\otimes$$

as functors $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_\Phi} \to (\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\Psi\wr\Phi}}$. If $\mathcal{C}$ is the underlying ∞-category of $\mathrm{Alg}^{\Phi,\Psi\wr\Phi}_{\mathcal{O}^\otimes}(\mathcal{Q}^\otimes)$, we thus have equivalences

$$\mathcal{C} \simeq \mathrm{Alg}^{\Psi}_{*}(\mathrm{Alg}^{\Phi,\Psi\wr\Phi}_{\mathcal{O}^\otimes}(\mathcal{Q}^\otimes)) \simeq \mathrm{Alg}^{\Psi\wr\Phi}_{*\wr\mathcal{O}^\otimes}(\mathcal{Q}^\otimes) \simeq \mathrm{Alg}^{\Psi\wr\Phi}_{i_{\Phi,!}\mathcal{O}^\otimes}(\mathcal{Q}^\otimes) \simeq \mathrm{Alg}^{\Phi}_{\mathcal{O}^\otimes}(i_\Phi^*\mathcal{Q}^\otimes).$$

Similarly $\mathcal{P}^\otimes \wr * = j_{\Psi,!}\mathcal{P}^\otimes$, which gives the underlying ∞-category of $\mathrm{Alg}^{\Psi,\Psi\wr\Phi}_{\mathcal{P}^\otimes}(\mathcal{Q}^\otimes)$ in the same way. □

Theorem 3.2.6.3 thus gives a "universal property" for the wreath product of ∞-operads.

Proposition 3.2.6.6 (Barwick, [Bar13, Proposition 9.3])**.** Suppose $F\colon \Phi' \to \Phi$ and $G\colon \Psi' \to \Psi$ are operator morphisms, $\mathcal{O}^\otimes$ is a Φ'-∞-operad and $\mathcal{P}^\otimes$ is a Ψ'-∞-operad. There is a natural equivalence

$$(G \wr F)_!(\mathcal{O}^\otimes \wr \mathcal{P}^\otimes) \simeq G_!\mathcal{O}^\otimes \wr F_!\mathcal{P}^\otimes.$$

Definition 3.2.6.7. If $\mathcal{O}^\otimes$ and $\mathcal{P}^\otimes$ are $\mathbb{F}$-∞-operads, we write $\mathcal{O}^\otimes \otimes \mathcal{P}^\otimes$ for $u_!^{\mathbb{F}\wr\mathbb{F}}(\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)$.

This is the *Boardman-Vogt tensor product* of $\mathbb{F}$-∞-operads; it is proved in [Lur11, §2.2.5] that this extends to a symmetric monoidal structure on $\mathrm{Opd}_\infty^{\mathbb{F}}$.

Corollary 3.2.6.8. Suppose Φ and Ψ are perfect operator categories. Then we have a natural equivalence $\mathcal{U}^\otimes_{\Psi\wr\Phi} \simeq \mathcal{U}^\otimes_\Psi \otimes \mathcal{U}^\otimes_\Phi$.

Proof. By definition $\mathcal{U}^\otimes_\Psi \otimes \mathcal{U}^\otimes_\Phi \simeq u_!^{\mathbb{F}\wr\mathbb{F}}(\mathcal{U}^\otimes_\Psi \wr \mathcal{U}^\otimes_\Phi)$. Now Proposition 3.2.6.6 gives an equivalence

$$u_!^{\mathbb{F}\wr\mathbb{F}}(\mathcal{U}^\otimes_\Psi \wr \mathcal{U}^\otimes_\Phi) \simeq u_!^{\mathbb{F}\wr\mathbb{F}}(u^\Psi \wr u^\Phi)_!(\mathcal{L}^\Psi \wr \mathcal{L}^\Phi) \simeq u_!^{\Psi\wr\Phi}(\mathcal{L}^{\Psi\wr\Phi}) \simeq \mathcal{U}^\otimes_{\Psi\wr\Phi}. \qquad \square$$

Corollary 3.2.6.9 (Barwick, [Bar13, Proposition 11.5])**.** There are equivalences $\mathcal{U}^\otimes_{\mathbb{O}(n)} \simeq \mathbb{E}_n^\otimes$.

Proof. Combining [Lur11, Proposition 4.1.2.10] and [Lur11, Example 5.1.0.7] gives an equivalence $\mathcal{U}^\otimes_{\mathbb{O}} \simeq \mathbb{E}_1^\otimes$. Now Corollary 3.2.6.8 and [Lur11, Theorem 5.1.2.2] give equivalences

$$\mathcal{U}^\otimes_{\mathbb{O}(n)} \simeq (\mathcal{U}^\otimes_{\mathbb{O}})^{\otimes n} \simeq (\mathbb{E}_1^\otimes)^{\otimes n} \simeq \mathbb{E}_n^\otimes. \qquad \square$$

Definition 3.2.6.10. An operator morphism $f\colon \Phi \to \Psi$ between perfect operator categories is *étale* if the induced functor

$$f_!\colon \mathrm{Opd}_\infty^\Phi \to (\mathrm{Opd}_\infty^\Psi)_{/f_!\mathcal{L}^\Phi}$$

is an equivalence. We say a perfect operator category Φ is *étale* if the unique operator morphism $u^\Phi\colon \Phi \to \mathbb{F}$ is étale, i.e. induces an equivalence $\mathrm{Opd}_\infty^\Phi \xrightarrow{\sim} (\mathrm{Opd}_\infty^{\mathbb{F}})_{/\mathcal{U}_\Phi^\otimes}$.

Theorem 3.2.6.11 (Lurie)**.** $\mathbb{O}$ is an étale operator category.

Proof. Combine [Lur11, Proposition 4.1.2.10] and [Lur11, Theorem 6.1.1.10]. □

Conjecture 3.2.6.12. The operator categories $\mathbb{O}(n)$ are étale for all n.

In other words, the ∞-category $\mathrm{Opd}_\infty^{\mathbb{O}(n)}$ of $\mathbb{O}(n)$-∞-operads should be equivalent to the ∞-category $(\mathrm{Opd}_\infty^{\mathbb{F}})_{/\mathbb{E}_n^\otimes}$ of symmetric ∞-operads over $\mathbb{E}_n^\otimes$. This may well follow from the results of [Lur11, §5.1.2], or variants thereof, but we will not consider this further here.

3.2.7 Colimits of Algebras

Ideally we would like to prove that colimits of algebras exist by generalizing Lurie's theory of operadic colimits and operadic Kan extensions from symmetric ∞-operads to general ∞-operads, but unfortunately it is not obvious how to carry out such a generalization. In the next section we will summarize the construction for non-symmetric ∞-operads, where trivial variants of Lurie's proofs work. Here, we restrict ourselves to what we can deduce from Lurie's results using adjunctions.

We first consider the symmetric case, i.e. the existence of colimits in the ∞-category $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{P}^\otimes/\mathcal{O}^\otimes}(\mathcal{C}^\otimes)$ where $\mathcal{C}^\otimes$ is an $\mathcal{O}^\otimes$-monoidal ∞-category and $\mathcal{P}^\otimes$ is a symmetric ∞-operad over the symmetric ∞-operad $\mathcal{O}^\otimes$. For this we need slight generalizations of the results of [Lur11, §3.2.3]. We first consider the case of sifted colimits:

Lemma 3.2.7.1 ([Lur11, Lemma 3.2.3.7])**.** Suppose K is a sifted simplicial set and $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is an $\mathcal{O}^\otimes$-monoidal ∞-category that is compatible with K-indexed colimits. Then for every morphism $\phi\colon X \to Y$ in $\mathcal{O}^\otimes$ the associated functor $\phi_!\colon \mathcal{C}^\otimes_X \to \mathcal{C}^\otimes_Y$ preserves K-indexed colimits.

Proposition 3.2.7.2. Suppose K is a sifted simplicial set and $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is an $\mathcal{O}^\otimes$-monoidal ∞-category that is compatible with K-indexed colimits. Then for any morphism $p\colon \mathcal{M} \to \mathcal{O}^\otimes$ of generalized symmetric ∞-operads, we have:

(i) The ∞-category $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ admits K-indexed colimits.

(ii) A map $K^\triangleright \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ is a colimit diagram if and only if for every $X \in \mathcal{M}$ the induced diagram $K^\triangleright \to \mathcal{C}^\otimes_{p(X)}$ is a colimit diagram.

(iii) The full subcategory $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{M}/\mathcal{O}^\otimes}(\mathcal{C}^\otimes)$ of $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ is stable under K-indexed colimits.

(iv) A map $K^\triangleright \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ is a colimit diagram if and only if, for every $X \in \mathcal{O}^\otimes_{\langle 1\rangle}$ and $Y \in \mathcal{M}_X$, the induced diagram $K^\triangleright \to \mathcal{C}^\otimes_X$ is a colimit diagram.

(v) The restriction functor $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{M}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathcal{O}^{\otimes}_{\langle 1\rangle}}(\mathcal{M}_{\langle 1\rangle}, \mathcal{C}^{\otimes}_{\langle 1\rangle})$ detects K-indexed colimits.

Proof. Sifted simplicial sets are weakly contractible by [Lur09a, Proposition 5.5.8.7] so (i)–(iii) follow from Theorem 2.1.13.1 (which is implicit in the proof of [Lur11, Proposition 3.2.3.1]). Then (iv) and (v) follow as in the proof [Lur11, Proposition 3.2.3.1]. □

Lemma 3.2.7.3. Suppose $\mathcal{C}^{\otimes}$ is an $\mathcal{O}^{\otimes}$-monoidal ∞-category and $p\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ is a morphism of generalized symmetric ∞-operads. Then the forgetful functor

$$\tau^*_{\mathcal{M}}\colon \mathrm{Alg}^{\mathbb{F}}_{\mathcal{M}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathbb{F}}_{\mathcal{M}_{\mathrm{triv}}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}) \simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}_{\langle 1\rangle}}(\mathcal{M}_{\langle 1\rangle}, \mathcal{C})$$

is conservative.

Proof. The ∞-category $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{M}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ is a full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$. Therefore a map of algebras $f\colon A \to B$ is an equivalence in $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{M}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ if and only if it is an equivalence in $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$. Applying Proposition 3.2.7.2 to Δ^0-indexed colimits, we see that a morphism $f\colon A \to B$ in $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is an equivalence if and only if $f_X\colon A(X) \to B(X)$ is an equivalence in $\mathcal{C}^{\otimes}$ for all $X \in \mathcal{M}$. Thus equivalences are detected after restricting to $\mathcal{M}_{\mathrm{triv}}$. □

Corollary 3.2.7.4. Suppose $\mathcal{C}^{\otimes}$ is an $\mathcal{O}^{\otimes}$-monoidal ∞-category compatible with small colimits, and $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of symmetric ∞-operads. Then the adjunction

$$\tau_{\mathcal{P}^{\otimes},!}\colon \mathrm{Alg}^{\mathbb{F}}_{\mathcal{P}^{\otimes}_{\mathrm{triv}}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}) \rightleftarrows \mathrm{Alg}^{\mathbb{F}}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})\colon \tau^*_{\mathcal{P}^{\otimes}}$$

is monadic.

Proof. We showed that the functor $\tau^*_{\mathcal{P}^{\otimes}}$ is conservative in Lemma 3.2.7.3, and that it preserves sifted colimits in Proposition 3.2.7.2. The adjunction $\tau_{\mathcal{P}^{\otimes},!} \dashv \tau^*_{\mathcal{P}^{\otimes}}$ is therefore monadic by Theorem 2.1.9.5. □

Corollary 3.2.7.5. Suppose $\mathcal{C}^{\otimes}$ is an $\mathcal{O}^{\otimes}$-monoidal ∞-category compatible with small colimits and $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of symmetric ∞-operads. Then $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ has all small colimits. Moreover, if $\mathcal{C}$ is presentable, so is $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$.

Proof. Apply Lemma 2.1.9.6 and Proposition 2.1.9.7 to the monadic adjunction $\tau_{\mathcal{P}^{\otimes},!} \dashv \tau^*_{\mathcal{P}^{\otimes}}$. □

Corollary 3.2.7.6. Let Φ be a perfect operator category, and suppose $\mathcal{C}^{\otimes}$ is a $\mathcal{U}^{\otimes}_{\Phi}$-monoidal ∞-category compatible with small colimits. If $\mathcal{M}$ is a generalized Φ-∞-operad, then the ∞-category $\mathrm{Alg}^{\Phi}_{\mathcal{M}}(u^{\Phi,*}\mathcal{C}^{\otimes})$ has small colimits. Moreover, if $\mathcal{C}$ is presentable, then so is $\mathrm{Alg}^{\Phi}_{\mathcal{M}}(u^{\Phi,*}\mathcal{C}^{\otimes})$.

Proof. Let $\overline{\mathcal{M}}$ denote the image of $\mathcal{M}$ under the left adjoint of the inclusion $\mathrm{Opd}^{\Phi}_{\infty} \hookrightarrow \mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty}$. Then the result follows from Corollary 3.2.7.5 since we have an equivalence

$$\mathrm{Alg}^{\Phi}_{\mathcal{M}}(u^{\Phi,*}\mathcal{C}^{\otimes}) \simeq \mathrm{Alg}^{\mathbb{F}}_{u^{\Phi}_{!}\overline{\mathcal{M}}/\mathcal{U}^{\otimes}_{\Phi}}(\mathcal{C}^{\otimes}),$$

where L denotes the localization functor $\mathrm{Opd}^{\Phi,\mathrm{gen}}_{\infty} \to \mathrm{Opd}^{\Phi}_{\infty}$. □

Of course, since we in general have no way of accessing $u_!^{\Phi}\overline{\mathcal{M}}$, this does not give us any description of the colimits in $\mathrm{Alg}^{\Phi}_{\mathcal{M}}(u^{\Phi,*}\mathcal{C}^{\otimes})$. Although every Φ-monoidal ∞-category is of the form $u^{\Phi,*}\mathcal{C}$ for some $\mathcal{U}_{\Phi}^{\otimes}$-monoidal ∞-category $\mathcal{C}$ by Proposition 3.2.4.7, it is also unsatisfying that we need a hypothesis on $\mathcal{C}$ that we do not know how to express in terms of $u^{\Phi,*}\mathcal{C}$.

3.2.8 The Algebra Fibration

We now construct an ∞-category of algebras for all Φ-∞-operads in a given Φ-monoidal ∞-category, and then consider its properties as we vary this Φ-monoidal ∞-category. Since we do not have a theory of operadic colimits for Φ-∞-operads, for most of our results we are forced to only consider Φ-monoidal ∞-categories that are pulled back from $\mathcal{U}_{\Phi}^{\otimes}$-monoidal ∞-categories compatible with small colimits.

Definition 3.2.8.1. Let $\mathcal{O}^{\otimes}$ be a Φ-∞-operad. By Remark 2.1.12.10 $(\mathrm{Set}_{\Delta}^{+})_{\mathfrak{O}_{\Phi}}$ is a marked simplicial model category, so we have a functor

$$(\mathrm{Set}_{\Delta}^{+})^{\mathrm{op}}_{\mathfrak{O}_{\Phi}} \to \mathrm{Set}_{\Delta}^{+}$$

represented by $\mathcal{O}^{\otimes}$. This restricts to a functor between the fibrant objects in these marked simplicial model categories; forgetting from the marked simplicial enrichment down to enrichment in simplicial sets (by forgetting the unmarked 1-simplices) and taking nerves we get a functor

$$(\mathrm{Opd}_{\infty}^{\Phi})^{\mathrm{op}} \to \mathrm{Cat}_{\infty};$$

this sends a non-symmetric ∞-operad $\mathcal{P}^{\otimes}$ to $\mathrm{Alg}^{\Phi}_{\mathcal{P}^{\otimes}}(\mathcal{O}^{\otimes})$. We define

$$\mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes}) \to \mathrm{Opd}_{\infty}^{\Phi}$$

to be a Cartesian fibration corresponding to this functor. If $\mathcal{V}^{\otimes}$ is a $\mathcal{U}_{\Phi}^{\otimes}$-monoidal ∞-category we will abbreviate $\mathrm{Alg}^{\Phi}(u^{\Phi,*}\mathcal{V}^{\otimes})$ to $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes})$.

Remark 3.2.8.2. Similarly, if $\mathcal{O}^{\otimes}$ is a Φ-∞-operad and $\mathcal{P}^{\otimes}$ is a Φ-∞-operad over $\mathcal{O}^{\otimes}$, we can define a *relative algebra fibration* $\mathrm{Alg}^{\Phi}_{/\mathcal{O}^{\otimes}}(\mathcal{P}^{\otimes}) \to (\mathrm{Opd}_{\infty}^{\Phi})_{/\mathcal{O}^{\otimes}}$ whose fibre at $\mathcal{Q}^{\otimes} \to \mathcal{O}^{\otimes}$ is $\mathrm{Alg}^{\Phi}_{\mathcal{Q}^{\otimes}/\mathcal{O}^{\otimes}}(\mathcal{P}^{\otimes})$.

Moreover, if $p\colon \mathcal{M} \to \mathcal{L}^{\Phi}$ is a generalized Φ-∞-operad and $\mathcal{N}$ is a generalized Φ-∞-operad over $\mathcal{M}$ we can define $\mathrm{Alg}^{\Phi}_{/\mathcal{M}}(\mathcal{N}) \to (\mathrm{Opd}_{\infty}^{\Phi,\mathrm{gen}})_{/\mathcal{M}}$ with fibre $\mathrm{Alg}^{\Phi}_{\mathcal{Q}/\mathcal{M}}(\mathcal{N})$ over $\mathcal{Q} \to \mathcal{M}$. If $\mathcal{O}^{\otimes}$ is a Φ-∞-operad we abbreviate $\mathrm{Alg}^{\Phi}_{/\mathcal{M}}(p^{*}\mathcal{O}^{\otimes})$ to $\mathrm{Alg}^{\Phi}_{/\mathcal{M}}(\mathcal{O}^{\otimes})$.

Definition 3.2.8.3. For $\mathcal{O}^{\otimes}$ a Φ-∞-operad, let

$$\mathrm{Alg}^{\Phi}_{\mathrm{triv}}(\mathcal{O}^{\otimes}) \to \mathrm{Opd}_{\infty}^{\Phi}$$

be the pullback of $\mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes})$ along the functor $\gamma_!^{\Phi}\gamma^{\Phi,*}$ from $\mathrm{Opd}_{\infty}^{\Phi}$ to itself that sends $\mathcal{P}^{\otimes}$ to $\mathcal{P}^{\otimes}_{\mathrm{triv}}$. The natural maps $\tau_{\mathcal{P}^{\otimes}}\colon \mathcal{P}^{\otimes}_{\mathrm{triv}} \to \mathcal{P}^{\otimes}$ then induce a functor

$$\tau^{*}\colon \mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes}) \to \mathrm{Alg}^{\Phi}_{\mathrm{triv}}(\mathcal{O}^{\otimes}).$$

Remark 3.2.8.4. Similarly, if $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of Φ-∞-operads, we can define $\mathrm{Alg}^{\Phi}_{/\mathcal{O}^{\otimes},\mathrm{triv}}(\mathcal{O}^{\otimes})$ as the pullback of $\mathrm{Alg}^{\Phi}_{/\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}) \to (\mathrm{Opd}_{\infty}^{\Phi})_{/\mathcal{O}^{\otimes}}$ along the functor that sends $\mathcal{Q}^{\otimes} \to \mathcal{O}^{\otimes}$ to $\mathcal{Q}^{\otimes}_{\mathrm{triv}} \to \mathcal{O}^{\otimes}$.

Lemma 3.2.8.5. Suppose $\mathcal{V}^{\otimes}$ is a $\mathcal{U}_{\Phi}^{\otimes}$-monoidal ∞-category compatible with small colimits. Then the projection $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes}) \to \mathrm{Opd}_{\infty}^{\Phi}$ is both Cartesian and coCartesian.

Proof. By [Lur09a, Corollary 5.2.2.5] it suffices to prove that for each $f\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ in $\mathrm{Opd}_{\infty}^{\Phi}$ the map $f^{*}\colon \mathrm{Alg}^{\Phi}_{\mathcal{P}^{\otimes}}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}^{\Phi}_{\mathcal{O}^{\otimes}}(\mathcal{V}^{\otimes})$ has a left adjoint. This follows from [Lur11, Corollary 3.1.3.4] after passing to the equivalent ∞-categories of relative algebras for $\mathbb{F}$-∞-operads. □

Lemma 3.2.8.6. Suppose $\mathcal{V}^{\otimes}$ is a $\mathcal{U}_{\Phi}^{\otimes}$-monoidal ∞-category compatible with small colimits. Then the functor τ^{*} has a left adjoint

$$\tau_{!}\colon \mathrm{Alg}^{\Phi}_{\mathrm{triv}}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes})$$

relative to $\mathrm{Opd}_{\infty}^{\Phi}$.

Proof. By [Lur11, Proposition 8.3.2.6] it suffices to prove that τ^{*} admits fibrewise left adjoints, which follows from [Lur11, Corollary 3.1.3.4] after passing to the equivalent ∞-categories of relative algebras for $\mathbb{F}$-∞-operads, and that τ^{*} preserves Cartesian arrows, which is clear since it is the functor associated to a natural transformation between the corresponding functors to Cat_{∞}. □

Lemma 3.2.8.7. The functor $\mathrm{Alg}^{\Phi}_{(-)}(\mathcal{V}^{\otimes})\colon (\mathrm{Opd}_{\infty}^{\Phi})^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ takes colimits in $\mathrm{Opd}_{\infty}^{\Phi}$ to limits.

Proof. For any categorical pattern $\mathfrak{P}$, the product

$$\mathrm{Set}_{\Delta}^{+} \times (\mathrm{Set}_{\Delta}^{+})_{\mathfrak{P}} \to (\mathrm{Set}_{\Delta}^{+})_{\mathfrak{P}}$$

is a left Quillen bifunctor by [Lur11, Remark B.2.5]. Thus the induced functor of ∞-categories preserves colimits in each variable. In particular, the tensor functor

$$\mathrm{Cat}_{\infty} \times \mathrm{Opd}_{\infty}^{\Phi} \to \mathrm{Opd}_{\infty}^{\Phi}$$

preserves colimits in each variable. Now $\mathrm{Alg}^{\Phi}_{(-)}(-)$ is defined as a right adjoint to this, so for any ∞-category $\mathcal{C}$ we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, \mathrm{Alg}^{\Phi}_{\mathrm{colim}_{\alpha} \mathcal{O}_{\alpha}^{\otimes}}(\mathcal{P}^{\otimes})) &\simeq \mathrm{Map}_{\mathrm{Opd}_{\infty}^{\Phi}}(\mathcal{C} \times \operatorname*{colim}_{\alpha} \mathcal{O}_{\alpha}^{\otimes}, \mathcal{P}^{\otimes}) \\
&\simeq \mathrm{Map}_{\mathrm{Opd}_{\infty}^{\Phi}}(\operatorname*{colim}_{\alpha}(\mathcal{C} \times \mathcal{O}_{\alpha}^{\otimes}), \mathcal{P}^{\otimes}) \\
&\simeq \lim_{\alpha} \mathrm{Map}_{\mathrm{Opd}_{\infty}^{\Phi}}(\mathcal{C} \times \mathcal{O}_{\alpha}^{\otimes}, \mathcal{P}^{\otimes}) \\
&\simeq \lim_{\alpha} \mathrm{Map}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, \mathrm{Alg}^{\Phi}_{\mathcal{O}_{\alpha}^{\otimes}}(\mathcal{P}^{\otimes})) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\infty}}(\mathcal{C}, \lim_{\alpha} \mathrm{Alg}^{\Phi}_{\mathcal{O}_{\alpha}^{\otimes}}(\mathcal{P}^{\otimes})).
\end{aligned}$$

Thus $\mathrm{Alg}^{\Phi}_{\mathrm{colim}\, \mathcal{O}_{\alpha}^{\otimes}}(\mathcal{P}^{\otimes}) \simeq \lim_{\alpha} \mathrm{Alg}^{\Phi}_{\mathcal{O}_{\alpha}^{\otimes}}(\mathcal{P}^{\otimes})$. □

Proposition 3.2.8.8. Suppose $\mathcal{V}^{\otimes}$ is a $\mathcal{U}_{\Phi}^{\otimes}$-monoidal ∞-category compatible with small colimits. Then $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes})$ admits small colimits.

Proof. By Lemma 3.2.8.5, the fibration $\pi\colon \mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes}) \to \mathrm{Opd}_{\infty}^{\Phi}$ is coCartesian. Moreover, its fibres have all colimits and the functors $f_!$ induced by morphisms f in $\mathrm{Opd}_{\infty}^{\Phi}$ preserve colimits, being left adjoints. Thus π satisfies the conditions of Lemma 2.1.5.10, which implies that $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes})$ has small colimits. □

Proposition 3.2.8.9. Let $\mathcal{P}^{\otimes}$ be a $\mathbb{F}$-∞-operad and suppose $\mathcal{V}^{\otimes}$ is a $\mathcal{P}^{\otimes}$-monoidal ∞-category compatible with small colimits. Then the forgetful functor

$$\tau^*\colon \mathrm{Alg}^{\mathbb{F}}_{/\mathcal{P}^{\otimes}}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}^{\mathbb{F}}_{/\mathcal{P}^{\otimes},\mathrm{triv}}(\mathcal{V}^{\otimes})$$

preserves filtered colimits.

Proof. Suppose $\phi\colon \mathcal{I} \to \mathrm{Alg}^{\mathbb{F}}_{/\mathcal{P}^{\otimes}}(\mathcal{V}^{\otimes})$ is a filtered diagram, sending $\alpha \in \mathcal{I}$ to

$$(\mathcal{O}^{\otimes}_{\alpha}, A_{\alpha}\colon \mathcal{O}^{\otimes}_{\alpha} \to \mathcal{V}^{\otimes}).$$

Let $\mathcal{O}^{\otimes}$ be the colimit of the non-symmetric ∞-operads $\mathcal{O}^{\otimes}_{\alpha}$ and write $f_{\alpha}\colon \mathcal{O}^{\otimes}_{\alpha} \to \mathcal{O}^{\otimes}$ for the canonical maps. Then the colimit A of ϕ in $\mathrm{Alg}^{\mathbb{F}}_{/\mathcal{P}^{\otimes}}(\mathcal{V}^{\otimes})$ can be described as the colimit of $f_{\alpha,!}A_{\alpha}$ in $\mathrm{Alg}^{\mathbb{F}}_{\mathcal{O}^{\otimes}/\mathcal{P}^{\otimes}}(\mathcal{V}^{\otimes})$. Since $\tau^*_{\mathcal{O}^{\otimes}}$ preserves sifted colimits, we have

$$\tau^* A \simeq \operatorname*{colim}_{\mathcal{I}} \tau^*_{\mathcal{O}^{\otimes}}(f_{\alpha})_! A_{\alpha},$$

On the other hand, $\operatorname{colim} \tau^* A_{\alpha}$ can be described as

$$\operatorname*{colim}_{\mathcal{I}} f^{\mathrm{triv}}_{\alpha,!} \tau^*_{\mathcal{O}^{\otimes}_{\alpha}} A_{\alpha},$$

where $f^{\mathrm{triv}}_{\alpha}$ denotes the map $\mathcal{O}^{\otimes}_{\alpha,\mathrm{triv}} \to \mathcal{O}^{\otimes}_{\mathrm{triv}}$ induced by f_{α}.

To show that the natural map $\operatorname{colim} \tau^* A_{\alpha} \to \tau^* A$ is an equivalence, it suffices to show that for each $x \in \mathcal{O}$ the map

$$\operatorname*{colim}_{\mathcal{I}} f^{\mathrm{triv}}_{\alpha,!} A_{\alpha}(x) \to \operatorname*{colim}_{\mathcal{I}} f_{\alpha,!} A_{\alpha}(x)$$

is an equivalence, where the colimits are now occurring in $\mathcal{V}$. The functor $f^{\mathrm{triv}}_{\alpha,!}$ is just a left Kan extension, so the source of this map can be described as

$$\operatorname*{colim}_{\alpha\in\mathcal{I}} \operatorname*{colim}_{y\in(\mathcal{O}_{\alpha})_{/x}} A_{\alpha}(y)$$

and from [Lur11, Proposition 3.1.1.16] and the definition of free algebras in terms of operadic Kan extensions we know that the target can be described as

$$\operatorname*{colim}_{\alpha\in\mathcal{I}} \operatorname*{colim}_{Y\in(\mathcal{O}^{\otimes}_{\alpha})^{\mathrm{act}}_{/x}} A^{\otimes}_{\alpha}(Y)$$

where we write $A^{\otimes}_{\alpha}(Y)$ for the coCartesian pushforward of $A_{\alpha}(Y)$ in $\mathcal{V}^{\otimes}_{X}$ along the given active map in $\mathcal{P}^{\otimes}$.

We have functors $\mathcal{I} \to \mathrm{Cat}_{\infty}$ sending α to $(\mathcal{O}_{\alpha})_{/x}$ and $(\mathcal{O}^{\otimes}_{\alpha})^{\mathrm{act}}_{/x}$, with natural transformations to the constant functor at $\mathcal{V}$. Let $\mathcal{J}, \mathcal{K} \to \mathcal{I}$ denote coCartesian fibrations associated to these functors, then by Proposition 2.1.13.2 the map we are interested in is the map on colimits induced by the obvious functor $\mathcal{J} \to \mathcal{K}$. It therefore suffices to prove that this functor is cofinal.

By [Lur09a, Theorem 4.1.3.1] ("Quillen's Theorem A" for ∞-categories) it suffices to show that for each $Y \in \mathcal{K}$, the ∞-category $\mathcal{J}_{Y/}$ is weakly contractible. We will show that this ∞-category is in fact filtered. To see this we must prove that given a diagram $p\colon K \to \mathcal{J}_{Y/}$, where K is a finite simplicial set, there exists an extension $\bar{p}\colon K^{\triangleright} \to \mathcal{J}_{Y/}$ of p.

Since $\mathcal{I}$ is filtered, the composite $K \to \mathcal{I}_{\alpha/}$ extends to $K^{\triangleright} \to \mathcal{I}_{\alpha/}$; let β be the image of the cone point ∞. Choosing a coCartesian lift along the maps to β, we may therefore suppose that p factors through $p'\colon K \to ((\mathcal{O}^{\otimes}_{\beta})_{\mathrm{act}})_{g_!(Y)//x}$ where $g\colon \alpha \to \beta$. On the other hand, the composite $K \to \mathcal{J}_{Y/} \to ((\mathcal{O}^{\otimes})_{\mathrm{act}})_{f_\alpha Y//x}$ corresponds to a diagram $K^{\triangleleft\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$. Since filtered colimits of symmetric ∞-operads are computed in Cat_∞ by Lemma 3.2.5.2, this map factors through $\mathcal{O}^{\otimes}_{\gamma,\mathrm{act}}$ for some γ, giving a map $K \to ((\mathcal{O}^{\otimes}_{\gamma})_{\mathrm{act}})_{Y'//x'}$. Since this ∞-category has a final object, there is an obvious extension $\bar{q}\colon K^{\triangleright} \to (\mathcal{O}^{\otimes}_{\gamma,\mathrm{act}})_{Y'//x}$. Now observe that there must exist some δ with maps $\gamma \to \delta$, $\beta \to \delta$ such that the pushforwards of $\bar{q}|_K$ and p' agree. Then the pushforward of $\bar{q}$ induces the desired extension $K^{\triangleright} \to \mathcal{J}_{Y/}$. □

Corollary 3.2.8.10. Suppose $\mathcal{V}^{\otimes}$ is a $\mathcal{U}^{\otimes}_{\Phi}$-monoidal ∞-category compatible with small colimits. Then the forgetful functor $\tau^*\colon \mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}^{\Phi}_{\mathrm{triv}}(\mathcal{V}^{\otimes})$ preserves filtered colimits.

Proof. We can identify $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\otimes})$ with the pullback of $\mathrm{Alg}^{\mathbb{F}}_{/\mathcal{U}^{\otimes}_{\Phi}}(\mathcal{V}^{\otimes})$ along $u^{\Phi}_!\colon \mathrm{Opd}^{\Phi}_{\infty} \to (\mathrm{Opd}^{\mathbb{F}}_{\infty})_{/\mathcal{U}^{\otimes}_{\Phi}}$, and similarly $\mathrm{Alg}^{\Phi}_{\mathrm{triv}}(\mathcal{V}^{\otimes})$ is the pullback of $\mathrm{Alg}^{\mathbb{F}}_{/\mathcal{U}^{\otimes}_{\Phi},\mathrm{triv}}(\mathcal{V}^{\otimes})$. Since $u^{\Phi}_!$ is colimit-preserving, it is easy to see that this follows from Proposition 3.2.8.9. □

Next we observe that the ∞-category $\mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes})$ is functorial in $\mathcal{O}^{\otimes}$:

Definition 3.2.8.11. Since the model category $(\mathrm{Set}^{+}_{\Delta})_{\mathfrak{O}_{\Phi}}$ is enriched in marked simplicial sets, the enriched Yoneda functor

$$\mathfrak{H}\colon (\mathrm{Set}^{+}_{\Delta})^{\mathrm{op}}_{\mathfrak{O}_{\Phi}} \times (\mathrm{Set}^{+}_{\Delta})_{\mathfrak{O}_{\Phi}} \to \mathrm{Set}^{+}_{\Delta}$$

sending $(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ to $\mathrm{Alg}^{\Phi}_{\mathcal{O}^{\otimes}}(\mathcal{P}^{\otimes})$ induces a functor of ∞-categories $(\mathrm{Opd}^{\Phi}_{\infty})^{\mathrm{op}} \times \mathrm{Opd}^{\Phi}_{\infty} \to \mathrm{Cat}_\infty$. Let $\mathrm{Alg}^{\Phi}_{\mathrm{co}} \to \mathrm{Opd}^{\Phi}_{\infty} \times (\mathrm{Opd}^{\Phi}_{\infty})^{\mathrm{op}}$ be a Cartesian fibration corresponding to this functor.

The fibre of $\mathrm{Alg}^{\Phi}_{\mathrm{co}}$ at $\mathcal{O}^{\otimes}$ in the second component is $\mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes})$. Thus the composite $\mathrm{Alg}^{\Phi}_{\mathrm{co}} \to (\mathrm{Opd}^{\Phi}_{\infty})^{\mathrm{op}}$ with projection to the second factor is a Cartesian fibration corresponding to a functor $\mathrm{Opd}^{\Phi}_{\infty} \to \mathrm{Cat}_\infty$ that sends $\mathcal{O}^{\otimes}$ to $\mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes})$. Thus we see that $\mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes})$ is functorial in $\mathcal{O}^{\otimes}$.

Definition 3.2.8.12. Let $\mathrm{Alg}^{\Phi} \to \mathrm{Opd}^{\mathrm{O}}_{\infty}$ be a coCartesian fibration corresponding to the functor $\mathcal{O}^{\otimes} \mapsto \mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes})$.

Now we show that the algebra fibration is compatible with products of Φ-∞-operads:

Proposition 3.2.8.13. $\mathrm{Alg}^{\Phi}(-)$ is lax monoidal with respect to the Cartesian product of non-symmetric ∞-operads.

Proof. Observe that $\mathfrak{H}$ is lax monoidal with respect to the Cartesian product of marked simplicial sets over $\mathcal{L}^{\Phi}$. This induces an $((\mathrm{Opd}^{\Phi}_{\infty})^{\mathrm{op}} \times \mathrm{Opd}^{\Phi}_{\infty})^{\times}$-monoid in Cat_∞, and so a Cartesian fibration $(\mathrm{Alg}^{\Phi}_{\mathrm{co}})^{\times} \to (((\mathrm{Opd}^{\Phi}_{\infty})^{\mathrm{op}} \times \mathrm{Opd}^{\Phi}_{\infty})^{\times})^{\mathrm{op}}$. Projecting to the second factor gives a Cartesian fibration that corresponds to a monoid $(\mathrm{Opd}^{\Phi}_{\infty})^{\times} \to \mathrm{Cat}_\infty$, and so a lax monoidal functor $(\mathrm{Opd}^{\Phi}_{\infty})^{\times} \to \mathrm{Cat}^{\times}_{\infty}$. This shows that $\mathrm{Alg}^{\Phi}(-)$ is a lax monoidal functor. □

This construction gives an "external product"

$$\boxtimes \colon \mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes}) \times \mathrm{Alg}^{\Phi}(\mathcal{P}^{\otimes}) \to \mathrm{Alg}^{\Phi}(\mathcal{O}^{\otimes} \times_{\mathcal{L}^{\Phi}} \mathcal{P}^{\otimes}),$$

which sends an $\mathcal{A}^{\otimes}$-algebra A in $\mathcal{O}^{\otimes}$ and a $\mathcal{B}^{\otimes}$-algebra B in $\mathcal{P}^{\otimes}$ to the fibre product

$$\mathcal{A}^{\otimes} \times_{\mathcal{L}^{\Phi}} \mathcal{B}^{\otimes} \xrightarrow{A \times_{\mathcal{L}^{\Phi}} B} \mathcal{O}^{\otimes} \times_{\mathcal{L}^{\otimes}} \mathcal{P}^{\otimes}.$$

When $\mathcal{V}$ is Cartesian monoidal, we can equivalently work with the analogous *monoid fibration*:

Definition 3.2.8.14. Suppose $\mathcal{V}$ is an ∞-category with finite products. Let $\mathrm{Mon}^{\Phi}(\mathcal{V}) \to \mathrm{Opd}_{\infty}^{\Phi}$ be the Cartesian fibration with fibre $\mathrm{Mon}^{\Phi}_{\mathcal{O}^{\otimes}}(\mathcal{V})$ at $\mathcal{O}^{\otimes} \in \mathrm{Opd}_{\infty}^{\Phi}$. This is equivalent to $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\times})$ over $\mathrm{Opd}_{\infty}^{\Phi}$.

Proposition 3.2.8.15. Suppose $\mathcal{V}$ is an ∞-category with finite products. Then the natural symmetric monoidal structure on $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\times})$ is Cartesian.

Proof. By [Lur11, Corollary 2.4.1.8] it suffices to prove that the unit for this monoidal structure is the final object, and for each pair of objects A, B the canonical maps

$$A \simeq A \otimes * \leftarrow A \otimes B \to * \otimes B \simeq B$$

exhibit $A \otimes B$ as a product of A and B. The equivalence $\mathrm{Alg}^{\Phi}(\mathcal{V}^{\times}) \xrightarrow{\sim} \mathrm{Mon}^{\Phi}(\mathcal{V})$ takes the lax monoidal structure on $\mathrm{Alg}^{\Phi}(-)$ to the natural lax monoidal structure on $\mathrm{Mon}^{\Phi}(-)$. Suppose given Φ-∞-operads $\mathcal{O}^{\otimes}$, $\mathcal{P}^{\otimes}$, and $\mathcal{Q}^{\otimes}$ and monoids $A \in \mathrm{Mon}^{\Phi}_{\mathcal{O}^{\otimes}}(\mathcal{V})$, $B \in \mathrm{Mon}^{\Phi}_{\mathcal{P}^{\otimes}}(\mathcal{V})$, and $C \in \mathrm{Mon}^{\Phi}_{\mathcal{Q}^{\otimes}}(\mathcal{V})$. If $\mu \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is the Cartesian product functor, the natural transformation from μ to the projections on the two factors of $\mathcal{V} \times \mathcal{V}$ induce morphisms $B \boxtimes C \to B, C$. We must prove that the induced map

$$\mathrm{Map}(A, B \boxtimes C) \to \mathrm{Map}(A, B) \times \mathrm{Map}(A, C)$$

is an equivalence. It suffices to show that it induces an equivalence on fibres over each $(f, g) \in \mathrm{Map}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \times_{\mathcal{L}^{\Phi}} \mathcal{Q}^{\otimes}) \simeq \mathrm{Map}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) \times \mathrm{Map}(\mathcal{O}^{\otimes}, \mathcal{Q}^{\otimes})$, i.e. we must show

$$\mathrm{Map}(A, (f, g)^*(B \boxtimes C) \to \mathrm{Map}(A, f^*B) \times \mathrm{Map}(A, f^*C)$$

is an equivalence. It is clear that $(f, g)^*(B \boxtimes C) \simeq \Delta^*(f^*B \boxtimes g^*C)$, where $\Delta \colon \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{L}^{\Phi}} \mathcal{O}^{\otimes}$ is the diagonal map. It follows that the map in question is an equivalence, since maps of $\mathcal{O}^{\otimes}$-monoids are just natural transformations, and $\Delta^*(f^*B \boxtimes g^*C)$ is the product of the functors f^*B and $g^*C \colon \mathcal{O}^{\otimes} \to \mathcal{V}$. □

3.3 Non-Symmetric ∞-Operads

In this section we discuss versions of some results from [Lur11] for non-symmetric ∞-operads that we do not know how to extend to more general ∞-operads. We then use these to say a bit more about algebra fibrations in this context.

3.3.1 Monoidal Envelopes

Definition 3.3.1.1. Let $\mathrm{Act}(\mathbb{\Delta}^{\mathrm{op}})$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathbb{\Delta}^{\mathrm{op}})$ spanned by the active morphisms. If $\mathcal{M}$ is a generalized non-symmetric ∞-operad, we define $\mathrm{Env}(\mathcal{M})$ to be the fibre product

$$\mathcal{M} \times_{\mathrm{Fun}(\{0\}, \mathbb{\Delta}^{\mathrm{op}})} \mathrm{Act}(\mathbb{\Delta}^{\mathrm{op}}).$$

Proposition 3.3.1.2. The map $\mathrm{Env}(\mathcal{M}) \to \mathbb{\Delta}^{\mathrm{op}}$ induced by evaluation at 1 in Δ^1 is a double ∞-category.

Proof. As [Lur11, Proposition 2.2.4.4]. □

Proposition 3.3.1.3. Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad and $\mathcal{N}$ is a double ∞-category. The inclusion $\mathcal{M} \to \mathrm{Env}(\mathcal{M})$ induces an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{M}), \mathcal{N}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{N}).$$

Proof. As [Lur11, Proposition 2.2.4.9]. □

Corollary 3.3.1.4. Suppose $\mathcal{O}^{\otimes}$ is a non-symmetric ∞-operad. Then $\mathrm{Env}(\mathcal{O}^{\otimes})$ is a monoidal ∞-category, and if $\mathcal{C}^{\otimes}$ is a monoidal ∞-category then

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}^{\otimes}), \mathcal{C}^{\otimes}) \simeq \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}).$$

Proof. The only object of $\mathbb{\Delta}$ that admits an active map from $[0]$ is $[0]$, hence for any generalized non-symmetric ∞-operad $\mathcal{M}$ we have $\mathrm{Env}(\mathcal{M})_{[0]} \simeq \mathcal{M}_{[0]}$. In particular $\mathrm{Env}(\mathcal{O}^{\otimes})_{[0]} \simeq *$, so the result follows from Proposition 3.3.1.2 and Proposition 3.3.1.3. □

Definition 3.3.1.5. If $\mathcal{O}^{\otimes}$ is a non-symmetric ∞-operad, the monoidal ∞-category $\mathrm{Env}(\mathcal{O}^{\otimes})$ is the *monoidal envelope* of $\mathcal{O}^{\otimes}$. This gives a monoidal structure on the subcategory $\mathcal{O}^{\otimes}_{\mathrm{act}}$ of $\mathcal{O}^{\otimes}$ determined by the active morphisms. We denote this tensor product on $\mathcal{O}^{\otimes}_{\mathrm{act}}$ by $\oplus$.

3.3.2 Operadic Colimits

Definition 3.3.2.1. Suppose $q\colon \mathcal{O}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ is a non-symmetric ∞-operad. Given a diagram $p\colon K \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ we write $\mathcal{O}^{\mathrm{act}}_{p/} := \mathcal{O} \times_{\mathcal{O}^{\otimes}} (\mathcal{O}^{\otimes}_{\mathrm{act}})_{p/}$. A diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ is a *weak operadic colimit diagram* if the induced map $\mathcal{O}^{\mathrm{act}}_{\bar{p}/} \to \mathcal{O}^{\mathrm{act}}_{p/}$ is a categorical equivalence

A diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ is an *operadic colimit diagram* if the composite functors

$$K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}} \xrightarrow{-\oplus X} \mathcal{O}^{\otimes}_{\mathrm{act}}$$

$$K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}} \xrightarrow{X\oplus -} \mathcal{O}^{\otimes}_{\mathrm{act}}$$

are weak operadic colimit diagrams for all $X \in \mathcal{O}^{\otimes}$.

Remark 3.3.2.2. By [Lur09a, Proposition 2.1.2.1], the map $\mathcal{O}^{\mathrm{act}}_{\bar{p}/} \to \mathcal{O}^{\mathrm{act}}_{p/}$ in the definition of weak operadic colimits is always a left fibration, hence it is a categorical equivalence if and only if it is a trivial Kan fibration.

Proposition 3.3.2.3. Let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad, and suppose given finitely many operadic colimit diagrams $\bar{p}_i\colon K_i^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$, $i = 0, \ldots, n$. Let $K = \prod_i K_i$, and let $\bar{p}$ be the composite

$$K^{\triangleright} \to \prod_i K_i^{\triangleright} \to \prod_i \mathcal{O}^{\otimes}_{\mathrm{act}} \simeq \mathrm{Env}(\mathcal{O}^{\otimes})_{[n]} \xrightarrow{\oplus} \mathcal{O}^{\otimes}_{\mathrm{act}}.$$

Then $\bar{p}$ is an operadic colimit diagram.

Proof. As [Lur11, Proposition 3.1.1.8]. □

Definition 3.3.2.4. Suppose $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ is a monoidal ∞-category. If K is a simplicial set, we say that $\mathcal{V}^{\otimes}$ is *compatible with K-indexed colimits* if

(1) the ∞-category $\mathcal{V}^{\otimes}_{[1]}$ has K-indexed colimits (hence so does $\mathcal{V}^{\otimes}_{[n]} \simeq \prod \mathcal{V}^{\otimes}_{[1]}$ and $\phi_!$ preserves them for any inert map ϕ)

(2) for all (active) maps $\phi\colon [n] \to [m]$ in Δ^{op}, the map

$$\phi_!\colon \prod \mathcal{V}^{\otimes}_{[1]} \simeq \mathcal{V}^{\otimes}_{[n]} \to \mathcal{V}^{\otimes}_{[m]}$$

preserves K-indexed colimits separately in each variable

Lemma 3.3.2.5. Suppose K is a sifted simplicial set, and $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ is a monoidal ∞-category that is compatible with K-indexed colimits. Then $\phi_!\colon \mathcal{V}^{\otimes}_{[n]} \to \mathcal{V}^{\otimes}_{[m]}$ preserves K-indexed colimits for all ϕ in Δ^{op}.

Proof. As [Lur11, Lemma 3.2.3.7]. □

Proposition 3.3.2.6. Let $\mathcal{V}^{\otimes}$ be a monoidal ∞-category, and let $\bar{p}\colon K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]}$ be a diagram. Then $\bar{p}$ is a weak operadic colimit diagram if and only if the composite

$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \xrightarrow{r_!} \mathcal{V}$$

is a colimit diagram, where r is the unique active map $[m] \to [1]$.

Proof. This follows as in the proof of [Lur11, Proposition 3.1.1.6]. □

Corollary 3.3.2.7. Let $\mathcal{V}^{\otimes}$ be a monoidal ∞-category, and let $\bar{p}\colon K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]}$ be a diagram. Then $\bar{p}$ is an operadic colimit diagram if and only if for every object $Y \in \mathcal{V}^{\otimes}$ with image $[n]$ in Δ^{op} the composites

$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \xrightarrow{-\oplus Y} \mathcal{V}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{C}$$

$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \xrightarrow{Y\oplus -} \mathcal{V}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{V}$$

are colimit diagrams in $\mathcal{V}$, where r is the unique active map $[n+m] \to [1]$.

Proposition 3.3.2.8. Let $q\colon \mathcal{O}^{\otimes} \to \Delta^{\mathrm{op}}$ be a non-symmetric ∞-operad, and suppose given a map $\bar{h}\colon \Delta^1 \times K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$; write $\bar{h}_i := \bar{h}|_{\{i\}\times K^{\triangleright}}$, $i = 0, 1$. Suppose that

(a) For every vertex x of $K^{\triangleright}$, the restriction $\bar{h}|_{\Delta^1\times\{x\}}$ is a q-coCartesian edge of $\mathcal{O}^{\otimes}$.

(b) The composite map
$$\Delta^1 \times \{\infty\} \hookrightarrow \Delta^1 \times K^{\triangleright} \xrightarrow{\bar{h}} \mathcal{O}^{\otimes} \xrightarrow{q} \mathbb{\Delta}^{\mathrm{op}}$$
is an equivalence in $\mathbb{\Delta}^{\mathrm{op}}$.

Then $\bar{h}_0$ is a weak operadic colimit diagram if and only if $\bar{h}_1$ is a weak operadic colimit diagram. Moreover, if $\mathcal{O}^{\otimes}$ is a monoidal ∞-category, then $\bar{h}_0$ is an operadic colimit diagram if and only if $\bar{h}_1$ is an operadic colimit diagram.

Proof. As [Lur11, Proposition 3.1.1.15] □

Corollary 3.3.2.9. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be monoidal ∞-categories compatible with small colimits, and suppose $F^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a strong monoidal functor such that $F \colon \mathcal{C} \to \mathcal{D}$ preserves colimits. Then composition with F preserves operadic colimit diagrams.

Proof. Suppose $\bar{p} \colon K^{\triangleright} \to \mathcal{C}^{\otimes}$ is an operadic colimit diagram. We wish to show that the composite map $K^{\triangleright} \to \mathcal{D}^{\otimes}$ is also an operadic colimit diagram. By Proposition 3.3.2.8 we may assume that $\bar{p}$ lands in a fibre $\mathcal{C}^{\otimes}_{[m]}$. We now apply Corollary 3.3.2.7 to conclude that it suffices to show that the composites
$$K^{\triangleright} \to \mathcal{C}^{\otimes}_{[m]} \to \mathcal{D}^{\otimes}_{[m]} \xrightarrow{-\oplus Y} \mathcal{D}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{D}$$
$$K^{\triangleright} \to \mathcal{C}^{\otimes}_{[m]} \to \mathcal{D}^{\otimes}_{[m]} \xrightarrow{Y\oplus -} \mathcal{D}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{D}$$
are colimit diagrams. Observe that the functors $r_!(-\oplus Y)$ and $r_!(Y \oplus -)$ are equivalently given by $m_!(r_!(-) \oplus Y)$ and $m_!(Y \oplus r_!(-))$, where $m \colon [2] \to [1]$ is the unique active map. Since $m_!$ preserves colimits in each variable in both $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$, it suffices to show that
$$K^{\triangleright} \to \mathcal{D}^{\otimes}_{[m]} \xrightarrow{r_!} \mathcal{D}$$
is a colimit diagram. But we have a commutative diagram
$$\begin{tikzcd}
\mathcal{C}^{\otimes}_{[m]} \arrow[r, "F^{\otimes}_{[m]}"] \arrow[d, "r_!"'] & \mathcal{D}^{\otimes}_{[m]} \arrow[d, "r_!"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$
so this is true since $K^{\triangleright} \to \mathcal{C}^{\otimes}_{[m]} \to \mathcal{C}$ is a colimit diagram and F preserves colimits. □

Proposition 3.3.2.10. Let $q \colon \mathcal{C}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ be a monoidal ∞-category compatible with K-indexed colimits for some simplicial set K. Suppose given a diagram $\bar{p} \colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ that sends the cone point ∞ to an object in $\mathcal{C}^{\otimes}_{[1]}$. Let $\bar{q} \colon K^{\triangleright} \to \mathcal{C}^{\otimes}$ be a coCartesian lift of $\bar{p}$ along the active maps to $[1]$. Then $\bar{p}$ is an operadic colimit diagram if and only if $\bar{q}$ is a colimit diagram. In particular, given a diagram $p \colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ there exists an operadic colimit diagram $\bar{p} \colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ extending p that sends ∞ to an object of $\mathcal{C}^{\otimes}_{[1]}$.

Proof. As [Lur11, Proposition 3.1.1.20]. □

3.3.3 Operadic Kan Extensions

In this section we work in slightly more generality than for the corresponding results in [Lur11] — the proof of Lurie's existence result for operadic Kan extensions can also be used to construct operadic Kan extensions along a restricted class of morphisms of generalized non-symmetric ∞-operads that we will now define:

Definition 3.3.3.1. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-family of (generalized) non-symmetric ∞-operads* is a categorical fibration $\pi\colon \mathcal{O}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}} \times \mathcal{C}$ such that:

(i) For $c \in \mathcal{C}$, $x \in \mathcal{O}^{\otimes}_c$, and α an inert morphism in $\mathbb{\Delta}^{\mathrm{op}}$ from the image of x, there exists a coCartesian morphism $x \to y$ over α in $\mathcal{O}^{\otimes}_c$.

(ii) For $x \in \mathcal{O}^{\otimes}_c$ with image $[n] \in \mathbb{\Delta}^{\mathrm{op}}$ let $p_x\colon K^{\triangleleft}_{[n]} \to \mathcal{O}^{\otimes}$ be a coCartesian lift of $K^{\triangleleft}_{[n]} \to \mathbb{\Delta}^{\mathrm{op}}$ (or consider a lift of $\mathcal{G}^{\mathcal{O}}_{[n]/} \to \mathbb{\Delta}^{\mathrm{op}}$ for a generalized non-symmetric ∞-operad). Then p_x is a π-limit diagram.

(iii) For each $c \in \mathcal{C}$, the induced map $\mathcal{O}^{\otimes}_c \to \mathbb{\Delta}^{\mathrm{op}}$ is a (generalized) non-symmetric ∞-operad.

A Δ^1-family will also be referred to as a *correspondence of (generalized) non-symmetric ∞-operads*.

Definition 3.3.3.2. A Δ^1-family of generalized non-symmetric ∞-operads $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}} \times \Delta^1$ has the *Kan extension property* if given $B \in \mathcal{M} \times_{\Delta^1} \{1\}$ and coCartesian morphisms $B \to B_i$ over the inert maps $[n] \to [1]$, the induced map $(\mathcal{M}_{\mathrm{act}} \times_{\Delta^1} \{0\})_{/B} \to \prod_i (\mathcal{M}_{\mathrm{act}} \times_{\Delta^1} \{0\})_{/B_i}$ is a categorical equivalence.

Lemma 3.3.3.3.

(i) Every Δ^1-family of non-symmetric ∞-operads has the Kan extension property.

(ii) Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is a morphism of generalized non-symmetric ∞-operads such that $\mathcal{A}_{[0]}$ is a Kan complex and $\pi_0\mathcal{A}_{[0]} \to \pi_0\mathcal{B}_{[0]}$ is an injection. Then the associated correspondence $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}} \times \Delta^1$ has the Kan extension property.

Proof. (i) is clear, so we suppose the hypothesis of (ii) holds. Given $B \in \mathcal{B}_{[n]}$ choose coCartesian maps $B \to B_i$ and $B \to B_{i(i+1)}$ along the inert maps $[n] \to [1]$ and $[n] \to [0]$. These induce an equivalence

$$(\mathcal{A}_{\mathrm{act}})_{/B} \simeq (\mathcal{A}_{\mathrm{act}})_{/B_{01}} \times_{(\mathcal{A}_{\mathrm{act}})_{/B_1}} \cdots \times_{(\mathcal{A}_{\mathrm{act}})_{/B_{n-1}}} (\mathcal{A}_{\mathrm{act}})_{/B_{(n-1)n}}.$$

But the only active map to $[0]$ is the identity, so $(\mathcal{A}_{\mathrm{act}})_{/X}$ is $(\mathcal{A}_{[0]})_{/X}$ for $X \in \mathcal{B}_{[0]}$. This is contractible if $\mathcal{A}_{[0]}$ is a Kan complex and there's only one component that hits X. □

Definition 3.3.3.4. Let $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}} \times \Delta^1$ be a correspondence of generalized non-symmetric ∞-operads from $\mathcal{A}$ to $\mathcal{B}$ satisfying the Kan extension property, let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad, and let $\bar{F}\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ be a map of generalized non-symmetric ∞-operads. The map $\bar{F}$ is an *operadic left Kan extension* of $F = \bar{F}|_{\mathcal{A}}$ if for every $B \in \mathcal{B}_{[1]}$ the composite map

$$((\mathcal{M}_{\mathrm{act}})_{/B} \times_{\mathcal{M}} \mathcal{A})^{\triangleright} \to (\mathcal{M}_{/B})^{\triangleright} \to \mathcal{M} \xrightarrow{\bar{F}} \mathcal{O}^{\otimes}$$

is an operadic colimit diagram.

Theorem 3.3.3.5.

(i) Suppose given a Δ^1-family of generalized non-symmetric ∞-operads $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}} \times \Delta^1$ satisfying the Kan extension property, a non-symmetric ∞-operad $\mathcal{O}^{\otimes}$ and a commutative diagram of generalized non-symmetric ∞-operad family maps

$$\begin{tikzcd} \mathcal{M} \times_{\Delta^1} \{0\} \arrow[r, "f"] \arrow[d] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathcal{M} \arrow[r] & \mathbb{\Delta}^{\mathrm{op}}. \end{tikzcd}$$

Then there exists an operadic left Kan extension $\bar{f}$ of f if and only if for every B in $\mathcal{M} \times_{\Delta^1} \{1\}$, the diagram

$$(\mathcal{M}_{\mathrm{act}})_{/B} \times_{\Delta^1} \{0\} \to \mathcal{M} \times_{\Delta^1} \{0\} \xrightarrow{f} \mathcal{O}^{\otimes}$$

can be extended to an operadic colimit diagram lifting

$$((\mathcal{M}_{\mathrm{act}})_{/B} \times_{\Delta^1} \{0\})^{\triangleright} \to \mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}}.$$

(ii) Suppose given a Δ^n-family of generalized non-symmetric ∞-operads $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}} \times \Delta^n$ with $n \geq 1$ such that all sub-Δ^1-families have the Kan extension property, a non-symmetric ∞-operad $\mathcal{O}^{\otimes}$ and a commutative diagram of generalized non-symmetric ∞-operad family maps

$$\begin{tikzcd} \mathcal{M} \times_{\Delta^n} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathcal{M} \arrow[r] & \mathbb{\Delta}^{\mathrm{op}} \end{tikzcd}$$

such that the restriction of f to $\mathcal{M} \times_{\Delta^n} \Delta^{\{0,1\}}$ is an operadic left Kan extension of $f|_{\mathcal{M} \times_{\Delta^n} \{0\}}$. Then there exists a morphism $\bar{f} \colon \mathcal{M} \to \mathcal{O}^{\otimes}$ extending f.

Proof. As [Lur11, Theorem 3.1.2.3]. □

3.3.4 Free Algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be generalized non-symmetric ∞-operads, let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad, and let $i \colon \mathcal{A} \to \mathcal{B}$ be a map of generalized non-symmetric ∞-operads. Then i induces by composition a functor $i^* \colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{B}}(\mathcal{O}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{A}}(\mathcal{O}^{\otimes})$. In this section we will prove that when $\mathcal{O}^{\otimes}$ is a monoidal ∞-category compatible with small colimits and i has the Kan extension property, then this has a left adjoint $i_!$ given by forming *free* algebras:

Definition 3.3.4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be generalized non-symmetric ∞-operads, let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad, and let $i \colon \mathcal{A} \to \mathcal{B}$ be a map of generalized non-symmetric ∞-operads with the Kan extension property. Suppose $A \in \mathrm{Alg}^{\mathbf{O}}_{\mathcal{A}}(\mathcal{O}^{\otimes})$, $B \in \mathrm{Alg}^{\mathbf{O}}_{\mathcal{B}}(\mathcal{O}^{\otimes})$, and $\phi \colon A \to i^*B$ is a map of $\mathcal{A}$-algebras in $\mathcal{O}^{\otimes}$. For $b \in \mathcal{B}_{[1]}$, let $(\mathcal{A}_{\mathrm{act}})_{/b} := \mathcal{A} \times_{\mathcal{B}} (\mathcal{B}_{\mathrm{act}})_{/b}$. Then

A and B induce maps $\alpha, \beta\colon (\mathcal{A}_{\mathrm{act}})_{/b} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ and ϕ determines a natural transformation $\eta\colon \alpha \to \beta$. The map β clearly extends to $\bar{\beta}\colon (\mathcal{A}_{\mathrm{act}})_{/b} \to (\mathcal{O}^{\otimes}_{\mathrm{act}})_{/B(b)}$. Since the projection $(\mathcal{O}^{\otimes}_{\mathrm{act}})_{/B(b)} \to \mathcal{O}^{\otimes}_{\mathrm{act}} \times_{\Delta^{\mathrm{op}}_{\mathrm{act}}} (\Delta^{\mathrm{op}}_{\mathrm{act}})_{/[n]}$ (where b lies over $[n] \in \Delta^{\mathrm{op}}$) is a right fibration, we can lift η to an essentially unique $\bar{\eta}\colon \bar{\alpha} \to \bar{\beta}$ (over Δ^{op}). We say that ϕ *exhibits B as a free $\mathcal{B}$-algebra generated by A* if for every $b \in \mathcal{B}_{[1]}$ the map $\bar{\alpha}$ determines an operadic q-colimit diagram $(\mathcal{A}_{\mathrm{act}})^{\triangleright}_{/b} \to \mathcal{O}^{\otimes}$.

Remark 3.3.4.2. The map $\phi\colon A \to i^*B$ above determines a map

$$H\colon (\mathcal{A} \times \Delta^1) \amalg_{\mathcal{A} \times \{1\}} \mathcal{B} \to \mathcal{O}^{\otimes} \times \Delta^1.$$

Choose a factorization of H as

$$H\colon (\mathcal{A} \times \Delta^1) \amalg_{\mathcal{A} \times \{1\}} \mathcal{B} \xrightarrow{H'} \mathcal{M} \xrightarrow{H''} \mathcal{O}^{\otimes} \times \Delta^1,$$

where H' is a categorical equivalence and $\mathcal{M}$ is an ∞-category. The composite map $\mathcal{M} \to \Delta^{\mathrm{op}} \times \Delta^1$ exhibits $\mathcal{M}$ as a correspondence of non-symmetric ∞-operads. Then the map ϕ exhibits B as a free $\mathcal{B}$-algebra generated by A if and only if the composite $\mathcal{M} \to \mathcal{O}^{\otimes}$ is an operadic left Kan extension.

Proposition 3.3.4.3. Suppose $\phi\colon A \to i^*B$ exhibits B as a free $\mathcal{B}$-algebra in $\mathcal{O}^{\otimes}$ generated by A. Then for every $B' \in \mathrm{Alg}^{\mathcal{O}}_{\mathcal{B}}(\mathcal{O}^{\otimes})$ composition with ϕ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathcal{B}}(\mathcal{O}^{\otimes})}(B, B') \to \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathcal{A}}(\mathcal{O}^{\otimes})}(A, i^*B').$$

Proof. As [Lur11, Proposition 3.1.3.2]. □

Proposition 3.3.4.4. Suppose $A \in \mathrm{Alg}^{\mathcal{O}}_{\mathcal{A}}(\mathcal{O}^{\otimes})$. Then there exists a free $\mathcal{B}$-algebra B generated by A if and only if for every $b \in \mathcal{B}_*$ the induced map

$$(\mathcal{A}_{\mathrm{act}})_{/b} \to \mathcal{A}_{\mathrm{act}} \xrightarrow{A} \mathcal{O}^{\otimes}$$

can be extended to an operadic colimit diagram lying over

$$(\mathcal{A}_{\mathrm{act}})^{\triangleright}_{/b} \to \mathcal{B}_{\mathrm{act}} \to \Delta^{\mathrm{op}}_{\mathrm{act}}.$$

Proof. As [Lur11, Proposition 3.1.3.3]. □

Corollary 3.3.4.5. Let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad, and suppose $i\colon \mathcal{A} \to \mathcal{B}$ is a map of generalized non-symmetric ∞-operads with the Kan extension property. The functor $i^*\colon \mathrm{Alg}^{\mathcal{O}}_{\mathcal{B}}(\mathcal{O}^{\otimes}) \to \mathrm{Alg}^{\mathcal{O}}_{\mathcal{A}}(\mathcal{O}^{\otimes})$ admits a left adjoint $i_!$, provided that for every $\mathcal{A}$-algebra A in $\mathcal{O}^{\otimes}$ and every $b \in \mathcal{B}_*$, the diagram

$$(\mathcal{A}_{\mathrm{act}})_{/b} \to \mathcal{A}_{\mathrm{act}} \xrightarrow{A} \mathcal{O}^{\otimes}$$

can be extended to an operadic colimit diagram lying over

$$(\mathcal{A}_{\mathrm{act}})^{\triangleright}_{/b} \to \mathcal{B}_{\mathrm{act}} \to \Delta^{\mathrm{op}}_{\mathrm{act}}.$$

Proof. As [Lur11, Corollary 3.1.3.4]. □

Combining this with Proposition 3.3.2.10 gives the following:

Theorem 3.3.4.6. Suppose $\mathcal{C}^{\otimes}$ is a monoidal ∞-category compatible with κ-small colimits for some uncountable regular cardinal κ, and $i\colon \mathcal{A} \to \mathcal{B}$ is a map of generalized non-symmetric ∞-operads satisfying the Kan extension property, with $\mathcal{A}$ and $\mathcal{B}$ essentially κ-small. Then the functor $i^*\colon \mathrm{Alg}^{\mathcal{O}}_{\mathcal{B}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathcal{O}}_{\mathcal{A}}(\mathcal{C}^{\otimes})$ admits a left adjoint $i_!$.

Lemma 3.3.4.7. Suppose $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are monoidal ∞-categories compatible with small colimits, and let $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a strong monoidal functor such that $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits. Then the induced functor

$$F^{\otimes}_{*}\colon \mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{D}^{\otimes})$$

preserves free algebras, i.e. for all maps of generalized non-symmetric ∞-operads $f\colon \mathcal{N} \to \mathcal{M}$ with the Kan extension property the natural map $f_! F^{\otimes}_{*} \to F^{\otimes}_{*} f_!$ (adjoint to $F^{\otimes}_{*} \to F^{\otimes}_{*} f^* f_! \simeq f^* F^{\otimes}_{*} f_!$) is an equivalence.

Proof. This follows immediately from Corollary 3.3.2.9. □

Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad such that the inclusion

$$\tau_{\mathcal{M}}\colon \mathcal{M}_{\mathrm{triv}} \hookrightarrow \mathcal{M}$$

has the Kan extension property; by Lemma 3.3.3.3 this is true if $\mathcal{M}_{[0]}$ is a Kan complex. Then we can give a more explicit description of the left adjoint $(\tau_{\mathcal{M}})_!$. Recall that by Proposition 3.2.3.5, if $\mathcal{O}^{\otimes}$ is a non-symmetric ∞-operad then we have $\mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{O}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{O})$. We can therefore regard $(\tau_{\mathcal{M}})_!$ as a functor

$$\mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{O}) \to \mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{O}^{\otimes}).$$

Definition 3.3.4.8. For $[n] \in \Delta^{\mathrm{op}}$ and $x \in \mathcal{M}_{[1]}$, let $\mathcal{P}^{\mathcal{M}}_{x,n}$ be the full subcategory of $\mathcal{M}_{\mathrm{triv}} \times_{\mathcal{M}} \mathcal{M}_{/y}$ of morphisms $y \to x$ over the active map $[n] \to [1]$.

Suppose $\mathcal{C}^{\otimes}$ is a monoidal ∞-category and $F\colon \mathcal{M}_{[1]} \to \mathcal{C}^{\otimes}_{[1]}$ is a functor. Let $\bar{F}$ be the associated $\mathcal{M}_{\mathrm{triv}}$-algebra in $\mathcal{C}^{\otimes}$. We have a canonical map $h\colon \mathcal{P}^{\mathcal{M}}_{x,n} \times \Delta^1 \to \mathcal{M}$, a natural transformation from $\mathcal{P}^{\mathcal{M}}_{x,n} \to \mathcal{M}_{\mathrm{triv}} \hookrightarrow \mathcal{M}$ to the constant functor at x. Since $\mathcal{C}^{\otimes} \to \Delta^{\mathrm{op}}$ is coCartesian, from $\bar{F} \circ h$ we get a coCartesian natural transformation $\bar{h}$ from a functor $g\colon \mathcal{P}^{\mathcal{M}}_{x,n} \to \mathcal{C}^{\otimes}_{[1]}$ to the constant functor at $F(x)$. We let $\mathrm{P}^{n}_{\mathcal{M},x}(F)$ denote a colimit of g, if it exists.

Proposition 3.3.4.9. Suppose $\mathcal{C}^{\otimes}$ is a monoidal ∞-category compatible with κ-small colimits, and $\mathcal{M}$ is a κ-small generalized non-symmetric ∞-operad such that $\tau_{\mathcal{M}}$ satisfies the Kan extension property. Suppose moreover that A is an $\mathcal{M}$-algebra in $\mathcal{C}^{\otimes}$ and $F\colon \mathcal{M}_{[1]} \to \mathcal{C}^{\otimes}_{[1]}$ is a functor. Then a map $F \to (\tau_{\mathcal{M}})^* A$ is adjoint to an equivalence $\tau_{\mathcal{M},!}F \xrightarrow{\sim} A$ if and only if for every $x \in \mathcal{M}_{[1]}$ the maps $\mathrm{P}^{n}_{\mathcal{M},x}(F) \to A(x)$ exhibit $A(x)$ as a coproduct

$$\coprod_{[n] \in \Delta^{\mathrm{op}}} \mathrm{P}^{n}_{\mathcal{M},x}(F) \to A(x)$$

Proof. As [Lur11, Proposition 3.1.3.11]. □

3.3.5 Colimits of Algebras in Monoidal ∞-Categories

In this subsection we show that colimits exist in the ∞-categories $\mathrm{Alg}^{\mathcal{O}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ for all small non-symmetric ∞-operads $\mathcal{O}^{\otimes}$ when $\mathcal{C}^{\otimes}$ is a monoidal ∞-category compatible with small colimits. We first consider the case of sifted colimits:

Lemma 3.3.5.1. Suppose K is a sifted simplicial set and $\mathcal{C}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ is a monoidal ∞-category that is compatible with K-indexed colimits. Then for every $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}^{\mathrm{op}}$ the associated functor $\phi_{!}\colon \mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{[m]}$ preserves K-indexed colimits.

Proof. As [Lur11, Lemma 3.2.3.7]. □

Proposition 3.3.5.2. Suppose K is a sifted simplicial set and $\mathcal{C}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ is a monoidal ∞-category that is compatible with K-indexed colimits. Then for any generalized non-symmetric ∞-operad $p\colon \mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}}$, we have:

(i) The ∞-category $\mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$ admits K-indexed colimits.

(ii) A map $K^{\triangleright} \to \mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is a colimit diagram if and only if for every $X \in \mathcal{M}$ the induced diagram $K^{\triangleright} \to \mathcal{C}^{\otimes}_{p(X)}$ is a colimit diagram.

(iii) The full subcategory $\mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})$ of $\mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is stable under K-indexed colimits.

(iv) A map $K^{\triangleright} \to \mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is a colimit diagram if and only if, for every $X \in \mathcal{M}_{[1]}$, the induced diagram $K^{\triangleright} \to \mathcal{C}^{\otimes}_{[1]}$ is a colimit diagram.

(v) The restriction functor $\mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes}) \to \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{C}^{\otimes}_{[1]})$ detects K-indexed colimits.

Proof. Sifted simplicial sets are weakly contractible by [Lur09a, Proposition 5.5.8.7] so (i)–(iii) follow from Theorem 2.1.13.1 (which is implicit in the proof of [Lur11, Proposition 3.2.3.1]). Then (iv) and (v) follow as in the proof [Lur11, Proposition 3.2.3.1]. □

We now use this to that show the adjunction $\tau_{\mathcal{M},!} \dashv \tau^{*}_{\mathcal{M}}$ is monadic; we first check $\tau^{*}_{\mathcal{M}}$ is conservative:

Lemma 3.3.5.3. Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad and $\mathcal{C}^{\otimes}$ is a monoidal ∞-category. Then the forgetful functor

$$\tau^{*}_{\mathcal{M}}\colon \mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{C}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{C})$$

is conservative.

Proof. The ∞-category $\mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})$ is a full subcategory of $\mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$. Therefore a map of algebras $f\colon A \to B$ is an equivalence in $\mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})$ if and only if it is an equivalence in $\mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$. Applying Proposition 3.3.5.2 to Δ^{0}-indexed colimits, we see that a morphism $f\colon A \to B$ in $\mathrm{Fun}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is an equivalence if and only if $f_X\colon A(X) \to B(X)$ is an equivalence in $\mathcal{C}^{\otimes}$ for all $X \in \mathcal{M}$. Thus equivalences are detected after restricting to $\mathcal{M}_{\mathrm{triv}}$. □

Corollary 3.3.5.4. Suppose $\mathcal{C}^{\otimes}$ is a monoidal ∞-category compatible with small colimits, and $\mathcal{M}$ is a small generalized non-symmetric ∞-operad such that $\tau_{\mathcal{M}}$ satisfies the Kan extension property. Then the adjunction

$$(\tau_{\mathcal{M}})_{!}\colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{C}^{\otimes}) \rightleftarrows \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})\colon (\tau_{\mathcal{M}})^{*}$$

is monadic.

Proof. We showed that the functor $\tau^{*}_{\mathcal{M}}$ is conservative in Lemma 3.3.5.3, and that it preserves sifted colimits in Proposition 3.3.5.2. The adjunction $(\tau_{\mathcal{M}})_{!} \dashv \tau^{*}_{\mathcal{M}}$ is therefore monadic by [Lur11, Theorem 6.2.2.5]. □

Corollary 3.3.5.5. Suppose $\mathcal{C}^{\otimes}$ is a monoidal ∞-category compatible with small colimits and $\mathcal{M}$ is a small generalized non-symmetric ∞-operad such that $\tau_{\mathcal{M}}$ satisfies the Kan extension property. Then $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})$ has all small colimits. Moreover, if $\mathcal{C}$ is presentable, so is $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})$.

Proof. Apply Lemma 2.1.9.6 and Proposition 2.1.9.7 to the monadic adjunction $\tau_{\mathcal{M},!} \dashv \tau^{*}_{\mathcal{M}}$. □

Proposition 3.3.5.6. Let $\mathcal{M}$ be a generalized non-symmetric ∞-operad such that $\tau_{\mathcal{M}}$ satisfies the Kan extension property, and let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be monoidal ∞-categories compatible with small colimits. Suppose $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a strong monoidal functor such that $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits. Then the induced functor

$$F^{\otimes}_{*}\colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{D}^{\otimes})$$

preserves colimits.

Proof. Write $F^{\otimes,\mathrm{triv}}_{*}$ for the induced functor $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{D}^{\otimes})$. Under the equivalences $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{C}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{C})$ and $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{D}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{D})$ this corresponds to composition with F, and so preserves colimits. Clearly $\tau^{*}_{\mathcal{M}}F^{\otimes}_{*} \simeq F^{\otimes,\mathrm{triv}}_{*}\tau^{*}_{\mathcal{M}}$. Since $\tau^{*}_{\mathcal{M}}$ detects sifted colimits, it follows that F_{*} preserves sifted colimits. To prove that it preserves all colimits, it remains to prove F_{*} also preserves finite coproducts.

Since $F^{\otimes}$ is strong monoidal, by Lemma 3.3.4.7 the functor $F^{\otimes}_{*}$ preserves free algebras, i.e. $F^{\otimes}_{*}\tau_{\mathcal{M},!} \simeq \tau_{\mathcal{M},!}F^{\otimes,\mathrm{triv}}_{*}$. Therefore F_{*} preseves colimits of free algebras. Let A and B be objects of $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{C}^{\otimes})$ and let $A_{\bullet}$ and $B_{\bullet}$ be free resolutions of A and B. Then we have natural equivalences

$$\begin{aligned} F^{\otimes}_{*}(A \amalg B) &\simeq F^{\otimes}_{*}(|A_{\bullet} \amalg B_{\bullet}|) \simeq |F^{\otimes}_{*}(A_{\bullet} \amalg B_{\bullet})| \simeq |F^{\otimes}_{*}(A_{\bullet}) \amalg F^{\otimes}_{*}(B_{\bullet})| \\ &\simeq |F^{\otimes}_{*}(A_{\bullet})| \amalg |F^{\otimes}_{*}(B_{\bullet})| \simeq F^{\otimes}_{*}(|A_{\bullet}|) \amalg F^{\otimes}_{*}(|B_{\bullet}|) \simeq F^{\otimes}_{*}(A) \amalg F^{\otimes}_{*}(B), \end{aligned}$$

so $F^{\otimes}_{*}$ does indeed preserve coproducts. □

Proposition 3.3.5.7. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be presentably monoidal ∞-categories and suppose $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a strong monoidal functor such that the underlying functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a right adjoint of F. Then there exists a lax monoidal functor $G^{\otimes}\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ extending G such that for any small non-symmetric ∞-operad $\mathcal{O}^{\otimes}$ we have an adjunction

$$F^{\otimes}_{*}\colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}) \rightleftarrows \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{D}^{\otimes}) : G^{\otimes}_{*}.$$

Proof. By Proposition 3.3.5.6 the functor $F_*^{\otimes}$ is colimit-preserving, and by Corollary 3.3.5.5 these ∞-categories of $\mathcal{O}^{\otimes}$-algebras are presentable. It follows by Theorem 2.1.7.10 that $F_*^{\otimes}$ has a right adjoint

$$R_{\mathcal{O}^{\otimes}} \colon \mathrm{Alg}^{\mathbb{O}}_{\mathcal{O}^{\otimes}}(\mathcal{D}^{\otimes}) \to \mathrm{Alg}^{\mathbb{O}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}).$$

Moreover, since $F_*^{\otimes}$ is natural in $\mathcal{O}^{\otimes}$ so is $R^{\mathcal{O}^{\otimes}}$, by [Lur09a, Corollary 5.2.2.5]. Taking the underlying spaces of the ∞-categories of algebras, we see that $R_{(-)}$ induces a natural transformation $\rho \colon \mathrm{Map}(-, \mathcal{D}^{\otimes}) \to \mathrm{Map}(-, \mathcal{C}^{\otimes})$ of functors $(\mathrm{Opd}^{\mathbb{O}}_{\infty})^{\mathrm{op}} \to \mathcal{S}$.

Since $\mathcal{D}$ is presentable, it is the union of its full subcategories $\mathcal{D}^{\kappa}$ of κ-compact objects, and the ∞-categories $\mathcal{D}^{\kappa}$ are all small — i.e. $\mathcal{D}$ is the colimit of a (large) diagram of small ∞-categories $\mathcal{D}^{\kappa}$ indexed by cardinals κ. Similarly, if we write $\mathcal{D}^{\otimes}_{\kappa}$ for the full subcategory of $\mathcal{D}^{\otimes}$ of objects $X \in \mathcal{D}^{\otimes}_{[n]}$ whose components $X_i \in \mathcal{D}$, $i = 1, \ldots, n$ lie in $\mathcal{D}^{\kappa}$, then $\mathcal{D}^{\otimes}$ is the union of the small ∞-categories $\mathcal{D}^{\otimes}_{\kappa}$. The ∞-categories $\mathcal{D}^{\otimes}_{\kappa}$ are non-symmetric ∞-operads, though not necessarily monoidal ∞-categories.

Applying $R_{\mathcal{D}^{\otimes}_{\kappa}}$ to the inclusion $\mathcal{D}^{\otimes}_{\kappa} \to \mathcal{D}^{\otimes}$ gives compatible maps $G^{\otimes}_{\kappa} \colon \mathcal{D}^{\otimes}_{\kappa} \to \mathcal{C}^{\otimes}$. Combining these we get a map $G^{\otimes} \colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ from the colimit $\mathcal{D}^{\otimes}$, which is clearly a lax monoidal functor (since each inert map in $\mathcal{D}^{\otimes}$ lies in some $\mathcal{D}^{\otimes}_{\kappa}$).

Since every map $\mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ where $\mathcal{O}^{\otimes}$ is a small non-symmetric ∞-operad factors through $\mathcal{D}^{\otimes}_{\kappa}$ for some κ, we see that ρ is given by composition with $G^{\otimes}$. Moreover, the functor $R_{(-)}$ must also be given by composition with $G^{\otimes}$, since $\mathrm{Alg}^{\mathbb{O}}_{\mathcal{O}^{\otimes}}(\mathcal{D}^{\otimes})$ is the ∞-category associated to the simplicial space $\mathrm{Map}(\mathcal{O}^{\otimes} \otimes \Delta^{\bullet}, \mathcal{D}^{\otimes})$.

It remains to show that $G^{\otimes}$ is indeed a lax monoidal extension of G. This follows from taking $\mathcal{O}^{\otimes}$ to be the trivial non-symmetric ∞-operad $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{int}}$: then $\mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{int}}}(\mathcal{C}^{\otimes}) \simeq \mathcal{C}$ and $\mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{int}}}(\mathcal{D}^{\otimes}) \simeq \mathcal{D}$ and under these identifications $F_*^{\otimes}$ corresponds to F and $G_*^{\otimes}$ to the functor $G^{\otimes}_{[1]}$. Thus G and $G^{\otimes}_{[1]}$ are both right adjoint to F and so must be equivalent. □

Definition 3.3.5.8. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category. A *unit* for $\mathcal{V}^{\otimes}$ is an initial object of $\mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{V}^{\otimes})$.

Proposition 3.3.5.9. If $\mathcal{V}^{\otimes}$ is a monoidal ∞-category, then $\mathcal{V}^{\otimes}$ has a unit $\mathbb{\Delta}^{\mathrm{op}} \to \mathcal{V}^{\otimes}$.

Proof. As [Lur11, Proposition 3.2.1.8]. □

3.3.6 Approximations of ∞-Operads

In this subsection we use Lurie's theory of *approximations* to give a criterion for a map to be the operadic localization of a generalized non-symmetric ∞-operad. Here we write $L \colon \mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_{\infty} \to \mathrm{Opd}^{\mathbb{O}}_{\infty}$ for the left adjoint to the inclusion $\mathrm{Opd}^{\mathbb{O}}_{\infty} \hookrightarrow \mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_{\infty}$.

Definition 3.3.6.1. Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad, $\mathcal{O}^{\otimes}$ is a non-symmetric ∞-operad, and $f \colon \mathcal{M} \to \mathcal{O}^{\otimes}$ is a fibration of generalized non-symmetric ∞-operads. Then f is an *approximation* if for all $C \in \mathcal{M}$ and $\alpha \colon X \to f(C)$ active in $\mathcal{O}^{\otimes}$ there exists an f-Cartesian morphism $\bar{\alpha} \colon \bar{X} \to C$ lifting α, and a *weak approximation* if given $C \in \mathcal{M}$ and $\alpha \colon X \to f(C)$ an arbitrary morphism in $\mathcal{O}^{\otimes}$, the full subcategory of

$$\mathcal{M}_{/C} \times_{\mathcal{O}^{\otimes}_{/f(C)}} \mathcal{O}^{\otimes}_{X//f(C)}$$

corresponding to pairs $(\beta\colon C' \to C, \gamma\colon X \to f(C'))$ with γ inert is weakly contractible. More generally, a map $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ is a (*weak*) *approximation* if it factors as a composition

$$\mathcal{M} \xrightarrow{f'} \mathcal{M}' \xrightarrow{f''} \mathcal{O}^{\otimes}$$

where f' is an equivalence of generalized non-symmetric ∞-operads and f'' is a categorical fibration that is a (weak) approximation.

Proposition 3.3.6.2. An approximation is a weak approximation.

Proof. As [Lur11, Lemma 2.3.3.10]. □

Proposition 3.3.6.3. A fibration of generalized non-symmetric ∞-operads $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$, where $\mathcal{O}^{\otimes}$ is a non-symmetric ∞-operad, is a weak approximation if and only if for every object $C \in \mathcal{M}$ and every active morphism $\alpha\colon X \to f(C)$ in $\mathcal{O}^{\otimes}$, the ∞-category

$$\mathcal{M}_{/C} \times_{\mathcal{O}^{\otimes}_{/f(C)}} \{X\}$$

is weakly contractible.

Proof. As [Lur11, Proposition 2.3.3.11]. □

Proposition 3.3.6.4. Let $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ be a fibration of generalized non-symmetric ∞-operads, where $\mathcal{O}^{\otimes}$ is a non-symmetric ∞-operad. If $\mathcal{O}^{\otimes}_{[1]}$ is a Kan complex, then f is a weak approximation if and only if f is an approximation.

Proof. As [Lur11, Corollary 2.3.3.17]. □

Theorem 3.3.6.5. Suppose $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ is a weak approximation such that $f_{[1]}\colon \mathcal{M}_{[1]} \to \mathcal{O}^{\otimes}_{[1]}$ is a categorical equivalence. Then for any non-symmetric ∞-operad $\mathcal{P}^{\otimes}$, the induced map

$$f^*\colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{P}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{P}^{\otimes})$$

is an equivalence.

Proof. As [Lur11, Theorem 2.3.3.23]. □

Corollary 3.3.6.6. Suppose $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ is a weak approximation such that $f_{[1]}$ is a categorical equivalence. Then the induced map of non-symmetric ∞-operads $L\mathcal{M} \to \mathcal{O}^{\otimes}$ is an equivalence.

Proposition 3.3.6.7. Suppose $f\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a map of non-symmetric ∞-operads, and $\mathcal{P}^{\otimes}_{[1]}$ is a Kan complex. The commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}^{\mathbf{O}}_{\mathcal{P}^{\otimes}}(\mathcal{S}^{\times}) \arrow[r, "f^*"] \arrow[d, "\tau^*_{\mathcal{P}^{\otimes}}"'] & \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{S}^{\times}) \arrow[d, "\tau^*_{\mathcal{O}^{\otimes}}"] \\
\mathrm{Fun}(\mathcal{P}^{\otimes}_{[1]}, \mathcal{S}) \arrow[r, "f^*_{[1]}"'] & \mathrm{Fun}(\mathcal{O}^{\otimes}_{[1]}, \mathcal{S})
\end{tikzcd}$$

induces a natural transformation $\alpha\colon \tau_{\mathcal{O}^{\otimes},!} \circ f^*_{[1]} \to f^* \circ \tau_{\mathcal{P}^{\otimes},!}$. If α induces an equivalence $\tau_{\mathcal{O}^{\otimes},!} f^*_{[1]} A \xrightarrow{\sim} f^* \tau_{\mathcal{P}^{\otimes},!} A$ where A is the constant functor $\mathcal{P}^{\otimes}_{[1]} \to \mathcal{S}$ with value $*$, then f is an approximation.

Proof. As [Lur11, Proposition 2.3.4.8]. □

Corollary 3.3.6.8. Let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad such that $\mathcal{O}^{\otimes}_{[1]}$ is a Kan complex, and let $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ be a map of generalized non-symmetric ∞-operads such that the functor $f_{[1]}\colon \mathcal{M}_{[1]} \to \mathcal{O}^{\otimes}_{[1]}$ is an equivalence. Write A for the constant functor from $\mathcal{M}_{[1]} \simeq \mathcal{O}^{\otimes}_{[1]}$ to $\mathcal{S}$ with value $*$. If the natural map $\tau_{\mathcal{M},!}A \to f^*\tau_{\mathcal{O}^{\otimes},!}A$ is an equivalence, then f exhibits $\mathcal{O}^{\otimes}$ as the operadic localization of $\mathcal{M}$.

Proof. Applying Proposition 3.3.6.7 to the induced map $f'\colon L\mathcal{M} \to \mathcal{O}^{\otimes}$, we see that this map is an approximation and induces an equivalence $L\mathcal{M}_{[1]} \to \mathcal{O}^{\otimes}_{[1]}$. By Theorem 3.3.6.5, it follows that f' is an equivalence. □

Corollary 3.3.6.9. Let $\mathcal{O}^{\otimes}$ be a non-symmetric ∞-operad such that $\mathcal{O}^{\otimes}_{[1]}$ is a Kan complex, and suppose $f\colon \mathcal{M} \to \mathcal{O}^{\otimes}$ is a map of generalized non-symmetric ∞-operads such that $f_{[1]}\colon \mathcal{M}_{[1]} \to \mathcal{O}^{\otimes}_{[1]}$ is an equivalence. If the induced map $(\mathcal{M}_{\mathrm{act}})_{/x} \to (\mathcal{O}^{\otimes}_{\mathrm{act}})_{/x}$ is cofinal for all $x \in \mathcal{M}_{[1]} \simeq \mathcal{O}^{\otimes}_{[1]}$, then f exhibits $\mathcal{O}^{\otimes}$ as the operadic localization of $\mathcal{M}$.

Proof. By Corollary 3.3.6.8 it suffices to show that the natural map of $\mathcal{M}$-algebras $\tau_{\mathcal{M},!}A \to f^*\tau_{\mathcal{O}^{\otimes},!}A$ is an equivalence. Since $\tau^*_{\mathcal{M}}$ detects equivalences by Lemma 3.3.5.3, to see this it suffices to show that for all $x \in \mathcal{M}_{[1]}$ the map of spaces $(\tau_{\mathcal{M},!}A)(x) \to (\tau_{\mathcal{O}^{\otimes},!}A)(x)$ is an equivalence. Recalling the definition of an operadic Kan extension, we see that this is the map

$$\operatorname*{colim}_{(\mathcal{M}_{\mathrm{act}})_{/x}} * \to \operatorname*{colim}_{(\mathcal{O}^{\otimes}_{\mathrm{act}})_{/x}} *$$

of colimits induced by $(\mathcal{M}_{\mathrm{act}})_{/x} \to (\mathcal{O}^{\otimes}_{\mathrm{act}})_{/x}$. If this map is cofinal, then the induced map on colimits is an equivalence. □

Remark 3.3.6.10. The same argument shows that for any presentably monoidal ∞-category $\mathcal{V}^{\otimes}$ the natural map $\tau_{\mathcal{M},!}F \to f^*\tau_{\mathcal{O}^{\otimes},!}F$ is an equivalence for any functor $F\colon \mathcal{M}_{[1]} \to \mathcal{V}$. It follows that $\tau_{\mathcal{M},!}$ and $\tau_{\mathcal{O}^{\otimes},!}$ are given by the same monad on $\mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{V})$, hence the ∞-categories of algebras $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{V}^{\otimes})$ and $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{V}^{\otimes})$ must be equivalent, since they are both ∞-categories of algebras for this monad. An alternative proof of Corollary 3.3.6.9 (not using the notion of approximation) results by embedding any small non-symmetric ∞-operad $\mathcal{P}^{\otimes}$ in a presentably monoidal ∞-category $\widehat{\mathcal{P}}^{\otimes}$ and showing that $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\mathcal{P}^{\otimes})$ and $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{P}^{\otimes})$ are the same subcategory of $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{M}}(\widehat{\mathcal{P}}^{\otimes}) \simeq \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\widehat{\mathcal{P}}^{\otimes})$.

3.3.7 More on the Algebra Fibration

Here we use the results of this section to say a bit more about algebra fibrations in the non-symmetric case. First we observe that colimit-preserving strong monoidal functors induce colimit-preserving functors on algebra fibrations:

Proposition 3.3.7.1. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be monoidal ∞-categories compatible with small colimits, and suppose $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a strong monoidal functor such that $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits. Then $F^{\otimes}_*\colon \mathrm{Alg}^{\mathbf{O}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}(\mathcal{D}^{\otimes})$ preserves colimits.

Proof. Since $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are compatible with small colimits, the projections

$$\mathrm{Alg}^{\mathbf{O}}(\mathcal{C}^{\otimes}), \mathrm{Alg}^{\mathbf{O}}(\mathcal{D}^{\otimes}) \to \mathrm{Opd}_{\infty}^{\mathbf{O}}$$

are coCartesian fibrations by Lemma 3.2.8.5. Thus a diagram in $\mathrm{Alg}^{\mathbf{O}}(\mathcal{D}^{\otimes})$ is a colimit diagram if and only if it is a relative colimit diagram whose projection to $\mathrm{Opd}_{\infty}^{\mathbf{O}}$ is a colimit diagram.

It therefore suffices to prove that F_* preserves coCartesian arrows and preserves colimits fibrewise. The former follows from Lemma 3.3.4.7, and the latter we proved in Proposition 3.3.5.6. □

Our next goal is to prove that for algebras in monoidal ∞-categories the external product ⊠ preserves colimits in each variable:

Lemma 3.3.7.2. Suppose $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are monoidal ∞-categories compatible with small colimits. Then the external product ⊠ preserves free algebras, i.e. given non-symmetric ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$, algebras $A \in \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ and$B \in \mathrm{Alg}^{\mathbf{O}}_{\mathcal{P}^{\otimes}}(\mathcal{D}^{\otimes})$, and morphisms of non-symmetric ∞-operads $f\colon \mathcal{O}^{\otimes} \to \mathcal{Q}^{\otimes}$ and $g\colon \mathcal{P}^{\otimes} \to \mathcal{R}^{\otimes}$, we have $f_! A \boxtimes g_! B \simeq (f \times g)_!(A \boxtimes B)$ in $\mathrm{Alg}^{\mathbf{O}}_{\mathcal{Q}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{R}^{\otimes}}(\mathcal{C}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}})$.

Proof. This follows from considering operadic colimits in $\mathcal{C}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{D}^{\otimes}$. □

Proposition 3.3.7.3. Suppose $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are monoidal ∞-categories compatible with small colimits, and let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be non-symmetric ∞-operads and $A \in \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes})$ an $\mathcal{O}^{\otimes}$-algebra. Then

$$A \boxtimes (-)\colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{P}^{\otimes}}(\mathcal{D}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{P}^{\otimes}}(\mathcal{C}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{D}^{\otimes})$$

preserves colimits.

Proof. First we consider the case of trival non-symmetric ∞-operads. Suppose A' is an $\mathcal{O}^{\otimes}_{\mathrm{triv}}$-algebra. Then

$$A' \boxtimes -\colon \mathrm{Alg}^{\mathbf{O}}_{\mathcal{P}^{\otimes}_{\mathrm{triv}}}(\mathcal{D}^{\otimes}) \to \mathrm{Alg}^{\mathbf{O}}_{\mathcal{O}^{\otimes}_{\mathrm{triv}} \times_{\Delta^{\mathrm{op}}} \mathcal{P}^{\otimes}_{\mathrm{triv}}}(\mathcal{C}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{D}^{\otimes})$$

clearly preserves colimits, since it is equivalent to the the functor

$$A'|_{\mathcal{O}^{\otimes}_{[1]}} \times -\colon \mathrm{Fun}(\mathcal{P}^{\otimes}_{[1]}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{O}^{\otimes}_{[1]} \times \mathcal{P}^{\otimes}_{[1]}, \mathcal{C} \times \mathcal{D}).$$

Since we have $\tau^*_{\mathcal{C}\times\mathcal{D}}(A \boxtimes B) \simeq \tau^*_{\mathcal{C}} A \boxtimes \tau^*_{\mathcal{D}} B$ and $\tau^*_{\mathcal{C}\times\mathcal{D}}$ detects sifted colimits, it follows that $A \boxtimes -$ preserves sifted colimits.

Next we consider the case where A is a free algebra $\tau_{\mathcal{C},!}A'$ where A' is an $\mathcal{O}^{\otimes}_{\mathrm{triv}}$-algebra in $\mathcal{C}$. By Lemma 3.3.7.2 we have

$$\tau_{\mathcal{C},!}A' \boxtimes \tau_{\mathcal{D},!}B' \simeq \tau_{\mathcal{C}\times\mathcal{D},!}(A' \boxtimes B'),$$

so the functor $\tau_{\mathcal{C},!}A \boxtimes -$ preserves colimits of free algebras. Thus it must preserve all colimits, by monadicity.

Finally, suppose $A_\bullet$ is a free resolution of A, and $\alpha \mapsto B_\alpha$ is any diagram. Then since ⊠ preserves sifted colimits we have

$$A \boxtimes \operatorname{colim} B_\alpha \simeq |A_\bullet| \boxtimes \operatorname{colim} B_\alpha \simeq |A_\bullet \boxtimes \operatorname{colim} B_\alpha|.$$

From the case of free algebras we then see that this is equivalent to

$$|\operatorname{colim}(A_\bullet \boxtimes B_\alpha)| \simeq \operatorname{colim} |A_\bullet \boxtimes B_\alpha|.$$

But since $\boxtimes$ preserves sifted colimits in each variable, this is

$$\operatorname{colim}(|A_\bullet| \boxtimes B_\alpha) \simeq \operatorname{colim}(A \boxtimes B_\alpha). \qquad \square$$

Remark 3.3.7.4. The Cartesian product of non-symmetric ∞-operads does not preserve colimits, so it is not possible for the external product $A \boxtimes (-)$ to preserve colimits as a functor $\mathrm{Alg}^{\mathcal{O}}(\mathcal{D}^{\otimes}) \to \mathrm{Alg}^{\mathcal{O}}(\mathcal{C}^{\otimes} \times_{\Delta^{\mathrm{op}}} \mathcal{D}^{\otimes})$.

3.3.8 Modules

Here we briefly introduce a definition of modules over associative algebras, to motivate our definition of correspondences between enriched ∞-categories. Our definition is a little different from that used by Lurie, but we will not bother to compare them here.

Definition 3.3.8.1. For $[n]$ in Δ^{op} the functor $\mathrm{Hom}_{\Delta^{\mathrm{op}}}([n], -) \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ satisfies the Segal conditions. Let $\Delta^{\mathrm{op}}[n] \to \Delta^{\mathrm{op}}$ be an associated coGrothendieck fibration — then $\Delta^{\mathrm{op}}[n]$ is a double ∞-category. If $\phi \colon [m] \to [n]$ is a morphism in Δ^{op}, then there is clearly an induced functor $\phi \colon \Delta^{\mathrm{op}}[m] \to \Delta^{\mathrm{op}}[n]$.

Remark 3.3.8.2. The objects of $\Delta^{\mathrm{op}}[n]$ can be described as sequences $(i_0, \ldots, i_k)$ where $0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n$. There is a unique morphism $(i_0, \ldots, i_k) \to (i_{\phi(0)}, \ldots, i_{\phi(m)})$ over every map $\phi \colon [m] \to [k]$ in Δ.

Definition 3.3.8.3. Let $\mathcal{M}$ be a generalized non-symmetric ∞-operad. A *bimodule* in $\mathcal{M}$ is a $\Delta^{\mathrm{op}}[1]$-algebra in $\mathcal{M}$. We write $\mathrm{Bimod}(\mathcal{M}) := \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}[1]}(\mathcal{M})$ for the ∞-category of bimodules in $\mathcal{M}$. If M is a bimodule in $\mathcal{M}$ then $A = d_1^* M$ and $B = d_0^* M$ are associative algebras in $\mathcal{M}$, and we say that M is an *A-B-bimodule*.

Remark 3.3.8.4. Let $M \colon \Delta^{\mathrm{op}}[1] \to \mathcal{M}$ be a bimodule; then we see that M is determined by an object $M(0,1) \in \mathcal{M}$ with compatible actions of associative algebras $M(0,0)$ on the left and $M(1,1)$ on the right.

Lemma 3.3.8.5. The projection $(d_1^*, d_0^*) \colon \mathrm{Bimod}(\mathcal{M}) \to \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{M}) \times \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{M})$ is a Cartesian fibration.

Definition 3.3.8.6. If A, B are associative algebra objects in a generalized non-symmetric ∞-operad $\mathcal{M}$, we write $\mathrm{Bimod}_{A,B}(\mathcal{M})$ for the fibre of $\mathrm{Bimod}(\mathcal{M})$ at (A, B).

Definition 3.3.8.7. For $[n] \in \Delta^{\mathrm{op}}$, the inert maps $[n] \to [1], [0]$ determine a map of generalized non-symmetric ∞-operads $\kappa_n \colon \Delta^{\mathrm{op}}_{\amalg}[n] := \Delta^{\mathrm{op}}[1] \amalg_{\Delta^{\mathrm{op}}} \cdots \amalg_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}[1] \to \Delta^{\mathrm{op}}[n]$. We say a $\Delta^{\mathrm{op}}[n]$-algebra X in a generalized non-symmetric ∞-operad $\mathcal{M}$ is a *tensor product* if X is a left operadic Kan extension of its restriction $\kappa_n^* X$, i.e. the natural map $\kappa_{n,!}\kappa_n^* X \to X$ is an equivalence.

Remark 3.3.8.8. A $\Delta^{\mathrm{op}}[2]$-algebra X is a tensor product if and only if the map

$$\operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} r_{n,!} X(0, 1, \ldots, 1, 2) \to X(0, 2),$$

where r_n is the unique active map $[n] \to [1]$, is an equivalence (because this copy of Δ^{op} is cofinal in the category of active maps to $(0,2)$). If $\mathcal{M}$ is a monoidal ∞-category this says that $X(0,2)$ is given by the bar construction $|X(0,1) \otimes X(1,1)^{\otimes \bullet} \otimes X(1,2)|$, which is the usual definition of the (derived) tensor product of modules.

Definition 3.3.8.9. Let $\mathrm{BIMOD}_n(\mathcal{M})$ be the full subcategory of $\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}[n]}(\mathcal{M})$ spanned by the $\Delta^{\mathrm{op}}[n]$-algebras that are tensor products. Then $\mathrm{BIMOD}_\bullet(\mathcal{M})$ is a simplicial ∞-category.

Proposition 3.3.8.10. Suppose $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category. Then for any $\Delta^{\mathrm{op}}_{\amalg}[n]$-algebra X in $\mathcal{V}^{\otimes}$ the adjunction morphism $X \to \kappa_n^* \kappa_{n,!} X$ is an equivalence.

Proof. This is a special case of Corollary 4.6.2.6; as the proof is rather complicated (although slightly simpler than the general case), we will not prove this case separately. □

Corollary 3.3.8.11. Suppose $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category. Then $\mathrm{BIMOD}_\bullet(\mathcal{V}^{\otimes})$ is a double ∞-category — the *double ∞-category of bimodules*.

Remark 3.3.8.12. Given associative algebras A, B, C in $\mathcal{V}^{\otimes}$, looking at fibres the functor $d_1 \colon \mathrm{BIMOD}_2(\mathcal{V}^{\otimes}) \to \mathrm{BIMOD}_1(\mathcal{V}^{\otimes})$ gives a tensor product functor

$$\otimes_B \colon \mathrm{Bimod}_{A,B}(\mathcal{V}^{\otimes}) \times \mathrm{Bimod}_{B,C}(\mathcal{V}^{\otimes}) \to \mathrm{Bimod}_{A,C}(\mathcal{V}^{\otimes}).$$

The remaining structure of the double ∞-category $\mathrm{BIMOD}_\bullet(\mathcal{V}^{\otimes})$ shows that these relative tensor products are coherently associative.

Definition 3.3.8.13. Suppose $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category, and let A be an associative algebra object in $\mathcal{V}^{\otimes}$. Let $\mathrm{Bimod}_A(\mathcal{V}^{\otimes})^{\otimes}$ be the full subcategory of $\mathrm{BIMOD}_\bullet(\mathcal{V}^{\otimes})$ of objects over $A \in \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^{\otimes}) \simeq \mathrm{BIMOD}_0(\mathcal{V}^{\otimes})$. This is a monoidal ∞-category — the monoidal ∞-category of A-bimodules.

Definition 3.3.8.14. Let LM be the full subcategory of $\Delta^{\mathrm{op}}[1]$ spanned by objects $(0, \dots, 0, 1)$ and $(0, \dots, 0)$. This is a double ∞-category. A *left module* in a generalized non-symmetric ∞-operad $\mathcal{M}$ is an LM-algebra in $\mathcal{M}$; we write $\mathrm{LMod}(\mathcal{M}) := \mathrm{Alg}^{\mathcal{O}}_{\mathrm{LM}}(\mathcal{M})$ for the ∞-category of left modules in $\mathcal{M}$. The inclusion $l \colon \Delta^{\mathrm{op}} \to \mathrm{LM}$ that sends $[n]$ to $(1, \dots, 1)$ is a morphism of generalized non-symmetric ∞-operads. If M is a left module in $\mathcal{M}$ and $A = l^*M$, then we say that M is a *left A-module*.

Lemma 3.3.8.15. Let $\mathcal{M}$ be a generalized non-symmetric ∞-operad. The functor

$$l^* \colon \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{M})$$

is a Cartesian fibration. We write $\mathrm{LMod}_A(\mathcal{M})$ for the fibre of l^* at $A \in \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{M})$ — this is the ∞-category of left A-modules in $\mathcal{M}$.

Definition 3.3.8.16. Let RM be the full subcategory of $\Delta^{\mathrm{op}}[1]$ spanned by objects $(0, 1, \dots, 1)$ and $(1, \dots, 1)$. This is a double ∞-category. A *right module* in a generalized non-symmetric ∞-operad $\mathcal{M}$ is an RM-algebra in $\mathcal{M}$; we write $\mathrm{RMod}(\mathcal{M}) := \mathrm{Alg}^{\mathcal{O}}_{\mathrm{RM}}(\mathcal{M})$ for the ∞-category of right modules in $\mathcal{M}$. The inclusion $r \colon \Delta^{\mathrm{op}} \to \mathrm{RM}$ that sends $[n]$ to $(0, \dots, 0)$ is a morphism of generalized non-symmetric ∞-operads. If M is a right module in $\mathcal{M}$ and $A = r^*M$, then we say that M is a *right A-module*.

Lemma 3.3.8.17. Let $\mathcal{M}$ be a generalized non-symmetric ∞-operad. The functor

$$r^* \colon \mathrm{RMod}(\mathcal{M}) \to \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{M})$$

is a Cartesian fibration. We write $\mathrm{RMod}_A(\mathcal{M})$ for the fibre of r^* at $A \in \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{M})$ — this is the ∞-category of right A-modules in $\mathcal{M}$.

Chapter 4

Enriched ∞-Categories

In this, the main chapter of this thesis, we introduce our theory of enriched ∞-categories. In §4.1 we define these objects as algebras for certain non-symmetric ∞-operads and construct an "algebraic" ∞-category of ∞-categories enriched in a fixed monoidal ∞-category $\mathcal{V}$ using the ∞-categories of algebras for these ∞-operads. Then in §4.2 we construct the correct ∞-category of $\mathcal{V}$-∞-categories by localizing this at the fully faithful and essentially surjective functors; our main result here is that this is an accessible localization, given by restricting to certain "complete" objects. §4.3 contains some simple applications of our setup. We next compare our ∞-categories of enriched ∞-categories to those coming from model categories of enriched categories and Segal categories in §4.4, where we also show that iterated enrichment in spaces gives an ∞-category equivalent to that of complete n-fold Segal spaces. In §4.5 we study natural transformations and functor categories, and construct an $(\infty,2)$-category of $\mathcal{V}$-∞-categories, and in §4.6 we introduce correspondences between $\mathcal{V}$-∞-categories.

4.1 Categorical Algebras

In this section we use the theory of generalized ∞-operads developed in Chapter 3 to define *categorical algebras* and construct ∞-categories of these.

In §4.1.1 we construct double ∞-categories $\Delta^{\mathrm{op}}_{\mathcal{C}}$, where $\mathcal{C}$ is an ∞-category; we then define ∞-categories enriched in a monoidal ∞-category $\mathcal{V}$ to be Δ^{op}_{X}-algebras in $\mathcal{V}$ when X is a space. Next, in §4.1.2 we identify the non-symmetric ∞-operad associated to Δ^{op}_{X} with that arising from a certain simplicial multicategory. Then in §4.1.3 we use the algebra fibration from §3.3.7 to construct "algebraic" ∞-categories of enriched ∞-categories. Finally in §4.1.4 we prove that ∞-categories enriched in spaces are equivalent to Segal spaces.

4.1.1 The Double ∞-Categories $\Delta^{\mathrm{op}}_{\mathcal{C}}$

Here we introduce double ∞-categories $\Delta^{\mathrm{op}}_{\mathcal{C}}$, observe some of their basic properties, and define enriched ∞-categories to be algebras for these when $\mathcal{C}$ is a space.

Definition 4.1.1.1. Let i denote the inclusion $\{[0]\} \hookrightarrow \Delta^{\mathrm{op}}$. Taking right Kan extensions along i gives a functor $i_*\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_\infty)$. If $\mathcal{C}$ is an ∞-category we write $\Delta^{\mathrm{op}}_{\mathcal{C}} \to \Delta^{\mathrm{op}}$ for a coCartesian fibration corresponding to the functor $i_*\mathcal{C}$.

Remark 4.1.1.2. If $\mathcal{C}$ is an ∞-category, then $i_*\mathcal{C}$ is the simplicial ∞-category with nth space $\mathcal{C}^{\times n+1}$, face maps given by the appropriate projections, and degeneracies by the appropriate diagonal maps.

Lemma 4.1.1.3. Let $\mathcal{C}$ be an ∞-category. The coCartesian fibration $\Delta^{\mathrm{op}}_{\mathcal{C}} \to \Delta^{\mathrm{op}}$ is a double ∞-category.

Proof. It is clear that $i_*\mathcal{C}$ is a category object, hence $\Delta^{\mathrm{op}}_{\mathcal{C}}$ is a double ∞-category by Proposition 3.2.4.6. $\square$

Definition 4.1.1.4. Let $\mathcal{V}^{\otimes}$ be a monoidal ∞-category. A *categorical algebra* in $\mathcal{V}^{\otimes}$, or $\mathcal{V}$-*enriched ∞-category*, or $\mathcal{V}$-*∞-category*, with *underlying space of objects* X, is a Δ^{op}_X-algebra in $\mathcal{V}^{\otimes}$.

Remark 4.1.1.5. This definition clearly does not require $\mathcal{V}^{\otimes}$ to be a monoidal ∞-category — we can define ∞-categories with space of objects X enriched in any generalized non-symmetric ∞-operad as Δ^{op}_X-algebras. This gives an ∞-categorical version of Leinster's notion of enrichment in an **fc**-multicategory [Lei02]. However, as there are technical obstacles in the theory of ∞-operads to extending most of our results below beyond the case of monoidal ∞-categories we will not consider this generalization here.

Definition 4.1.1.6. If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories with spaces of objects X and Y, a $\mathcal{V}$-*functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of a map of spaces $f\colon X \to Y$ and a map of Δ^{op}_X-algebras from $\mathcal{C}$ to $(\Delta^{\mathrm{op}}_f)^*\mathcal{D}$, i.e. a natural transformation $\mathcal{C} \to \mathcal{D} \circ \Delta^{\mathrm{op}}_f$ of functors $\Delta^{\mathrm{op}}_X \to \mathcal{V}^{\otimes}$.

Remark 4.1.1.7. The functor

$$\Delta^{\mathrm{op}}_{(-)}\colon \mathrm{Cat}_\infty \to \mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_\infty$$

is a right adjoint to the functor $\mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_\infty \to \mathrm{Cat}_\infty$ that sends a generalized non-symmetric ∞-operad $\mathcal{M}$ to its fibre $\mathcal{M}_{[0]}$ at $[0]$: it is a composite of the right Kan extension functor $i_*\colon \mathrm{Cat}_\infty \to \mathrm{Dbl}_\infty$, which is right adjoint to the fibre-at-$[0]$ functor, and the inclusion $\mathrm{Dbl}_\infty \hookrightarrow \mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_\infty$, right adjoint to the monoidal envelope functor, which preserves fibres at $[0]$ (cf. §3.3.1).

Proposition 4.1.1.8. The functor $\Delta^{\mathrm{op}}_{(-)}\colon \mathrm{Cat}_\infty \to \mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_\infty$ preserves sifted colimits.

Proof. Suppose we have a sifted diagram of ∞-categories $p\colon \mathcal{I} \to \mathrm{Cat}_\infty$ with colimit $\mathcal{C}$. Since $\Delta^{\mathrm{op}}_{\mathcal{C}}$ is a generalized non-symmetric ∞-operad, by Lemma 3.2.5.1 it suffices to show that $\Delta^{\mathrm{op}}_{\mathcal{C}}$ is the colimit of $\Delta^{\mathrm{op}}_{p(-)}$ in Cat_∞. Now this composite functor

$$\mathrm{Cat}_\infty \xrightarrow{\Delta^{\mathrm{op}}_{(-)}} \mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_\infty \to \mathrm{Cat}_\infty$$

factors as

$$\mathrm{Cat}_\infty \xrightarrow{i_*} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{CoCart}(\Delta^{\mathrm{op}}) \xrightarrow{q} \mathrm{Cat}_\infty,$$

where the rightmost functor q is the forgetful functor that sends a fibration $\mathcal{E} \to \Delta^{\mathrm{op}}$ to the ∞-category $\mathcal{E}$. Since the ∞-category $\mathrm{CoCart}(\Delta^{\mathrm{op}})$ is the ∞-category associated to the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{P}^{\mathrm{coCart}}_{\Delta^{\mathrm{op}}}}$, it follows from Example 2.1.12.15 that q preserves colimits. It thus remains to prove that i_* preserves sifted colimits. Colimits in functor categories are computed pointwise, so to see this it suffices to show that for each $[n]$ the composite functor

$\mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ induced by composing with evaluation at $[n]$ preserves sifted colimits. This functor sends $\mathcal{D}$ to the product $\mathcal{D}^{\times(n+1)}$, and so preserves sifted colimits by [Lur09a, Proposition 5.5.8.6], since the Cartesian product of ∞-categories preserves colimits separately in each variable. $\square$

4.1.2 The ∞-Operad Associated to $\mathbb{\Delta}^{\mathrm{op}}_X$

By Corollary 3.2.2.16 there is a universal non-symmetric ∞-operad $L\mathbb{\Delta}^{\mathrm{op}}_X$ receiving a map from the double ∞-category $\mathbb{\Delta}^{\mathrm{op}}_X$. In this subsection we describe a concrete model for $L\mathbb{\Delta}^{\mathrm{op}}_X$ as a simplicial multicategory; this will allow us to conclude that the functor that sends X to $L\mathbb{\Delta}^{\mathrm{op}}_X$ preserves products.

Remark 4.1.2.1. Although it is obvious that the functor $\mathbb{\Delta}^{\mathrm{op}}_{(-)}$ preserves products, since it's a right adjoint by Remark 4.1.1.7, it is not clear that the localization functor $L\colon \mathrm{Opd}_\infty^{\mathbb{O},\mathrm{gen}} \to \mathrm{Opd}_\infty^{\mathbb{O}}$ preserves products (and this may well be false in general).

First we define simplicial categories $\mathcal{D}(\mathcal{C})$ that model $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ when $\mathcal{C}$ is a simplicial category:

Definition 4.1.2.2. Given a simplicial category $\mathcal{C}$, the simplicial category $\mathcal{D}(\mathcal{C})$ has objects finite sequences $(c_0, \ldots, c_n)$ of objects of $\mathcal{C}$; morphisms are given by

$$\mathcal{D}(\mathcal{C})((c_0, \ldots, c_n), (d_0, \ldots, d_m)) := \coprod_{\phi\colon [m]\to[n]} \prod_{i=0}^{m} \mathcal{C}(c_{\phi(i)}, d_i),$$

with the obvious composition maps induced by those in $\mathcal{C}$.

Proposition 4.1.2.3. Suppose $\mathcal{C}$ is a fibrant simplicial category. Then:

(i) The projection $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathrm{N}\mathbb{\Delta}^{\mathrm{op}}$ is a coCartesian fibration.

(ii) The fibre $\mathrm{N}\mathcal{D}(\mathcal{C})_{[0]}$ is equivalent to $\mathrm{N}\mathcal{C}$.

(iii) There is a natural map $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$.

(iv) This map is an equivalence of ∞-categories.

Proof.

(i) It is clear that $\mathcal{D}(\mathcal{C}) \to \mathbb{\Delta}^{\mathrm{op}}$ is a fibration in the model structure on simplicial categories; since N is a right Quillen functor, it follows that $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathrm{N}\mathbb{\Delta}^{\mathrm{op}}$ is a categorical fibration. It therefore suffices to check that $\mathrm{N}\mathcal{D}(\mathcal{C})$ has coCartesian morphisms. Given an object $C = (c_0, \ldots, c_n)$ in $\mathcal{D}(\mathcal{C})$ and a map $\phi\colon [m] \to [n]$ in $\mathbb{\Delta}$, let $\phi_!$ denote the obvious map $C \to C' = (c_{\phi(0)}, \ldots, c_{\phi(m)})$ in $\mathcal{D}(\mathcal{C})$. We apply the criterion of [Lur11, Proposition 2.4.1.10] to see that $\phi_!$ is coCartesian in $\mathrm{N}\mathcal{D}(\mathcal{C})$; thus we need to show that for every $X \in \mathcal{D}(\mathcal{C})$ over $[k] \in \mathbb{\Delta}^{\mathrm{op}}$ the commutative diagram

$$\begin{tikzcd}
\mathcal{D}(\mathcal{C})(C', X) \arrow[r] \arrow[d] & \mathcal{D}(\mathcal{C})(C, X) \arrow[d] \\
\mathrm{Hom}_{\mathbb{\Delta}^{\mathrm{op}}}([m], [k]) \arrow[r] & \mathrm{Hom}_{\mathbb{\Delta}^{\mathrm{op}}}([n], [k])
\end{tikzcd}$$

is a homotopy Cartesian square of simplicial sets. Since the simplicial category $\mathcal{C}$ is fibrant, so is $\mathcal{D}(\mathcal{C})$, hence the vertical maps are Kan fibrations. It therefore suffices to show that the induced maps on fibres are equivalences, which is clear from the definition of $\mathcal{D}(\mathcal{C})$.

(ii) We have a pullback diagram of simplicial categories

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D}(\mathcal{C}) \arrow[d] \\ \{[0]\} \arrow[r] & \mathbb{\Delta}^{\mathrm{op}}. \end{tikzcd}$$

Since the simplicial nerve is a right adjoint, it follows that $\mathrm{N}\mathcal{C}$ is the fibre of the map of simplicial sets $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathbb{\Delta}^{\mathrm{op}}$ at $[0]$. Since this map is a coCartesian fibration, by [Lur09a, Corollary 3.3.1.4] $\mathrm{N}\mathcal{C}$ is also the homotopy fibre in the Joyal model structure.

(iii) By definition $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ corresponds to the right Kan extension $i_*\mathrm{N}\mathcal{C}$ of $\mathrm{N}\mathcal{C}$ along the inclusion $i\colon \{[0]\} \hookrightarrow \mathbb{\Delta}^{\mathrm{op}}$. The functor i_* is right adjoint to the fibre-at-$[0]$ functor i^*, and from (ii) we know that $i^*\mathcal{D}(\mathcal{C}) \simeq \mathrm{N}\mathcal{C}$. The adjunction $i^* \dashv i_*$ then gives the required map $\mathcal{D}(\mathcal{C}) \to \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$.

(iv) By [Lur09a, Corollary 2.4.4.4] it suffices to show that for each $[n]$ in $\mathbb{\Delta}^{\mathrm{op}}$ the induced map on fibres

$$(\mathrm{N}\mathcal{D}(\mathcal{C}))_{[n]} \to (\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}})_{[n]}$$

is a categorical equivalence. As in (ii) we can identify the fibre $(\mathrm{N}\mathcal{D}(\mathcal{C}))_{[n]}$ with $\mathrm{N}\mathcal{C}^{\times n}$, via the Segal maps, so by naturality we have a commutative diagram

$$\begin{tikzcd} (\mathrm{N}\mathcal{D}(\mathcal{C}))_{[n]} \arrow[r] \arrow[d] & (\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{N}})_{[n]} \arrow[d] \\ \mathrm{N}\mathcal{C}^{\times n} \arrow[r] & \mathrm{N}\mathcal{C}^{\times n}, \end{tikzcd}$$

where all but the top horizontal map are known to be categorical equivalences. Hence this must also be a categorical equivalence, by the 2-out-of-3 property. □

Definition 4.1.2.4. Let $\mathcal{C}$ be a simplicial category. The simplicial multicategory $\mathcal{O}_{\mathcal{C}}$ has objects $\mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{C}$ and multimorphism spaces defined by

$$\mathcal{O}_{\mathcal{C}}((x_0, y_1), \ldots, (x_{n-1}, y_n); (y_0, x_n)) := \mathcal{C}(y_0, x_0) \times \mathcal{C}(y_1, x_1) \times \cdots \times \mathcal{C}(y_{n-1}, x_{n-1}) \times \mathcal{C}(y_n, x_n).$$

Composition is defined in the obvious way, using composition in $\mathcal{C}$. Write $\mathcal{O}^{\otimes}_{\mathcal{C}}$ for the associated simplicial May-Thomason category over $\mathbb{\Delta}^{\mathrm{op}}$, defined as in Remark 3.1.6.2.

If $\mathcal{C}$ is a fibrant simplicial category, then $\mathcal{O}_{\mathcal{C}}$ is a fibrant simplicial multicategory in the sense of Definition 3.2.1.15, and so $\mathrm{N}\mathcal{O}^{\otimes}_{\mathcal{C}}$ is a non-symmetric ∞-operad by Lemma 3.2.1.16.

The simplicial multicategory $\mathcal{O}_{\mathcal{C}}$ is only a model for $\Delta^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ when $\mathrm{N}\mathcal{C}$ is a space, but is easier to define than the version that works more generally. Indeed there is not even a natural map from $\mathcal{D}(\mathcal{C})$ to $\mathcal{O}_{\mathcal{C}}$ in general; however, we can construct one if $\mathcal{C}$ is a simplicial groupoid. By Remark 2.1.2.10 we may regard a simplicial groupoid $\mathcal{C}$ as a simplicial category equipped with an involution i that sends a morphism to its inverse. Using this we can define a functor $\mathcal{D}(\mathcal{C}) \to \mathcal{O}_{\mathcal{C}}^{\otimes}$:

Definition 4.1.2.5. Suppose $\mathcal{C}$ is a simplicial groupoid. Let $\Phi\colon \mathcal{D}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}^{\otimes}$ be the functor sending an object $(c_0, \ldots, c_n)$ of $\mathcal{D}(\mathcal{C})$ to $((c_0, c_1), (c_1, c_2), \ldots, (c_{n-1}, c_n))$ and given on morphisms by applying i on the first factor and inserting identities into the factors that are missing in $\mathcal{D}(\mathcal{C})$ in the obvious way.

Theorem 4.1.2.6. Let $\mathcal{C}$ be a fibrant simplicial groupoid. Then the map

$$\mathrm{N}\Phi\colon \mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathrm{N}\mathcal{O}_{\mathcal{C}}^{\otimes}$$

exhibits $\mathrm{N}\mathcal{O}_{\mathcal{C}}^{\otimes}$ as the operadic localization of $\mathrm{N}\mathcal{D}(\mathcal{C})$.

Proof. By Corollary 3.3.6.9 it suffices to show that for all $(x, y) \in \mathcal{C} \times \mathcal{C}$ the induced map $g\colon (\mathrm{N}\mathcal{D}(\mathcal{C})_{\mathrm{act}})_{/(x,y)} \to (\mathrm{N}(\mathcal{O}_{\mathcal{C}}^{\otimes})_{\mathrm{act}})_{/(x,y)}$ is cofinal. We will prove that g is a categorical equivalence; to see this we show that g is essentially surjective and induces equivalences on mapping spaces.

We first observe that g is essentially surjective: an active morphism to (x, y) in $\mathcal{O}_{\mathcal{C}}^{\otimes}$ is determined by an object $T = ((t_0, s_1), (t_1, s_2), \ldots, (t_{n-1}, s_n))$ and morphisms $\alpha\colon x \to t_0$, $\beta_1\colon s_1 \to t_1, \ldots, \beta_{n-1}\colon s_{n-1} \to t_{n-1}$, $\gamma\colon s_n \to y$ in $\mathcal{C}$. Such a morphism is in the image of g if and only if the β_i's are all identities. Since $\mathcal{C}$ is by assumption a simplicial groupoid all morphisms in $\mathcal{C}$ are equivalences, and so the morphism

$$((t_0, s_1), (s_1, s_2), \ldots, (s_{n-1}, s_n)) \to ((t_0, s_1), (t_1, s_2), \ldots, (t_{n-1}, s_n))$$

given by $(\mathrm{id}, \mathrm{id}, \beta_1, \mathrm{id}, \beta_2, \ldots, \mathrm{id})$ is an equivalence from an object in the image of g to T.

It remains to show that g is fully faithful. Given objects $Z = (z_0, \ldots, z_n)$ and $Z' = (z'_0, \ldots, z'_m)$ in $\mathcal{D}(\mathcal{C})$ we must show that for each active map $\phi\colon [m] \to [n]$ in Δ^{op} the map

$$\mathrm{Map}^{\phi}_{\mathrm{N}\mathcal{D}(\mathcal{C})_{/(x,y)}}(Z, Z') \to \mathrm{Map}^{\phi}_{(\mathrm{N}\mathcal{O}_{\mathcal{C}}^{\otimes})_{/(x,y)}}(g(Z), g(Z'))$$

is an equivalence, where the superscripts denote the fibres over ϕ in Δ^{op}. Let α be the unique active map $[1] \to [n]$ in Δ; then we can identify this as a map of homotopy fibres from the commutative square

$$\begin{tikzcd}
\mathcal{D}(\mathcal{C})^{\phi}(Z, Z') \arrow[r] \arrow[d] & \mathcal{D}(\mathcal{C})^{\alpha}(Z, (x, y)) \arrow[d] \\
(\mathcal{O}_{\mathcal{C}}^{\otimes})^{\phi}(g(Z), g(Z')) \arrow[r] & (\mathcal{O}_{\mathcal{C}}^{\otimes})^{\alpha}(g(Z), (x, y)),
\end{tikzcd}$$

where the superscripts again denote the fibres of these spaces over maps in Δ^{op}. To see that our map of homotopy fibres is an equivalence it suffices to show that this diagram is homotopy Cartesian.

We have equivalences

$$\begin{aligned}
\mathcal{D}(\mathcal{C})^{\phi}(Z,Z') &\simeq \prod_{i=0}^{m} \mathcal{C}(z_{\phi(i)}, z'_i),\\
\mathcal{D}(\mathcal{C})^{\alpha}(Z,(x,y)) &\simeq \mathcal{C}(z_0,x)\times\mathcal{C}(z_n,y),\\
(\mathcal{O}_{\mathcal{C}}^{\otimes})^{\phi}(g(Z),g(Z')) &\simeq \mathcal{C}(z'_0, z_{\phi(0)}) \times \mathcal{C}(z_{\phi(0)+1}, z_{\phi(0)+1}) \times \cdots \times \mathcal{C}(z_{\phi(1)-1}, z_{\phi(1)-1})\\
&\quad \times \mathcal{C}(z_{\phi(1)}, z'_1) \times \mathcal{C}(z'_1, z_{\phi(1)}) \times \cdots \times \mathcal{C}(z_{\phi(m)}, z'_m),\\
(\mathcal{O}_{\mathcal{C}}^{\otimes})^{\alpha}(g(Z),(x,y)) &\simeq \mathcal{C}(x,z_0)\times\mathcal{C}(z_1,z_1)\times\cdots\times\mathcal{C}(z_{n-1},z_{n-1})\times\mathcal{C}(z_n,y).
\end{aligned}$$

Under these equivalences our commutative square is the product of the squares

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d] \\
\mathcal{C}(z_j,z_j) \arrow[r] & \mathcal{C}(z_j,z_j)
\end{tikzcd}$$

for j not in the image of ϕ,

$$\begin{tikzcd}
\mathcal{C}(z_0,z'_0)\times\mathcal{C}(z_n,z'_m) \arrow[r] \arrow[d, "{(i,\mathrm{id})}"'] & \mathcal{C}(z_0,x)\times\mathcal{C}(z_n,y) \arrow[d, "{(i,\mathrm{id})}"] \\
\mathcal{C}(z'_0,z_0)\times\mathcal{C}(z_n,z'_m) \arrow[r] & \mathcal{C}(x,z_0)\times\mathcal{C}(z_n,y),
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C}(z_{\phi(i)},z'_i) \arrow[r] \arrow[d, "{(\mathrm{id},i)}"'] & * \arrow[d] \\
\mathcal{C}(z_{\phi(i)},z'_i)\times\mathcal{C}(z'_i,z_{\phi(i)}) \arrow[r] & \mathcal{C}(z_{\phi(i)},z_{\phi(i)})
\end{tikzcd}$$

for $i = 1,\dots,m-1$.

The first squares are obviously homotopy Cartesian, the second is homotopy Cartesian since the maps induced by the involution i are equivalences, and the last squares are homotopy Cartesian since $\mathcal{C}$ is a simplicial groupoid. □

Corollary 4.1.2.7. Let X be a space and $\mathcal{X}$ a fibrant simplicial groupoid such that the Kan complex $\mathrm{N}\mathcal{X}$ is equivalent to X. Then the composite map $\triangle_X^{\mathrm{op}} \simeq \mathrm{N}\mathcal{D}(\mathcal{X}) \to \mathrm{N}\mathcal{O}_{\mathcal{X}}^{\otimes}$ induces an equivalence of non-symmetric ∞-operads $L\triangle_X^{\mathrm{op}} \xrightarrow{\sim} \mathrm{N}\mathcal{O}_{\mathcal{X}}^{\otimes}$.

Corollary 4.1.2.8. $L(\triangle_{(-)}^{\mathrm{op}}) \colon \mathcal{S} \to \mathrm{Opd}_{\infty}^{\mathrm{O}}$ preserves products.

Proof. Given spaces X and Y, there exist fibrant simplicial groupoids $\mathcal{X}$ and $\mathcal{Y}$ such that

$\mathrm{N}\mathcal{X} \simeq X$ and $\mathrm{N}\mathcal{Y} \simeq Y$. Then by Corollary 4.1.2.7 we have a commutative diagram

$$\begin{tikzcd} L\mathbb{\Delta}^{\mathrm{op}}_{X\times Y} \arrow[r] \arrow[d] & L\mathbb{\Delta}^{\mathrm{op}}_{X} \times_{\mathbb{\Delta}^{\mathrm{op}}} L\mathbb{\Delta}^{\mathrm{op}}_{Y} \arrow[d] \\ \mathrm{N}\mathcal{O}^{\otimes}_{\mathcal{X}\times\mathcal{Y}} \arrow[r] & \mathrm{N}(\mathcal{O}^{\otimes}_{\mathcal{X}} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{O}^{\otimes}_{\mathcal{Y}}) \end{tikzcd}$$

where the vertical maps are equivalences. It is clear from the definition that $\mathcal{O}_{\mathcal{X}\times\mathcal{Y}} \simeq \mathcal{O}_{\mathcal{X}} \times \mathcal{O}_{\mathcal{Y}}$, so the natural map $\mathcal{O}^{\otimes}_{\mathcal{X}\times\mathcal{Y}} \to \mathcal{O}^{\otimes}_{\mathcal{X}} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{O}^{\otimes}_{\mathcal{Y}}$ is a weak equivalence of fibrant simplicial categories. By the 2-out-of-3 property the top horizontal map in the commutative square is therefore an equivalence of ∞-categories. ☐

4.1.3 The ∞-Category of Categorical Algebras

In this subsection we use the *algebra fibration*

$$\mathrm{Alg}^{\mathbb{O}}(\mathcal{V}^{\otimes}) \to \mathrm{Opd}^{\mathbb{O}}_{\infty}$$

from §3.2.8 to define an ∞-category of categorical algebras, and then show that this has various good properties.

Definition 4.1.3.1. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category. The ∞-category $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ is defined by the pullback square

$$\begin{tikzcd} \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathbb{O}}(\mathcal{V}^{\otimes}) \arrow[d] \\ \mathcal{S} \arrow[r, "L\mathbb{\Delta}^{\mathrm{op}}_{(-)}"'] & \mathrm{Opd}^{\mathbb{O}}_{\infty}, \end{tikzcd}$$

where the lower horizontal map sends a space X to the non-symmetric ∞-operad $L\mathbb{\Delta}^{\mathrm{op}}_{X}$ associated to the generalized non-symmetric ∞-operad $\mathbb{\Delta}^{\mathrm{op}}_{X}$. The objects of $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ are thus categorical algebras in $\mathcal{V}^{\otimes}$ and its 1-morphisms are $\mathcal{V}$-functors as defined above. We will refer to $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ as the *∞-category of categorical algebras*.

Remark 4.1.3.2. Since $\mathcal{V}^{\otimes}$ is a monoidal ∞-category, and so in particular a non-symmetric ∞-operad, we could equivalently have defined $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V})$ using the analogue of the algebra fibration over the base $\mathrm{Opd}^{\mathbb{O},\mathrm{gen}}_{\infty}$, since there is natural equivalence $\mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}_{X}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Alg}^{\mathbb{O}}_{L\mathbb{\Delta}^{\mathrm{op}}_{X}}(\mathcal{V}^{\otimes})$.

Our next goal is to prove that the ∞-category $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ is presentable if $\mathcal{V}^{\otimes}$ is presentably monoidal; to do this we first introduce the ∞-category of *graphs* in $\mathcal{V}$:

Definition 4.1.3.3. Let $\mathcal{V}^{\otimes}$ be a monoidal ∞-category. The ∞-category $\mathrm{Graph}^{\mathcal{V}}_{\infty}$ of *$\mathcal{V}$-graphs* is defined by the pullback

$$\begin{tikzcd}
\mathrm{Graph}_\infty^{\mathcal{V}} \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathbf{O}}_{\mathrm{triv}}(\mathcal{V}^\otimes) \arrow[d] \\
\mathcal{S} \arrow[r, "L\Delta^{\mathrm{op}}_{(-)}"'] & \mathrm{Opd}^{\mathbf{O}}_\infty.
\end{tikzcd}$$

Thus the fibre of $\mathrm{Graph}_\infty^{\mathcal{V}}$ at $X \in \mathcal{S}$ is $\mathrm{Fun}(X \times X, \mathcal{V})$.

Lemma 4.1.3.4. Suppose $\mathcal{V}$ is an accessible ∞-category. Then the ∞-category $\mathrm{Graph}_\infty^{\mathcal{V}}$ is accessible.

Proof. Let $\mathcal{F} \to \mathcal{S}$ be the Cartesian fibration associated to the functor $\mathcal{S} \to \mathrm{Cat}_\infty$ sending X to $\mathrm{Fun}(X, \mathcal{V})$. Then there is a pullback square

$$\begin{tikzcd}
\mathrm{Graph}_\infty^{\mathcal{V}} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathcal{S} \arrow[r, "\Delta"'] & \mathcal{S},
\end{tikzcd}$$

where the lower horizontal map is the diagonal functor, sending X to $X \times X$.

The ∞-category $\mathcal{F}$ is accessible, and the projection $\mathcal{F} \to \mathcal{S}$ is an accessible functor, by Theorem 2.1.11.1. Moreover, the functor Δ clearly preserves sifted colimits, and so is accesible. The pullback $\mathrm{Graph}_\infty^{\mathcal{V}}$ is therefore accesible and the projection $\mathrm{Graph}_\infty^{\mathcal{V}} \to \mathcal{S}$ is an accessible functor, by [Lur09a, Proposition 5.4.6.6]. $\square$

Proposition 4.1.3.5. Suppose $\mathcal{V}^\otimes$ is a monoidal ∞-category compatible with small colimits. Then $\mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ has all small colimits. Moreover, if $\mathcal{V}$ is presentable then so is $\mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$.

Proof. By Lemma 3.2.8.5, the fibration $\pi\colon \mathrm{Alg}^{\mathbf{O}}(\mathcal{V}^\otimes) \to \mathrm{Opd}^{\mathbf{O}}_\infty$ is both Cartesian and co-Cartesian, hence the same is true of its pullback $p\colon \mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes) \to \mathcal{S}$. Moreover, the fibres $\mathrm{Alg}^{\mathbf{O}}_{\Delta^{\mathrm{op}}_X}(\mathcal{V}^\otimes)$ have all colimits by Corollary 3.3.5.5 and the functors $f_!$ induced by morphisms f in $\mathcal{S}$ preserve colimits, being left adjoints. Thus p satisfies the conditions of Lemma 2.1.5.10, which implies that $\mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ has small colimits.

Since the functor $\tau^*\colon \mathrm{Alg}^{\mathbf{O}}(\mathcal{V}^\otimes) \to \mathrm{Alg}^{\mathbf{O}}_{\mathrm{triv}}(\mathcal{V}^\otimes)$ preserves filtered colimits by Corollary 3.2.8.10, it is clear that so does its pullback $U\colon \mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes) \to \mathrm{Graph}_\infty^{\mathcal{V}}$. Moreover, the pullback of the left adjoint $\tau_!$ of τ^* gives a functor $F\colon \mathrm{Graph}_\infty^{\mathcal{V}} \to \mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ left adjoint to U; this preserves compact objects by Lemma 2.1.7.11.

Every object of $\mathrm{Alg}^{\mathbf{O}}(\mathcal{V}^\otimes)$ is a (sifted) colimit of objects in the image of $\tau_!\colon \mathrm{Alg}^{\mathbf{O}}_{\mathrm{triv}}(\mathcal{V}^\otimes) \to \mathrm{Alg}^{\mathbf{O}}(\mathcal{V}^\otimes)$, hence every object of $\mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ is a (sifted) colimit of objects in the image of F. The ∞-category $\mathrm{Graph}_\infty^{\mathcal{V}}$ is accessible by Lemma 4.1.3.4; suppose it is generated under colimits by κ-compact objects. Since F preserves colimits it follows that every object of $\mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ is the colimit of objects that are the images of κ-compact objects of $\mathrm{Graph}_\infty^{\mathcal{V}}$ under F. As the functor F preserves κ-compact objects, this means there is a small subcategory of κ-compact objects of $\mathrm{Alg}^{\mathbf{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ — namely the images of κ-compact objects of

$\mathrm{Graph}_\infty^{\mathcal{V}}$ — such that every object of $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ is a colimit of objects in this ∞-category. In other words, the ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ is κ-accessible. $\square$

Now we show that $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ is functorial in $\mathcal{V}^\otimes$:

Definition 4.1.3.6. As in §3.2.8, let $\mathrm{Alg}_{\mathrm{co}}^{\mathcal{O}} \to \mathrm{Opd}_\infty^{\mathcal{O}} \times (\widehat{\mathrm{Opd}}_\infty^{\mathcal{O}})^{\mathrm{op}}$ be a Cartesian fibration classifying the functor $\mathrm{Alg}_{(-)}^{\mathcal{O}}(-)$. Let $\mathrm{Alg}_{\mathrm{cat,co}}^{\mathcal{O}}$ be the pullback

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{cat,co}}^{\mathcal{O}} \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathrm{co}}^{\mathcal{O}} \arrow[d] \\
\mathcal{S} \times (\widehat{\mathrm{Mon}}_\infty^{\mathcal{O},\mathrm{lax}})^{\mathrm{op}} \arrow[r] & \mathrm{Opd}_\infty^{\mathcal{O}} \times (\widehat{\mathrm{Opd}}_\infty^{\mathcal{O}})^{\mathrm{op}}.
\end{tikzcd}$$

Lemma 4.1.3.7. $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ is functorial in $\mathcal{V}^\otimes$ with respect to lax monoidal functors.

Proof. The composite $\mathrm{Alg}_{\mathrm{cat,co}}^{\mathcal{O}} \to (\widehat{\mathrm{Mon}}_\infty^{\mathcal{O},\mathrm{lax}})^{\mathrm{op}}$ is a Cartesian fibration classifying a functor $\mathcal{O}^\otimes \mapsto \mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{O}^\otimes)$. $\square$

Proposition 4.1.3.8. The restriction of $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(-)$ to the ∞-category $\mathrm{Mon}_\infty^{\mathcal{O},\mathrm{Pr}}$ of presentably monoidal ∞-categories factors through the ∞-category Pr^{L} of presentable ∞-categories and colimit-preserving functors.

Proof. If $\mathcal{V}^\otimes$ is presentably monoidal, then $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ is presentable by Proposition 4.1.3.5. Moreover, it follows by the same proof as that of Proposition 3.3.7.1 that a strong monoidal functor $F^\otimes \colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ such that $F_{[1]}^\otimes$ preserves colimits induces a colimit-preserving functor $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes) \to \mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{W}^\otimes)$. $\square$

Proposition 4.1.3.9. $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(-)$ is lax monoidal with respect to the Cartesian product of monoidal ∞-categories.

Proof. The functor $L\Delta_{(-)}^{\mathrm{op}}$ is strong monoidal with respect to the Cartesian products of spaces and non-symmetric ∞-operads, by Corollary 4.1.2.8. The result therefore follows by the same proof as that of Proposition 3.2.8.13. $\square$

Proposition 4.1.3.10. Suppose $\mathcal{V}$ is an ∞-category with finite products. Then the natural symmetric monoidal structure on $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\times)$ is Cartesian.

Proof. This follows from Proposition 3.2.8.15, since $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\times)$ is a full monoidal subcategory of $\mathrm{Alg}^{\mathcal{O}}(\mathcal{V}^\times)$. $\square$

Proposition 4.1.3.11. Let $\mathcal{V}^\otimes$ be a monoidal ∞-category, and suppose that $\mathcal{C}$ is a categorical algebra in $\mathcal{V}^\otimes$. Then $\mathcal{C} \boxtimes -\colon \mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{W}^\otimes) \to \mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes \times_{\Delta^{\mathrm{op}}} \mathcal{W}^\otimes)$ preserves colimits.

Proof. Since the Cartesian product of spaces preserves colimits in each variable, it suffices to prove that $\mathcal{C} \boxtimes (-)$ preserves colimits fibrewise, and preserves coCartesian arrows. This follows from Lemma 3.3.7.2 and Proposition 3.3.7.3. $\square$

Remark 4.1.3.12. This is where we need to know that $L\Delta^{\mathrm{op}}_{(-)}$ preserves products, since the Cartesian product of non-symmetric ∞-operads doesn't preserve colimits in each variable.

Corollary 4.1.3.13. The functor $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(-)\colon \mathrm{Mon}^{\mathbb{O},\mathrm{Pr}}_{\infty} \to \mathrm{Pr}^{\mathrm{L}}$ is lax monoidal with respect to the tensor product of presentable ∞-categories.

Proof. We have constructed a lax monoidal functor $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(-)\colon (\widehat{\mathrm{Mon}}^{\mathbb{O},\mathrm{lax}}_{\infty})^{\times} \to \widehat{\mathrm{Cat}}^{\times}_{\infty}$. By Proposition 4.1.3.11 and Propositon 4.1.3.8, the composite $(\mathrm{Mon}^{\mathbb{O},\mathrm{Pr}}_{\infty})^{\otimes} \to (\widehat{\mathrm{Mon}}^{\mathbb{O},\mathrm{lax}}_{\infty})^{\times} \to \widehat{\mathrm{Cat}}^{\times}_{\infty}$ factors through $(\mathrm{Pr}^{\mathrm{L}})^{\otimes}$. □

Corollary 4.1.3.14. If $\mathcal{V}^{\otimes}$ is presentably monoidal, then $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ is tensored and cotensored over $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{S}^{\times})$.

Definition 4.1.3.15. If $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category, $\mathcal{C}$ is a $\mathcal{V}$-∞-category, and $\mathcal{X}$ is an $\mathcal{S}$-∞-category, then we denote their tensor and cotensor by $\mathcal{X} \otimes \mathcal{C}$ and $\mathcal{C}^{\mathcal{X}}$, respectively.

4.1.4 Categorical Algebras in Spaces

In this subsection we prove that the ∞-category $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{S}^{\times})$ of categorical algebras in spaces is equivalent to the ∞-category $\mathrm{Seg}^{\mathbb{O}}_{\infty}$ of Segal spaces.

Definition 4.1.4.1. Suppose $\mathcal{V}$ is an ∞-category with finite products. The Cartesian fibration $\mathrm{Mnd}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$ is defined by pulling back $\mathrm{Mnd}^{\mathbb{O}}(\mathcal{V}) \to \mathrm{Opd}^{\mathbb{O}}_{\infty}$ along $L\Delta^{\mathrm{op}}_{(-)}$.

Remark 4.1.4.2. There is a natural equivalence over $\mathcal{S}$ between $\mathrm{Mnd}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V})$ and $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\times})$.

We can also define a Cartesian fibration $\mathrm{Mon}^{\mathbb{O},\mathrm{cat}}_{\infty} \to \mathcal{S}$ whose fibre at X is the ∞-category $\mathrm{Mon}^{\mathbb{O},\Delta^{\mathrm{op}}_X}_{\infty}$ of Δ^{op}_X-monoidal ∞-categories. Using the equivalence between functors to $\mathcal{S}$ and left fibrations, we can identify $\mathrm{Mnd}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{S})$ with the full subcategory $\mathrm{LMon}^{\mathbb{O},\mathrm{cat}}_{\infty}$ of $\mathrm{Mon}^{\mathbb{O},\mathrm{cat}}_{\infty}$ spanned by those Δ^{op}_X-monoidal ∞-categories that are left fibrations.

Similarly, we can identify the ∞-category $\mathrm{Seg}^{\mathbb{O}}_{\infty}$ of Segal spaces with the full subcategory LDbl_{∞} of Dbl_{∞} spanned by the double ∞-categories that are left fibrations.

There is an obvious functor

$$p\colon \mathrm{LMon}^{\mathbb{O},\mathrm{cat}}_{\infty} \to \mathrm{LDbl}_{\infty}$$

given by composing a Δ^{op}_X-monoidal ∞-category $\mathcal{C} \to \Delta^{\mathrm{op}}_X$ that is a left fibration with the map $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$, which is also a left fibration and a double ∞-category.

Proposition 4.1.4.3. This functor $p\colon \mathrm{LMon}^{\mathbb{O},\mathrm{cat}}_{\infty} \to \mathrm{LDbl}_{\infty}$ is an equivalence.

Proof. Let i denote the inclusion $\{[0]\} \hookrightarrow \Delta^{\mathrm{op}}$. Then there is an adjunction

$$i^{*}\colon \mathrm{Seg}^{\mathbb{O}}_{\infty} \rightleftarrows \mathcal{S}\colon i_{*},$$

and Δ^{op}_X is the object of LDbl_{∞} corresponding to $i_{*}X$. Moreover, i^{*} is a Cartesian fibration by Lemma 2.1.6.4; if $A \in \mathrm{Seg}^{\mathbb{O}}_{\infty}$, a Cartesian arrow with target A over $X \to i^{*}A$ is given by taking the pullback of $A \to i_{*}i^{*}A$ along $i_{*}X \to i_{*}i^{*}A$.

To prove that p is an equivalence, we must show that it is fully faithful and essentially surjective. We thus have to prove that the map

$$\mathrm{Map}_{\mathrm{LMon}_{\infty}^{\mathcal{O},\mathrm{cat}}}(A,B) \to \mathrm{Map}_{\mathrm{LDbl}_{\infty}}(p(A),p(B))$$

is an equivalence. Since the functor p clearly preserves Cartesian morphisms over $\mathcal{S}$, it suffices to show that the induced maps on fibres over $f\colon i^*p(A) \to i^*p(B)$ are equivalences. But this is clear: on both sides the fibre at f can be identified with the space of those maps over $\Delta^{\mathrm{op}}_{i^*A}$ from A to the pullback of B along Δ^{op}_{f} that preserve inert morphisms.

It remains to prove that p is essentially surjective. Suppose $\alpha\colon \mathcal{A} \to \Delta^{\mathrm{op}}$ is an object of LDbl_{∞}. The adjunction $i^* \dashv i_*$ induces a map $h\colon \mathcal{A} \to \Delta^{\mathrm{op}}_{\mathcal{A}_{[0]}}$; this is equivalent to a left fibration by Proposition 2.1.4.4 and so α is in the essential image of p. □

Corollary 4.1.4.4. The composite functor $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{S}^{\times}) \to \mathrm{Seg}^{\mathcal{O}}_{\infty}$ is an equivalence.

Remark 4.1.4.5. It is easy to see that the Segal space corresponding to an $\mathcal{S}$-∞-category $\mathcal{C}$ is $\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{S}^{\times})}([\bullet],\mathcal{C})$, cf. the more general discussion in §4.5.1.

4.2 The ∞-Category of Enriched ∞-Categories

Our goal in this section is to prove the first main result of this thesis: we can always localize the ∞-category of categorical algebras at the fully faithful and essentially surjective functors by restricting to the full subcategory of *complete* objects.

In §4.2.1 we define equivalences in enriched ∞-categories and study the classifying space for equivalences in an enriched ∞-category; the *complete* enriched ∞-categories are those whose classifying space of equivalences is equivalent to their underlying space of objects. Next we study three types of equivalences of $\mathcal{V}$-∞-categories: in §4.2.2 we introduce *fully faithful* and *essentially surjective* functors, in §4.2.3 we consider the *local equivalences* (those in the saturated class of a certain map) and finally in §4.2.4 we introduce *categorical equivalences* (those with an inverse up to natural equivalence). In §4.2.5 we prove that for ∞-categories enriched in a presentably monoidal ∞-category the fully faithful and essentially surjective functors are the same as the local equivalences, hence the full subcategory of complete objects gives the localization. In §4.2.6 we extend this result to ∞-categories enriched in a general large monoidal ∞-category by embedding this in a presentable ∞-category in a larger universe. Finally in §4.2.7 we prove that the localized ∞-category inherits the good functoriality properties of $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$.

4.2.1 Equivalences in Enriched ∞-Categories

In this subsection we study equivalences in enriched ∞-categories. In order to define these we must first introduce *trivial* enriched ∞-categories:

Definition 4.2.1.1. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category. By Proposition 3.3.5.9 $\mathcal{V}^{\otimes}$ has a unit, i.e. an initial associative algebra object $I_{\mathcal{V}}\colon \Delta^{\mathrm{op}} \to \mathcal{V}^{\otimes}$. For any space X, the *trivial $\mathcal{V}$-∞-category $E^{\mathcal{V}}_{X}$* with objects X is given by the composite

$$\Delta^{\mathrm{op}}_{X} \to \Delta^{\mathrm{op}} \xrightarrow{I_{\mathcal{V}}} \mathcal{V}^{\otimes}.$$

We will generally drop the $\mathcal{V}$ from the notation and just write E_X when the monoidal ∞-category in question is obvious from the context. The $\mathcal{V}$-∞-categories E_X are functorial in X. We abbreviate $E^n := E_{\{0,\ldots,n\}}$; restricting to order-preserving maps between the sets $\{0,\ldots,n\}$ $(n = 0,1,\ldots)$ we then have a cosimplicial $\mathcal{V}$-∞-category $E^\bullet$.

The identity map $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is the unique monoidal structure on the point $*$. This is the unit for the Cartesian product of monoidal ∞-categories, and so for every monoidal ∞-category $\mathcal{V}^\otimes$ the ∞-category $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ is tensored over $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\Delta^{\mathrm{op}})$, since $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(-)$ is lax monoidal by Proposition 4.1.3.9. Clearly the only $*$-∞-categories are of the form E^*_X for spaces X; we can identify the $\mathcal{V}$-∞-category $E^{\mathcal{V}}_X$ with the tensor $E^*_X \otimes I_{\mathcal{V}}$:

Lemma 4.2.1.2. For any monoidal ∞-category $\mathcal{V}^\otimes$ and space X, we have $E^{\mathcal{V}}_X \simeq E^*_X \otimes I_{\mathcal{V}}$. Moreover, if $\mathcal{V}^\otimes$ is presentably monoidal (so $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ is tensored over $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{S}^\times)$), then $E^{\mathcal{V}}_X \simeq E^{\mathcal{S}}_X \otimes I_{\mathcal{V}}$.

Proof. Considering the construction of the external product in $\mathrm{Alg}^{\mathbb{O}}$, we see that $E^*_X \otimes I_{\mathcal{V}}$ is given by

$$E^*_X \times_{\Delta^{\mathrm{op}}} I_{\mathcal{V}} \colon \Delta^{\mathrm{op}}_X \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times_{\Delta^{\mathrm{op}}} \mathcal{V}^\otimes \simeq \mathcal{V}^\otimes.$$

We can factor this as

$$\Delta^{\mathrm{op}}_X \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{E^*_X \times_{\Delta^{\mathrm{op}}} \mathrm{id}} \Delta^{\mathrm{op}} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{\mathrm{id} \times_{\Delta^{\mathrm{op}}} I_{\mathcal{V}}} \Delta^{\mathrm{op}} \times_{\Delta^{\mathrm{op}}} \mathcal{V}^\otimes,$$

which is clearly the same as $E^{\mathcal{V}}_X$.

In the presentable case, we have

$$E^{\mathcal{S}}_X \otimes I_{\mathcal{V}} \simeq (E^*_X \otimes I_{\mathcal{S}}) \otimes I_{\mathcal{V}} \simeq E^*_X \otimes (I_{\mathcal{S}} \otimes I_{\mathcal{V}}) \simeq E^*_X \otimes I_{\mathcal{V}} \simeq E^{\mathcal{V}}_X,$$

since it is easy to see that the tensorings with $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\Delta^{\mathrm{op}})$ and $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{S}^\times)$ are compatible. □

Definition 4.2.1.3. Suppose $\mathcal{C}$ is a $\mathcal{V}$-∞-category. An *equivalence* in $\mathcal{C}$ is a $\mathcal{V}$-functor $E^1 \to \mathcal{C}$.

Definition 4.2.1.4. Let $\mathcal{C}$ be a $\mathcal{V}$-∞-category. We write $\iota_n\mathcal{C} := \mathrm{Map}(E^n, \mathcal{C})$. Thus $\iota_1\mathcal{C} := \mathrm{Map}(E^1, \mathcal{C})$ is the *space of equivalences* in $\mathcal{C}$.

Lemma 4.2.1.5. Let $\mathcal{C} \colon \Delta^{\mathrm{op}}_X \to \mathcal{V}^\otimes$ be a $\mathcal{V}$-∞-category. Then the map

$$\iota_0\mathcal{C} := \mathrm{Map}_{\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)}(E^0, \mathcal{C}) \to \mathrm{Map}_{\mathcal{S}}(*, X) \simeq X$$

induced by the Cartesian fibration $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes) \to \mathcal{S}$ is an equivalence.

Proof. It suffices to check that the homotopy fibres of this map are contractible. By [Lur09a, Proposition 2.4.4.2] the homotopy fibre at a point $p \colon * \to X$ is

$$\mathrm{Map}_{\mathrm{Alg}^{\mathbb{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^\otimes)}(I_{\mathcal{V}}, p^*\mathcal{C}),$$

which is contractible since the unit $I_{\mathcal{V}}$ is the initial associative algebra object of $\mathcal{V}$. □

Definition 4.2.1.6. Let $\mathcal{C}$ be a $\mathcal{V}$-∞-category. The *classifying space of equivalences* $\iota\mathcal{C}$ of $\mathcal{C}$ is the geometric realization $|\iota_\bullet\mathcal{C}|$ of the simplicial space $\iota_\bullet\mathcal{C} := \mathrm{Map}(E^\bullet, \mathcal{C})$.

We regard $\iota\mathcal{C}$ as the "correct" space of objects of $\mathcal{C}$, and by analogy with Rezk's notion of complete Segal space we say that an enriched ∞-category is *complete* if its underlying space is the correct one:

Definition 4.2.1.7. A $\mathcal{V}$-∞-category $\mathcal{C}$ is *complete* if the natural map $\iota_0\mathcal{C} \to \iota\mathcal{C}$ is an equivalence.

Our next goal is to prove that the simplicial space $\iota_\bullet\mathcal{C}$ is always a groupoid object; we prove this by showing that the cosimplicial object $E^\bullet$ satisfies the dual condition of being a *cogroupoid* object:

Theorem 4.2.1.8. Let $\mathcal{V}^\otimes$ be a presentably monoidal ∞-category. Then the cosimplicial object $E^\bullet$ is a cogroupoid object.

Proof. We will show that $E^N \amalg_{E_{\{N\}}} E_{\{N,N+1\}} \to E^{N+1}$ is an equivalence; as the ordering of the objects is arbitrary, by induction this will imply that $E^\bullet$ is a cogroupoid object. Since $\mathcal{V}^\otimes$ is presentably monoidal, $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)$ is tensored over $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{S}^\times)$, and the tensoring is colimit-preserving in each variable; if therefore suffices to prove this when $\mathcal{V}^\otimes$ is $\mathcal{S}^\times$.

Under the equivalence $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{S}^\times) \simeq \mathrm{Seg}^{\mathcal{O}}_\infty$, the $\mathcal{S}$-∞-category E^X clearly corresponds to the Segal space i_*X. If S is a set it follows that in the model category structure on bisimplicial sets modelling Segal spaces, E^S corresponds to $\pi^*N\mathcal{I}_S$ where $\mathcal{I}_S$ is the ordinary category with objects S and a unique morphism between any pair of objects, and $\pi\colon \mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$ is the projection onto the first factor.

Define $G_N := \mathrm{N}\mathcal{I}_{\{0,\ldots,N\}}$. By [Rez01, Remark 10.2], for $0 < i < n$ the map $\pi^*\Lambda^n_k \to \pi^*\Delta^n$ is a Segal equivalence, so (since π^* is a left adjoint and thus preserves colimits) it suffices to prove that $G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} \hookrightarrow G_{N+1}$ is an inner anodyne morphism of simplicial sets. To prove this we consider a series of nested filtrations of the simplices of G_{N+1}. First we must introduce some notation:

An n-simplex σ of G_{N+1} can be described by a list $a_0 \cdots a_n$ of elements $a_i \in \{0,\ldots,N+1\}$; it is non-degenerate if $a_i \neq a_{i+1}$ for all i. If σ is a non-degenerate simplex, let $\beta(\sigma)$ be the number of times the sequence jumps between $\{0,\ldots,N\}$ and $\{N,N+1\}$.

Also let $\tau(\sigma)$ be the position of the first $N+1$ where the sequence jumps from $\{N,N+1\}$ to $\{0,\ldots,N\}$; if there is no such jump let $\tau(\sigma) = \infty$ and let $\tau'(\sigma)$ denote the position of the first jump from $\{0,\ldots,N\}$ to $\{N,N+1\}$. Then define

- If $t \neq \infty$, let $S^{b,t}_n$ be the set of non-degenerate n-simplices σ in G_{N+1} such that $\beta(\sigma) = b$, $\tau(\sigma) = t$, and $a_{t+1} \neq N$. Let $S^{1,\infty,t}_n$ be the set of non-degenerate n-simplices in G_{N+1} such that $\beta(\sigma) = 1$, $\tau(\sigma) = \infty$, $\tau'(\sigma) = t$, and $a_{t-1} \neq N$.
- If $t \neq \infty$, let $T^{b,t}_n$ be the set of non-degenerate $(n+1)$-simplices σ in G_{N+1} such that $\beta(\sigma) = b$, $\tau(\sigma) = t$ and $a_{t+1} = N$. Let $T^{1,\infty,t}_n$ be the set of non-degenerate $(n+1)$-simplices σ in G_{N+1} such that $\beta(\sigma) = 1$, $\tau(\sigma) = \infty$, $\tau'(\sigma) = t+1$, and $a_t = N$.

Define a filtration
$$G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} =: \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq G_{N+1}$$
by letting $\mathcal{F}_n$ be the subspace of G_{N+1} whose non-degenerate simplices are those of $\mathcal{F}_0$ together with all the non-degenerate i-simplices for $i \leq n$ and the $(n+1)$-simplices in $T^{b,t}_n$ and $T^{1,\infty,t}_n$ for all b,t. Then $G_{N+1} = \bigcup_n \mathcal{F}_n$ so to prove that $G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} \hookrightarrow G_{N+1}$ is inner anodyne it suffices to prove that the inclusions $\mathcal{F}_{n-1} \hookrightarrow \mathcal{F}_n$ are inner anodyne.

Next define a filtration

$$\mathcal{F}_{n-1} = \mathcal{F}_n^0 \subseteq \mathcal{F}_n^1 \subseteq \cdots \subseteq \mathcal{F}_n^{n-1} = \mathcal{F}_n$$

by setting $\mathcal{F}_n^b$ to be the subspace of $\mathcal{F}_n$ containing $\mathcal{F}_{n-1}$ together with the simplices in $S_n^{i,t}$ and $T_n^{i,t}$ for all $i \leq b$ together with $S_n^{1,\infty,t}$ and $T_n^{1,\infty,t}$ for all t. To prove that the inclusions $\mathcal{F}_{n-1} \hookrightarrow \mathcal{F}_n$ are inner anodyne it suffces to prove that the inclusions $\mathcal{F}_n^{b-1} \hookrightarrow \mathcal{F}_n^b$ are all inner anodyne.

Finally define a filtration

$$\mathcal{F}_n^{b-1} = \mathcal{F}_n^{b,n+1} \subseteq \mathcal{F}_n^{b,n} \subseteq \cdots \subseteq \mathcal{F}_n^{b,0} = \mathcal{F}_n^b,$$

by setting $\mathcal{F}_n^{b,t}$ to be the subspace of $\mathcal{F}_n^b$ containing $\mathcal{F}_n^{b-1}$ together with the simplices in $S_n^{b,j}$ and $T_n^{b,j}$ (as well as $S_n^{1,\infty,j}$ and $T_n^{1,\infty,j}$ if $b = 1$) for all $j \geq t$. To prove that the inclusions $\mathcal{F}_n^{b-1} \hookrightarrow \mathcal{F}_n^b$ are inner anodyne it suffices to prove that the inclusions $\mathcal{F}_n^{b,t-1} \hookrightarrow \mathcal{F}_n^{b,t}$ are all inner anodyne.

Now observe that (for $b > 1$) if $\sigma \in T_n^{b,t}$ then $d_t\sigma \in S_n^{b,t}$ and $d_i\sigma \in \mathcal{F}_n^{b,t-1}$ for $i \neq t$, and σ is uniquely determined by $d_t\sigma$. Thus we get a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in T_n^{b,t}} \Lambda_t^{n+1} \arrow[r] \arrow[d] & \coprod_{\sigma \in T_n^{b,t}} \Delta^{n+1} \arrow[d] \\
\mathcal{F}_n^{b,t-1} \arrow[r] & \mathcal{F}_n^{b,t}
\end{tikzcd}$$

where we always have $0 < t < n+1$. Thus the bottom horizontal map is inner anodyne. The proof is similar when $b = 1$, expect that we must also consider the simplices in $S_n^{1,\infty,t}$, so we conclude that $G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} \to G_{N+1}$ is indeed inner anodyne. $\square$

Remark 4.2.1.9. We can generalize this to the case of an arbitrary large monoidal ∞-category $\mathcal{V}^{\otimes}$ as follows: by [Lur11, Remark 6.3.1.8] there exists a presentably monoidal structure on the (very large) presentable ∞-category $\widehat{\mathcal{P}}(\mathcal{V})$ of presheaves of large spaces on $\mathcal{V}$, such that the Yoneda embedding $\mathcal{V} \to \widehat{\mathcal{P}}(\mathcal{V})$ is strong monoidal. This induces a fully faithful embedding $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes}) \to \widehat{\mathrm{Alg}}_{\mathrm{cat}}^{\mathcal{O}}(\widehat{\mathcal{P}}(\mathcal{V})^{\otimes})$; moreover, if X a small space then $E_X^{\widehat{\mathcal{P}}(\mathcal{V})}$ is clearly the image of $E_X^{\mathcal{V}}$. Thus if a diagram of $E_X^{\mathcal{V}}$'s is a colimit diagram in $\widehat{\mathrm{Alg}}_{\mathrm{cat}}^{\mathcal{O}}(\widehat{\mathcal{P}}(\mathcal{V})^{\otimes})$ it must also be a colimit diagram in $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$ — in particular $E_{\mathcal{V}}^{\bullet}$ is a cogroupoid object in $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$.

Corollary 4.2.1.10. The simplicial space $\iota_{\bullet}\mathcal{C}$ is a groupoid object in spaces for all $\mathcal{V}$-∞-categories $\mathcal{C}$.

Corollary 4.2.1.11. Let $\mathcal{C}$ be a $\mathcal{V}$-∞-category. The following are equivalent:

(i) $\mathcal{C}$ is complete.

(ii) The natural map $s_0 \colon \iota_0\mathcal{C} \to \iota_1\mathcal{C}$ is an equivalence.

(iii) The simplicial space $\iota_{\bullet}\mathcal{C}$ is constant (i.e. for every map $\phi \colon [n] \to [m]$ in Δ^{op} the induced map $\iota_n\mathcal{C} \to \iota_m\mathcal{C}$ is an equivalence).

Proof. Since $\mathcal{S}$ is an ∞-topos, the groupoid object $\iota_\bullet\mathcal{C}$ is effective (cf. [Lur09a, Corollary 6.1.3.20]). The result therefore follows from Lemma 2.1.10.4. $\square$

We end this subsection by showing that several other reasonable definitions of an equivalence in an enriched ∞-category are equivalent to the one we introduced above:

Definition 4.2.1.12. Suppose $\mathcal{V}$ is presentably monoidal, and let $[1]_{\mathcal{V}}$ be the $\mathcal{V}$-∞-category $[1] \otimes I_{\mathcal{V}}$.

The inclusion $[1]_{\mathcal{V}} \to E^1_{\mathcal{V}}$ corresponding to the map from 0 to 1 induces a map $\iota_1\mathcal{C} \to \mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})$. The two inclusions of $E^0_{\mathcal{V}}$ into $[1]_{\mathcal{V}}$ and $E^1_{\mathcal{V}}$ then give a commutative triangle

$$\begin{tikzcd} \iota_1\mathcal{C} \arrow[rr] \arrow[dr] & & \mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C}) \arrow[dl] \\ & \iota_0\mathcal{C} \times \iota_0\mathcal{C}. & \end{tikzcd}$$

Lemma 4.2.1.13. The fibre of $\mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})_{x,y}$ at points $x, y \in \iota_0\mathcal{C}$ is $\mathrm{Map}(I, \mathcal{C}(x,y))$.

Proof. The functor $t\colon \mathcal{S} \to \mathcal{V}$ given by tensoring with I is a strong monoidal colimit-preserving functor, and therefore the induced map $t_*\colon \mathrm{Alg}_{\Delta^{\mathrm{op}}_{\{0,1\}}}(\mathcal{S}^\times) \to \mathrm{Alg}_{\Delta^{\mathrm{op}}_{\{0,1\}}}(\mathcal{V}^\otimes)$ has by Proposition 3.3.5.7 a right adjoint, given by u_* where $u\colon \mathcal{V}^\otimes \to \mathcal{S}^\times$ is a canonical lax monoidal structure on the functor $\mathrm{Map}(I, –)$.

Thus $\mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})_{x,y} \simeq \mathrm{Map}([1]_{\mathcal{S}}, u_*\mathcal{C})_{x,y}$. Since $[1]_{\mathcal{S}}$ is the free $\mathcal{S}$-∞-category on the graph having a single edge from 0 to 1 this is given by $u_*\mathcal{C}(x,y) \simeq \mathrm{Map}(I, \mathcal{C}(x,y))$. $\square$

Definition 4.2.1.14. Suppose $\mathcal{C}$ is a $\mathcal{V}$-∞-category and x, y are objects of $\mathcal{C}$. We denote the subspace of $\mathrm{Map}(I, \mathcal{C}(x,y))$ consisting of the components that are in the image of $\iota_1\mathcal{C}_{x,y}$ under the induced map on fibres in the diagram above by $\mathrm{Map}(I, \mathcal{C}(x,y))_{\mathrm{eq}}$.

Proposition 4.2.1.15. The map $\iota_1\mathcal{C}_{x,y} \to \mathrm{Map}(I, \mathcal{C}(x,y))_{\mathrm{eq}}$ is an equivalence.

Proof. Observe (by Proposition 3.3.5.7 again) that it suffices to prove this for the $\mathcal{S}$-∞-category $u_*\mathcal{C}$ obtained by composing with the lax monoidal functor $u \simeq \mathrm{Map}(I, –)$. Using the identification of $\mathcal{S}$-∞-categories with Segal spaces, this therefore follows from the corresponding statement in that setting. The latter is a consequence of [Rez01, Theorem 6.2], since a map $I \to \mathcal{C}(x,y)$ is a "homotopy equivalence" in the sense of [Rez01, §5.5] if and only if it extends to a map from $E^1_{\mathcal{S}}$, by [Rez01, Proposition 11.1]) $\square$

Proposition 4.2.1.16. Suppose $\mathcal{C}$ is a $\mathcal{V}$-∞-category and $\alpha\colon I \to \mathcal{C}(x,y)$ is a morphism in $\mathcal{C}$. Then the following are equivalent:

(i) α is an equivalence (i.e. it extends to a functor $E^1 \to \mathcal{C}$).

(ii) For all $z \in \iota_0\mathcal{C}$, the composite map

$$\mathcal{C}(y,z) \to (I, \mathcal{C}(y,z)) \to (\mathcal{C}(x,y), \mathcal{C}(y,z)) \to \mathcal{C}(x,z)$$

given by composing with α is an equivalence.

(iii) For all $z \in \iota_0\mathcal{C}$, the composite map

$$\mathcal{C}(z,x) \to (\mathcal{C}(z,x), I) \to (\mathcal{C}(z,x), \mathcal{C}(x,y)) \to \mathcal{C}(x,y)$$

given by composing with α is an equivalence.

Proof. We will show that (i) is equivalent to (ii); the proof that (i) is equivalent to (iii) is similar.

Suppose (i) holds, and let $\hat{\alpha}\colon E^1 \to \mathcal{C}$ be an equivalence extending α. Composing with the inverse equivalence from y to x gives an inverse to composition with α, since the composite map is composing with the composite $x \to y \to x$, which is the identity.

Now suppose (ii) holds. Without loss of generality, we may assume that $\mathcal{V}^{\otimes}$ is presentably monoidal (by embedding in a presentably monoidal ∞-category of presheaves in a larger universe, if necessary); then a map $E^1_{\mathcal{V}} \simeq t_* E^1_{\mathcal{S}} \to \mathcal{C}$ is adjoint to a map $E^1_{\mathcal{S}} \to u_*\mathcal{C}$ where $u\colon \mathcal{V} \to \mathcal{S}$ is again the lax monoidal functor given by $\mathrm{Map}(I, –)$. Clearly if (ii) holds for α then the analogous condition holds for α considered as a morphism in $u_*\mathcal{C}$. It thus suffices to show that (ii) implies (i) in the case where $\mathcal{V}$ is $\mathcal{S}$. We again use the equivalence between $\mathcal{S}$-∞-categories and Segal spaces; the map α is clearly a "homotopy equivalence" in the sense of [Rez01, §5.5], and so extends to a map from E^1 by [Rez01, Theorem 6.2]. □

4.2.2 Fully Faithful and Essentially Surjective Functors

In this subsection we introduce the notions of *fully faithful* and *essentially surjective* functors between enriched ∞-categories, and prove their basic properties.

Definition 4.2.2.1. A $\mathcal{V}$-functor is *fully faithful* if it is a Cartesian morphism in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V})$ with respect to the projection $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$.

Lemma 4.2.2.2. A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if the maps $\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ are equivalences in $\mathcal{V}$ for all x, y in $\iota_0\mathcal{C}$.

Proof. If $f\colon X \to \iota_0\mathcal{D}$ is a map of spaces, then a Cartesian morphism over f with target $\mathcal{D}$ has source $f^*\mathcal{D} = \mathcal{D} \circ \Delta^{\mathrm{op}}_f$; in particular a Cartesian morphism induces equivalences $f^*\mathcal{D}(x,y) \to \mathcal{D}(f(x), f(y))$ for all $x, y \in X$.

Conversely, suppose $F\colon \mathcal{C} \to \mathcal{D}$ gives an equivalence on all mapping spaces. The functor F factors as

$$\mathcal{C} \xrightarrow{F'} (\iota_0 F)^*\mathcal{D} \xrightarrow{F''} \mathcal{D},$$

where F'' is Cartesian. The morphism F' induces an equivalence on underlying spaces and is given by equivalences $\mathcal{C}(x,y) \to \mathcal{D}(\iota_0F(x), \iota_0F(y))$ for all $x, y \in \iota_0\mathcal{C}$. By Lemma 3.3.5.3 it follows that F' is an equivalence in $\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}}(\mathcal{V}^{\otimes})$ and so in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$. In particular F' is a Cartesian morphism and hence so is the composite $F \simeq F'' \circ F'$. □

Definition 4.2.2.3. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if the induced map $\iota\mathcal{C} \to \iota\mathcal{D}$ is surjective on π_0.

Lemma 4.2.2.4. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is essentially surjective if and only if for every point $x \in \iota_0\mathcal{D}$ there exists an equivalence $E^1 \to \mathcal{D}$ from x to a point in the image of $\iota_0 F$.

Proof. Since $\iota_\bullet \mathcal{D}$ is a groupoid object, the set $\pi_0 \iota \mathcal{D}$ is the quotient of $\pi_0 \iota_0 \mathcal{D}$ where we identify two components of $\iota_0 \mathcal{D}$ if there exists a point of $\iota_1 \mathcal{D}$, i.e. an equivalence $E^1 \to \mathcal{D}$, connecting them. Thus $F\colon \mathcal{C} \to \mathcal{D}$ is essentially surjective if and only if every point of $\iota_0 \mathcal{D}$ is connected by an equivalence to a point in the image of $\iota_0 F$. □

Proposition 4.2.2.5. If $f\colon \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective, then the induced map $\iota f\colon \iota\mathcal{C} \to \iota\mathcal{D}$ is an equivalence.

Proof. The simplicial spaces $\iota_\bullet \mathcal{C}$ and $\iota_\bullet \mathcal{D}$ are groupoid objects by Corollary 4.2.1.10, and since f is essentially surjective the map ιf is by definition an effective epimorphism in the ∞-topos $\mathcal{S}$ (since these are precisely the maps of spaces that are surjective on π_0). By [Lur09b, Remark 1.2.17] it therefore suffices to show that the diagram

$$\begin{tikzcd}
\iota_1 \mathcal{C} \arrow[r] \arrow[d] & \iota_1 \mathcal{D} \arrow[d] \\
\iota_0 \mathcal{C} \times \iota_0 \mathcal{C} \arrow[r] & \iota_0 \mathcal{D} \times \iota_0 \mathcal{D}
\end{tikzcd}$$

is a pullback square. To prove this we must show that given points $x, y \in \iota_0 \mathcal{C}$ the map of fibres $\iota_1 \mathcal{C}_{x,y} \to \iota_1 \mathcal{D}_{fx,fy}$ is an equivalence. By Proposition 4.2.1.15, we can identify this with the map $\mathrm{Map}(I, \mathcal{C}(x,y))_{\mathrm{eq}} \to \mathrm{Map}(I, \mathcal{D}(fx,fy))_{\mathrm{eq}}$. Since f is fully faithful the map $\mathcal{C}(x,y) \to \mathcal{D}(fx,fy)$ is an equivalence in $\mathcal{V}$, hence $\mathrm{Map}(I, \mathcal{C}(x,y)) \to \mathrm{Map}(I, \mathcal{D}(x,y))$ is an equivalence in $\mathcal{S}$. To complete the proof it therefore suffices to show that

$$\mathrm{Map}(I, \mathcal{C}(x,y))_{\mathrm{eq}} \to \mathrm{Map}(I, \mathcal{D}(fx,fy))_{\mathrm{eq}}$$

is surjective on components — i.e. if $\alpha\colon I \to \mathcal{D}(fx,fy)$ is an equivalence then it is the image of an equivalence $\beta\colon I \to \mathcal{C}(x,y)$. We know that α is the image of some map β, so it suffices to show that such a β must be an equivalence. By Proposition 4.2.1.16 the map β is an equivalence if and only if for every $z \in \iota_0 \mathcal{C}$ the map $\mathcal{C}(z,x) \to \mathcal{C}(z,y)$ induced by composition with β is an equivalence. Consider the diagram

$$\begin{tikzcd}
\mathcal{C}(z,x) \arrow[r] \arrow[d] & \mathcal{D}(fz,fx) \arrow[d] \\
\mathcal{C}(z,y) \arrow[r] & \mathcal{C}(fz,fy).
\end{tikzcd}$$

Since f is fully faithful and α is an equivalence, all morphisms in this diagram except the left vertical map are known to be equivalences. By the 2-out-of-3 property this must also be an equivalence for all z, so β is indeed an equivalence. □

Corollary 4.2.2.6. A fully faithful functor F is essentially surjective if and only if ιF is an equivalence.

Corollary 4.2.2.7. A fully faithful and essentially surjective functor between complete $\mathcal{V}$-∞-categories is an equivalence in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$.

Proof. This follows by combining Lemma 4.2.2.2, Proposition 4.2.2.5, and Lemma 3.3.5.3. □

Proposition 4.2.2.8. Fully faithful and essentially surjective functors satisfy the 2-out-of-3 property.

Proof. Suppose we have functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ of $\mathcal{V}$-∞-categories. There are three cases to consider:

(1) Suppose F and G are fully faithful and essentially surjective. The map $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes}) \to \mathcal{S}$ is a Cartesian fibration, so composites of Cartesian morphisms are Cartesian and thus $G \circ F$ is fully faithful. Since $\pi_0 \iota F$ and $\pi_0 \iota G$ are surjective, so is their composite $\pi_0 \iota (G \circ F)$, thus $G \circ F$ is also essentially surjective.

(2) Suppose G and $G \circ F$ are fully faithful and essentially surjective. Then F is also Cartesian, i.e. fully faithful, by [Lur09a, Proposition 2.4.1.7]. By Proposition 4.2.2.5 the maps ιG and $\iota(G \circ F)$ are equivalences, hence so is ιF, thus F is also essentially surjective.

(3) Suppose F and $G \circ F$ are fully faithful and essentially surjective. By Proposition 4.2.2.5 the maps ιF and $\iota(G \circ F)$ are equivalences, hence so is ιG, thus G is essentially surjective. To see that G is fully faithful, we must show that for any x, y in $\iota_0 G$ the map $\mathcal{D}(x, y) \to \mathcal{E}(Gx, Gy)$ is an equivalence. But since F is essentially surjective there exist objects x', y' in $\iota_0 \mathcal{C}$ and equivalences $Fx' \simeq x$, $Fy' \simeq y$ in $\mathcal{D}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{D}(Fx', Fy') \arrow[r] \arrow[d] & \mathcal{E}(GFx', GFy') \arrow[d] \\
\mathcal{D}(x, y) \arrow[r] & \mathcal{E}(Gx, Gy),
\end{tikzcd}$$

where the vertical maps are equivalences by Proposition 4.2.1.16. The top horizontal map is also an equivalence, since in the commutative triangle

$$\begin{tikzcd}
\mathcal{C}(x', y') \arrow[rr] \arrow[dr] & & \mathcal{E}(GFx', GFy') \\
& \mathcal{D}(Fx', Fy') \arrow[ur] &
\end{tikzcd}$$

the other two maps are equivalences. Thus by the 2-out-of-3 property the bottom horizontal map $\mathcal{D}(x, y) \to \mathcal{E}(Gx, Gy)$ is also an equivalence, hence G is fully faithful by Lemma 4.2.2.2. □

Remark 4.2.2.9. Under the equivalence $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{S}^{\times}) \simeq \mathrm{Seg}^{\mathcal{O}}_{\infty}$, the fully faithful and essentially surjective functors correspond to the Dwyer-Kan equivalences in the sense of [Rez01, §7.4].

4.2.3 Local Equivalences

In this subsection we assume that $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category, so that the ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$ is presentable by Proposition 4.1.3.8.

Definition 4.2.3.1. The *local equivalences* in $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$ are the elements of the strongly saturated class of morphisms generated by the map $s^0 \colon E^1 \to E^0$.

Lemma 4.2.3.2. The following are equivalent, for a $\mathcal{V}$-∞-category $\mathcal{C}$:

(i) $\mathcal{C}$ is complete.

(ii) $\mathcal{C}$ is local with respect to $E^1 \to E^0$, i.e. the map $\mathrm{Map}(E^0, \mathcal{C}) \to \mathrm{Map}(E^1, \mathcal{C})$ is an equivalence.

(iii) For every local equivalence $\mathcal{A} \to \mathcal{B}$, the induced map $\mathrm{Map}(\mathcal{B}, \mathcal{C}) \to \mathrm{Map}(\mathcal{A}, \mathcal{C})$ is an equivalence.

Proof. (i) is equivalent to (ii) by Corollary 4.2.1.11, and (ii) is equivalent to (iii) by [Lur09a, Proposition 5.5.4.15(4)]. □

Definition 4.2.3.3. Write $\mathrm{Cat}_{\infty}^{\mathcal{V}}$ for the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V})$ spanned by the complete $\mathcal{V}$-∞-categories.

Lemma 4.2.3.4. The inclusion $\mathrm{Cat}_{\infty}^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V})$ has a left adjoint, which exhibits $\mathrm{Cat}_{\infty}^{\mathcal{V}}$ as the localization of $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V})$ with respect to the local equivalences.

Proof. The ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$ is presentable by Proposition 4.1.3.8, and the local equivalences are generated by a set of maps. The existence of the left adjoint therefore follows from [Lur09a, Proposition 5.5.4.15(4)] and Lemma 4.2.3.2. □

Lemma 4.2.3.5. The ∞-category $\mathrm{Cat}_{\infty}^{\mathcal{V}}$ is presentable.

Proof. This follows from [Lur09a, Proposition 5.5.4.15(3)]. □

Lemma 4.2.3.6. $\mathrm{Cat}_{\infty}^{\mathcal{S}}$ is equivalent to Cat_{∞}.

Proof. Under the equivalence $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{S}^{\times}) \simeq \mathrm{Seg}_{\infty}^{\mathcal{O}}$ of Corollary 4.1.4.4, the subcategory $\mathrm{Cat}_{\infty}^{\mathcal{S}}$ corresponds to the subcategory of *complete* Segal spaces. By Theorem 2.2.1.9 this is equivalent to Cat_{∞}. □

Lemma 4.2.3.7. The map $\mathrm{id} \otimes s^0 \colon E^1 \otimes E^1 \to E^1 \otimes E^0 \simeq E^1$ is a local equivalence.

Proof. It suffices to prove this when $\mathcal{V}^{\otimes}$ is $\mathcal{S}^{\times}$. We can identify $E^1 \otimes E^1$ with $E^{\{0,1\} \times \{0,1\}} \simeq E^3$; under this identification the map $E^1 \otimes E^1 \to E^1$ is induced by the map $\{0,1,2,3\} \to \{0,1\}$ sending $0,1$ to 0 and $2,3$ to 1. Using the equivalence

$$E^3 \simeq E^{\{0,1\}} \amalg_{E^{\{1\}}} E^{\{1,2\}} \amalg_{E^{\{2\}}} E^{\{2,3\}}$$

this corresponds to $s^0 \cup \mathrm{id} \cup s^0 \colon E^1 \amalg_{E_0} E^1 \amalg_{E^0} E^1 \to E^0 \amalg_{E^0} E^1 \amalg_{E^0} E^0$, which is clearly in the strongly saturated class generated by s^0. □

Lemma 4.2.3.8. If $\mathcal{C}$ is a complete $\mathcal{V}$-∞-category, then the $\mathcal{V}$-∞-category $\mathcal{C}^{E^1}$ is also complete.

Proof. We need to show that the natural map $\iota_0 \mathcal{C}^{E^1} \to \iota_1 \mathcal{C}^{E^1}$ is an equivalence. Using the adjunction between cotensoring and tensoring we can identify this with the map

$$\mathrm{Map}(E^1, \mathcal{C}) \to \mathrm{Map}(E^1 \otimes E^1, \mathcal{C})$$

induced by composition with $\mathrm{id} \otimes s^0$. This map is an equivalence since $\mathcal{C}$ is complete and $\mathrm{id} \otimes s^0$ is a local equivalence by Lemma 4.2.3.7. $\square$

4.2.4 Categorical Equivalences

In this subsection we study *categorical equivalences* between enriched ∞-categories, which are functors with an inverse up to natural equivalence. We show that categorical equivalences are always local equivalences as well as fully faithful and essentially surjective.

Definition 4.2.4.1. Suppose $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-∞-categories and $f, g\colon \mathcal{A} \to \mathcal{B}$ are $\mathcal{V}$-functors. A *natural equivalence* from f to g is a functor $H\colon \mathcal{A} \otimes E^1 \to \mathcal{B}$ such that $H \circ (\mathrm{id} \otimes d^1) \simeq f$ and $H \circ (\mathrm{id} \otimes d^0) \simeq g$. We say that f and g are *naturally equivalent* if there exists a natural equivalence from f to g.

Definition 4.2.4.2. A functor $f\colon \mathcal{A} \to \mathcal{B}$ is a *categorical equivalence* if there exists a functor $g\colon \mathcal{B} \to \mathcal{A}$ and natural equivalences from $f \circ g$ to $\mathrm{id}_{\mathcal{B}}$ and from $g \circ f$ to $\mathrm{id}_{\mathcal{A}}$. Such a functor g is called a *pseudo-inverse* of f.

Proposition 4.2.4.3. Categorical equivalences are fully faithful and essentially surjective.

Proof. Suppose $f\colon \mathcal{C} \to \mathcal{D}$ is a categorical equivalence, and let $g\colon \mathcal{D} \to \mathcal{C}$ be a pseudo-inverse with natural equivalences $\phi\colon \mathcal{C} \otimes E^1 \to \mathcal{C}$ from $g \circ f$ to $\mathrm{id}_{\mathcal{C}}$ and $\psi\colon \mathcal{D} \otimes E^1 \to \mathcal{D}$ from $f \circ g$ to $\mathrm{id}_{\mathcal{D}}$. For each object x in $\iota_0 \mathcal{D}$ the natural equivalence ψ supplies an equivalence between x and $fg(x)$, which is in the image of f, so f is essentially surjective by Lemma 4.2.2.4.

By Lemma 4.2.2.2, to prove that F is fully faithful it suffices to show that for all x, y in $\iota_0 \mathcal{C}$ the induced map $\alpha\colon \mathcal{C}(x, y) \to \mathcal{D}(fx, fy)$ is an equivalence.

The natural equivalence ϕ supplies an equivalence $\beta\colon \mathcal{C}(gfx, gfy) \xrightarrow{\sim} \mathcal{C}(x, y)$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[rr] \arrow[dr, "\mathrm{id}"'] & & \mathcal{C}(gfx, gfy) \arrow[dl, "\beta"] \\
& \mathcal{C}(x,y). &
\end{tikzcd}$$

The top map is the composite

$$\mathcal{C}(x,y) \xrightarrow{\alpha} \mathcal{D}(fx, fy) \xrightarrow{\gamma} \mathcal{C}(gfx, gfy),$$

and so we get that $\beta \circ \gamma \circ \alpha \simeq \mathrm{id}$.

From $f \circ \phi$ we likewise get an equivalence $\epsilon\colon \mathcal{D}(fgfx, fgfy) \xrightarrow{\sim} \mathcal{D}(fx, fy)$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{D}(fx, fy) \arrow[rr] \arrow[dr, "\mathrm{id}"'] & & \mathcal{D}(fgfx, fgfy) \arrow[dl, "\epsilon"] \\
& \mathcal{D}(fx, fy). &
\end{tikzcd}$$

where the top map is the composite

$$\mathcal{D}(fx, fy) \xrightarrow{\gamma} \mathcal{C}(gfx, gfy) \xrightarrow{\delta} \mathcal{D}(fgfx, fgfy),$$

and so $\epsilon \circ \delta \circ \gamma \simeq \mathrm{id}$. Moreover, we have a commutative square

$$\begin{tikzcd}
\mathcal{C}(gfx, gfy) \arrow[r, "\delta"] \arrow[d, "\beta"'] & \mathcal{D}(fgfx, fgfy) \arrow[d, "\epsilon"] \\
\mathcal{C}(x, y) \arrow[r, "\alpha"'] & \mathcal{D}(fx, fy),
\end{tikzcd}$$

and so we get $\alpha \circ \beta \circ \gamma \simeq \epsilon \circ \delta \circ \gamma \simeq \mathrm{id}$. This shows that $\beta \circ \gamma$ is an inverse of α, and so α is an equivalence in $\mathcal{V}$. □

Corollary 4.2.4.4. A categorical equivalence between complete $\mathcal{V}$-∞-categories is an equivalence.

Proof. Combine Proposition 4.2.4.3 and Corollary 4.2.2.7. □

Lemma 4.2.4.5. Categorical equivalences satisfy the 2-out-of-3 property.

Proof. Suppose we have functors $f: \mathcal{C} \to \mathcal{D}$ and $f': \mathcal{D} \to \mathcal{E}$. There are three cases to consider:

(1) Suppose f has a pseudo-inverse g with natural equivalences $\phi: \mathcal{C} \otimes E^1 \to \mathcal{C}$ and $\psi: \mathcal{D} \otimes E^1 \to \mathcal{D}$, and f' has a pseudo-inverse g' with natural equivalences $\phi': \mathcal{D} \otimes E^1 \to \mathcal{D}$ and $\psi': \mathcal{E} \otimes E^1 \to \mathcal{E}$. Then $g \circ \phi' \circ (f \otimes \mathrm{id})$ is a natural equivalence from $gg'f'f$ to gf. Combining this with ϕ gives a map $(\mathcal{C} \otimes E^1) \amalg_{\mathcal{C} \otimes E^0} (\mathcal{C} \otimes E^1) \to \mathcal{C}$. But tensoring with $\mathcal{C}$ preserves colimits, and $E^1 \amalg_{E^0} E^1 \simeq E^2$ by Theorem 4.2.1.8, so we get a map $\mathcal{C} \otimes E^2 \to \mathcal{C}$. Composing with $\mathrm{id} \otimes d^1 : \mathcal{C} \otimes E^1 \to \mathcal{C} \otimes E^2$ we get a natural equivalence from $gg'f'f$ to the identity. Using the same argument we can also combine $f' \circ \psi \circ (g' \otimes \mathrm{id})$ and ψ' to get a natural equivalence from $f'fgg'$ to the identity. Thus $f'f$ is a categorical equivalence with pseudo-inverse gg'.

(2) Suppose f has a pseudo-inverse g with natural equivalences $\phi: \mathcal{C} \otimes E^1 \to \mathcal{C}$ and $\psi: \mathcal{D} \otimes E^1 \to \mathcal{D}$, and $f'f$ has a pseudo-inverse h with natural equivalences $\alpha: \mathcal{C} \otimes E^1 \to \mathcal{C}$ and $\beta: \mathcal{E} \otimes E^1 \to \mathcal{E}$. We will show that fh is a pseudo-inverse of f'. Since β is a natural equivalence from $f'(fh)$ to id it remains to construct a natural equivalence from fhf' to id. Let $\bar{\psi}$ denote $\psi \circ (\mathrm{id} \otimes E_\sigma)$, where $\sigma: \{0, 1\} \to \{0, 1\}$ is the map that

interchanges 0 and 1 (thus $\bar{\psi}$ is ψ considered as a natural equivalence from id to fg). Combining $fhf' \circ \bar{\psi}$, $f \circ \alpha \circ (g \otimes \mathrm{id})$ and ψ we get a map

$$\mathcal{D} \otimes E^3 \simeq \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \to \mathcal{D}$$

and composing with $\mathcal{D} \otimes E_{\{0,3\}} \to \mathcal{D} \otimes E^3$ we get the required natural equivalence.

(3) Suppose f' has a pseudo-inverse g' with natural equivalences $\phi' \colon \mathcal{C} \otimes E^1 \to \mathcal{C}$ and $\psi' \colon \mathcal{D} \otimes E^1 \to \mathcal{D}$, and $f'f$ has a pseudo-inverse h with natural equivalences $\alpha \colon \mathcal{C} \otimes E^1 \to \mathcal{C}$ and $\beta \colon \mathcal{E} \otimes E^1 \to \mathcal{E}$. We will show that hf' is a pseudo-inverse of f. Since α is a natural equivalence from $(hf')f$ to id it remains to construct a natural equivalence from fhf' to id. Let $\bar{\phi}'$ denote $\phi \circ (\mathrm{id} \otimes E_\sigma)$; combining $\bar{\phi}' \circ (fhf' \otimes \mathrm{id})$, $g' \circ \beta \circ (f' \otimes \mathrm{id})$ and ϕ' we get a map

$$\mathcal{D} \otimes E^3 \simeq \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \to \mathcal{D},$$

and composing with $\mathcal{D} \otimes E_{\{0,3\}} \to \mathcal{D} \otimes E^3$ we get the required natural equivalence.

□

Our next goal is to prove that categorical equivalences are local equivalences; this will require some preliminary results:

Lemma 4.2.4.6. Suppose $f \colon S \to T$ is a map of sets. Then $E_f \colon E_S \to E_T$ is a categorical equivalence.

Proof. It suffices to prove this in $\mathcal{S}^\times$. First suppose f is surjective; let $g \colon T \hookrightarrow S$ be a section of f. We claim that E_g is a pseudo-inverse to E_f. We have $E_f \circ E_g \simeq E_{f \circ g} \simeq \mathrm{id}$, so it suffices to construct a natural equivalence $E_S \times E^1 \simeq E_{S \times \{0,1\}} \to E_S$ from $E_{g \circ f}$ to the identity. This is given by E_h where $h \colon S \times \{0,1\} \to S$ sends $(s,0)$ to $gf(s)$ and $(s,1)$ to s.

By the dual argument the result holds if f is injective. By Lemma 4.2.4.5 we can therefore conclude that it holds for a general f. □

Lemma 4.2.4.7. Suppose $\mathcal{V}^\otimes$ is a presentably monoidal ∞-category and $f \colon \mathcal{A} \to \mathcal{B}$ is a categorical equivalence of $\mathcal{S}$-∞-categories. Then for any $\mathcal{V}$-∞-category $\mathcal{C}$ the induced map $\mathcal{C}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{A}}$ is a categorical equivalence.

Proof. A natural equivalence $\mathcal{A} \otimes E^1 \to \mathcal{A}$ induces a natural equivalence $\mathcal{C}^{\mathcal{A}} \otimes E^1 \to \mathcal{C}^{\mathcal{A}}$ by taking the adjoint of the induced map $\mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathcal{A} \otimes E^1} \simeq (\mathcal{C}^{\mathcal{A}})^{E^1}$. □

Lemma 4.2.4.8. If $\mathcal{C}$ is a complete $\mathcal{V}$-∞-category, then the natural map $\mathcal{C} \simeq \mathcal{C}^{E^0} \to \mathcal{C}^{E^1}$ is an equivalence.

Proof. The map $E^1 \to E^0$ is a categorical equivalence by Lemma 4.2.4.6, so it follows by Lemma 4.2.4.7 that $\mathcal{C} \to \mathcal{C}^{E^1}$ is also a categorical equivalence. But $\mathcal{C}^{E^1}$ is complete by Lemma 4.2.3.8, and a categorical equivalence between complete objects is an equivalence by Corollary 4.2.4.4. □

Proposition 4.2.4.9. For any $\mathcal{V}$-∞-category $\mathcal{C}$, the map $\mathrm{id} \otimes s^0 \colon \mathcal{C} \otimes E^1 \to \mathcal{C} \otimes E^0 \simeq \mathcal{C}$ is a local equivalence.

Proof. We must show that for any complete $\mathcal{V}$-∞-category $\mathcal{D}$ the map

$$\mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}(\mathcal{C} \otimes E^1, \mathcal{D})$$

is an equivalence. Using the adjunction between tensoring and cotensoring with E^1, we see that this map is equivalent to the map

$$\mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}(\mathcal{C}, \mathcal{D}^{E^1})$$

given by composing with the map $\mathcal{D} \to \mathcal{D}^{E^1}$ induced by s^0. This is an equivalence by Lemma 4.2.4.8. □

Corollary 4.2.4.10. Suppose $\mathcal{D}$ is a complete $\mathcal{V}$-∞-category; then for any $\mathcal{V}$-∞-category $\mathcal{C}$ we have $|\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{D})| \simeq \mathrm{Map}(\mathcal{C}, \mathcal{D})$.

Proof. Since tensoring preserves colimits, and $E^\bullet$ is a cogroupoid object, the simplicial space $\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{D})$ is a groupoid object. By Lemma 2.1.10.4 it therefore suffices to show that $\mathrm{Map}(\mathcal{C} \otimes E^0, \mathcal{D}) \to \mathrm{Map}(\mathcal{C} \otimes E^1, \mathcal{D})$ is an equivalence, which is true by Proposition 4.2.4.9. □

Lemma 4.2.4.11. Suppose $\mathcal{D}$ is a complete $\mathcal{V}$-∞-category. Then for any $\mathcal{V}$-∞-category $\mathcal{C}$ the two maps

$$(\mathrm{id} \otimes d^0)^*, (\mathrm{id} \otimes d^1)^* \colon \mathrm{Map}(\mathcal{C} \otimes E^1, \mathcal{D}) \to \mathrm{Map}(\mathcal{C}, \mathcal{D})$$

are homotopic.

Proof. Clearly $(\mathrm{id} \otimes s^0)^* \circ (\mathrm{id} \otimes d^i)^* \colon \mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}(\mathcal{C}, \mathcal{D})$ is homotopic to the identity for $i = 0, 1$. But by Proposition 4.2.4.9, the map $(\mathrm{id} \otimes s^0)$ is a local equivalence, hence $(\mathrm{id} \otimes s^0)^*$ is an equivalence since $\mathcal{D}$ is complete. Composing with its inverse we get that $(\mathrm{id} \otimes d^0)^* \simeq (\mathrm{id} \otimes d^1)^*$. □

Theorem 4.2.4.12. Categorical equivalences are local equivalences.

Proof. Suppose $f \colon \mathcal{C} \to \mathcal{D}$ is a categorical equivalence with pseudo-inverse $g \colon \mathcal{D} \to \mathcal{C}$ and natural equivalences $\phi \colon \mathcal{C} \otimes E^1 \to \mathcal{C}$ from gf to id and $\psi \colon \mathcal{D} \otimes E^1 \to \mathcal{D}$ from fg to id. If $\mathcal{E}$ is a complete $\mathcal{V}$-∞-category we must show that the map

$$f^* \colon \mathrm{Map}(\mathcal{C}, \mathcal{E}) \to \mathrm{Map}(\mathcal{D}, \mathcal{E})$$

is an equivalence of spaces. By Lemma 4.2.4.11 we have equivalences

$$g^* f^* \simeq \phi^* \circ (\mathrm{id} \otimes d^1)^* \simeq \phi^* \circ (\mathrm{id} \otimes d^0)^* \simeq \mathrm{id},$$

$$f^* g^* \simeq \psi^* \circ (\mathrm{id} \otimes d^1)^* \simeq \psi^* \circ (\mathrm{id} \otimes d^0)^* \simeq \mathrm{id}.$$

Thus g^* is an inverse of f^*, and so f^* is indeed an equivalence. □

4.2.5 Completion in the Presentable Case

We will now construct an explicit completion functor, and use this to deduce that the local equivalences are precisely the fully faithful and essentially surjective functors. We again assume that $\mathcal{V}^\otimes$ is a presentably monoidal ∞-category.

Definition 4.2.5.1. If $\mathcal{C}$ is a $\mathcal{V}$-∞-category, the cosimplicial $\mathcal{S}$-∞-category $E^\bullet$ gives a simplicial $\mathcal{V}$-∞-category $\mathcal{C}^{E^\bullet}$. We let $\widehat{\mathcal{C}}$ denote its geometric realization $|\mathcal{C}^{E^\bullet}|$.

Theorem 4.2.5.2. The natural map $\mathcal{C} \to \widehat{\mathcal{C}}$ is both a local equivalence and fully faithful and essentially surjective. Moreover, the $\mathcal{V}$-∞-category $\widehat{\mathcal{C}}$ is complete.

Proof. The functors $E^n \to E^m$ induced by maps $[n] \to [m]$ in Δ are categorical equivalences by Lemma 4.2.4.6, so the induced functors $\mathcal{C}^{E^m} \to \mathcal{C}^{E^n}$ are also categorical equivalences by Lemma 4.2.4.7. These functors are therefore all fully faithful and essentially surjective by Proposition 4.2.4.3, and local equivalences by Theorem 4.2.4.12. Local equivalences are by definition closed under colimits, so it follows that the map $\mathcal{C} \to \widehat{\mathcal{C}}$ is a local equivalence.

By Lemma 3.2.8.7 we know that $\mathrm{Alg}^{\mathbb{O}}_{\Delta^{\mathrm{op}}_{\iota\mathcal{C}}}(\mathcal{V}^{\otimes}) \simeq \lim \mathrm{Alg}^{\mathbb{O}}_{\Delta^{\mathrm{op}}_{\iota_n\mathcal{C}}}(\mathcal{V}^{\otimes})$. Moreover, the maps $\mathcal{C}^{E^n} \to \mathcal{C}^{E^m}$ in the simplicial diagram $\mathcal{C}^{E^\bullet}$ are fully faithful, i.e. Cartesian. By Proposition 2.1.5.12 it follows that the induced maps $\mathcal{C}^{E^n} \to \widehat{\mathcal{C}}$ are also Cartesian. In particular, the map $\mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful, and since ι_0 preserves colimits this functor is also essentially surjective.

It remains to prove that $\widehat{\mathcal{C}}$ is complete, i.e. that the map $\iota_0\widehat{\mathcal{C}} \to \iota_1\widehat{\mathcal{C}}$ is an equivalence. We have a commutative diagram

$$\begin{tikzcd} {|\iota_0\mathcal{C}^{E^\bullet}|} \arrow[r] \arrow[d] & \iota_0\widehat{\mathcal{C}} \arrow[d] \\ {|\iota_1\mathcal{C}^{E^\bullet}|} \arrow[r] & \iota_1\widehat{\mathcal{C}}, \end{tikzcd}$$

where the top horizontal morphism is an equivalence since ι_0 preserves colimits. The left vertical map is also an equivalence: We have equivalences $\iota_1\mathcal{C}^{E^n} \simeq \mathrm{Map}(E^1 \otimes E^n, \mathcal{C}) \simeq \iota_n\mathcal{C}^{E^1}$, so $|\iota_1\mathcal{C}^{E^\bullet}| \simeq \iota\mathcal{C}^{E^1}$, and under this equivalence the left vertical map corresponds to that induced by the natural map $\mathcal{C} \to \mathcal{C}^{E^1}$; we know that this is fully faithful and essentially surjective, and so induces an equivalence on ι by Proposition 4.2.2.5. In order to show that $\widehat{\mathcal{C}}$ is complete, it thus suffices to show that the bottom vertical map $|\iota_1\mathcal{C}^{E^\bullet}| \to \iota_1\widehat{\mathcal{C}}$ is an equivalence.

To see this we consider the commutative diagram

$$\begin{tikzcd} {|\iota_1\mathcal{C}^{E^\bullet}|} \arrow[r] \arrow[d] & \iota_1\widehat{\mathcal{C}} \arrow[d] \\ {|\iota_0\mathcal{C}^{E^\bullet}|^{\times 2}} \arrow[r] & \iota_0\widehat{\mathcal{C}}^{\times 2}. \end{tikzcd}$$

Here the bottom horizontal map is an equivalence, so to prove that the top horizontal map is an equivalence it suffices to show that this is a pullback square. Since $\mathcal{C} \to \widehat{\mathcal{C}}$ is essentially surjective, to see this we need only show that for all $(x, y) \in \iota_0\mathcal{C}^{\times 2}$ the induced map on fibres $|\iota_1\mathcal{C}^{E^\bullet}|_{(x,y)} \to \iota_1\widehat{\mathcal{C}}_{(x,y)}$ is an equivalence.

Since $\mathcal{C}^{E^m} \to \mathcal{C}^{E^n}$ is fully faithful and essentially surjective for all $[n] \to [m]$ in Δ^{op}, the map $\iota\mathcal{C}^{E^m} \to \iota\mathcal{C}^{E^n}$ is an equivalence by Proposition 4.2.2.5. Since the groupoid objects $\iota_\bullet\mathcal{C}^{E^m}$

and $\iota_\bullet \mathcal{C}^{E^n}$ are effective, the diagram

$$\begin{tikzcd}
\iota_1 \mathcal{C}^{E^m} \arrow[r] \arrow[d] & \iota_1 \mathcal{C}^{E^n} \arrow[d] \\
(\iota_0 \mathcal{C}^{E^m})^{\times 2} \arrow[r] & (\iota_0 \mathcal{C}^{E^n})^{\times 2}
\end{tikzcd}$$

is therefore a pullback square. In other words, the natural transformation $\iota_1 \mathcal{C}^{E^\bullet} \to (\iota_0 \mathcal{C}^{E^\bullet})^{\times 2}$ is Cartesian. By [Lur09a, Theorem 6.1.3.9] the extended natural transformation of functors $(\Delta^{\mathrm{op}})^{\triangleright} \to \mathcal{S}$ that includes the colimits is also Cartesian. Thus we have a pullback square

$$\begin{tikzcd}
\iota_1 \mathcal{C} \arrow[r] \arrow[d] & |\iota_1 \mathcal{C}^{E^\bullet}| \arrow[d] \\
\iota_0 \mathcal{C}^{\times 2} \arrow[r] & |\iota_0 \mathcal{C}^{E^\bullet}|^{\times 2}.
\end{tikzcd}$$

In particular, for $(x, y) \in \iota_0 \mathcal{C}^{\times 2}$ the induced map on fibres $\iota_1 \mathcal{C}_{(x,y)} \to |\iota_1 \mathcal{C}^{E^\bullet}|_{(x,y)}$ is an equivalence. Since $\mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful and essentially surjective, the map $\iota_1 \mathcal{C}_{(x,y)} \to \iota_1 \widehat{\mathcal{C}}_{(x,y)}$ is also an equivalence. By the 2-out-of-3 property it then follows that $|\iota_1 \mathcal{C}^{E^\bullet}|_{(x,y)} \to \iota_1 \widehat{\mathcal{C}}_{(x,y)}$ is an equivalence too. This completes the proof that $\widehat{\mathcal{C}}$ is complete. $\square$

Remark 4.2.5.3. The proof that $\widehat{\mathcal{C}}$ is complete closely follows Rezk's proof in [Rez01, §14] of the equivalent statement for Segal spaces.

Corollary 4.2.5.4. The following are equivalent, for a functor $f : \mathcal{C} \to \mathcal{D}$ of $\mathcal{V}$-∞-categories:

(i) f is a local equivalence.

(ii) f is fully faithful and essentially surjective.

Proof. By Theorem 4.2.5.2 we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d] \\
\widehat{\mathcal{C}} \arrow[r, "\widehat{f}"'] & \widehat{\mathcal{D}},
\end{tikzcd}$$

where the vertical maps are both local equivalences and fully faithful and essentially surjective, and $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are complete.

Since local equivalences form a strongly saturated class of morphisms, it follows from the 2-out-of-3 property that f is a local equivalence if and only if $\widehat{f}$ is a local equivalence, i.e. if and only if $\widehat{f}$ is an equivalence, since $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are complete.

Fully faithful and essentially surjective functors also satisfy the 2-out-of-3 property, by Proposition 4.2.2.8, so f is fully faithful and essentially surjective if and only if $\widehat{f}$ is.

But by Corollary 4.2.2.7 $\widehat{f}$ is fully faithful and essentially surjective if and only if it is an equivalence, since $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are complete. Thus f is a local equivalence if and only if $\widehat{f}$ is an equivalence, which is true if and only if f is fully faithful and essentially surjective. □

Corollary 4.2.5.5. $\mathrm{Cat}_\infty^{\mathcal{V}}$ is the localization of $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ with respect to the fully faithful and essentially surjective functors.

Remark 4.2.5.6. We might expect that the fully faithful and essentially surjective functors also coincide with the categorical equivalences, but this turns out *not* to be the case when we allow spaces of objects. To see this, first observe that if $f\colon \mathcal{A} \to \mathcal{B}$ is a categorical equivalence, then for every $\mathcal{V}$-∞-category $\mathcal{C}$ the map $f_*\colon |\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{A})| \to |\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{B})|$ is surjective on π_0: suppose $g\colon \mathcal{B} \to \mathcal{A}$ is a pseudo-inverse to f, then given a functor $\phi\colon \mathcal{C} \to \mathcal{B}$ the natural equivalence from $f \circ g$ to id gives a natural equivalence from $f \circ g \circ \phi$ to ϕ, so up to natural equivalence ϕ is in the image of f_*. Now if $\mathcal{B} \to \widehat{\mathcal{B}}$ is a categorical equivalence where $\widehat{\mathcal{B}}$ is complete, then by Corollary 4.2.4.10 we have $|\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \widehat{\mathcal{B}})| \simeq \mathrm{Map}(\mathcal{C}, \widehat{\mathcal{B}})$, and since $\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{B})$ is a groupoid object the map $\mathrm{Map}(\mathcal{C}, \mathcal{B}) \to |\mathrm{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{B})|$ is surjective on π_0. Thus $\mathrm{Map}(\mathcal{C}, \mathcal{B}) \to \mathrm{Map}(\mathcal{C}, \widehat{\mathcal{B}})$ is surjective on π_0.

Now suppose $\iota_0\mathcal{B}$ is discrete and $\iota\mathcal{B}$ is not; then there clearly exists for some $n > 0$ a map from the n-sphere $S^n \to \iota\mathcal{B}$ that does not factor through $\iota_0\mathcal{B}$. But we have a $\mathcal{V}$-∞-category $S^n \otimes E^0$ such that $\mathrm{Map}(S^n \otimes E^0, \mathcal{B}) \simeq \mathrm{Map}(S^n, \iota_0\mathcal{B})$ — so if $\mathcal{B} \to \widehat{\mathcal{B}}$ were a categorical equivalence then $\mathrm{Map}(S^n, \iota_0\mathcal{B}) \to \mathrm{Map}(S^n, \iota\mathcal{B})$ would have to be surjective on π_0, a contradiction. This shows that completion maps $\mathcal{B} \to \widehat{\mathcal{B}}$ cannot be categorical equivalences in general.

4.2.6 The Non-Presentable Case

We now show that we can invert the fully faithful and essentially surjective functors of $\mathcal{V}$-∞-categories for a general large monoidal ∞-category $\mathcal{V}^\otimes$ by restricting to complete $\mathcal{V}$-∞-categories:

Theorem 4.2.6.1. Let $\mathcal{V}^\otimes$ be a large monoidal ∞-category. The inclusion of the full subcategory of complete $\mathcal{V}$-∞-categories $\mathrm{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ has a left adjoint that exhibits $\mathrm{Cat}_\infty^{\mathcal{V}}$ as the localization $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^\otimes)$ with respect to the fully faithful and essentially surjective functors.

Proof. Let $\widehat{\mathcal{P}}(\mathcal{V})$ be the ∞-category of presheaves of large spaces on $\mathcal{V}$. By [Lur11, Proposition 6.3.1.10] there exists a monoidal structure on $\widehat{\mathcal{P}}(\mathcal{V})$ such that the Yoneda embedding $j\colon \mathcal{V} \to \widehat{\mathcal{P}}(\mathcal{V})$ is a strong monoidal functor. Let $\widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V})^\otimes)$ be the (very large) ∞-category of large categorical algebras in $\widehat{\mathcal{P}}(\mathcal{V})$; this is a presentable ∞-category, and writing $\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})}$ for its subcategory of complete $\widehat{\mathcal{P}}(\mathcal{V})$-$\infty$-categories we know from Corollary 4.2.5.5 that the inclusion $\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})} \hookrightarrow \widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V}))$ has a left adjoint $\widehat{L}$ that exhibits $\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})}$ as the localization with respect to the fully faithful and essentially surjective functors.

If $\mathcal{C}$ is in the essential image of the fully faithful inclusion

$$\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}) \hookrightarrow \widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V})),$$

then the natural map $\mathcal{C} \to \widehat{L}\mathcal{C}$ is fully faithful and essentially surjective. But then $\iota_0\widehat{L}\mathcal{C} \simeq \iota\mathcal{C}$, so $\iota_0\widehat{L}\mathcal{C}$ is an (essentially) small space, and the mapping objects in $\widehat{L}\mathcal{C}$ are in the essential image of $\mathcal{V}$ in $\widehat{\mathcal{P}}(\mathcal{V})$. Thus $\widehat{L}\mathcal{C}$ is in the essential image of $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V})$, and so the functor $\widehat{L}$ restricts to a functor $L\colon \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Cat}^{\mathcal{V}}_{\infty}$, since $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ is equivalent to the full subcategory of $\widehat{\mathrm{Cat}}^{\widehat{\mathcal{P}}(\mathcal{V})}_{\infty}$ spanned by objects in the essential image of $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V})$. □

4.2.7 Properties of the Localized Category

In this subsection we prove that the ∞-category $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ inherits the naturality properties of $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$.

Proposition 4.2.7.1. Let $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}} \to \widehat{\mathrm{Mon}}^{\mathbb{O},\mathrm{lax}}_{\infty}$ be a coCartesian fibration corresponding to the functor $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(-)$. Define Enr_{∞} to be the full subcategory of $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}$ whose objects are the complete enriched ∞-categories. Then the restricted projection $\mathrm{Enr}_{\infty} \to \widehat{\mathrm{Mon}}^{\mathbb{O},\mathrm{lax}}_{\infty}$ is a coCartesian fibration, and the inclusion $\mathrm{Enr}_{\infty} \hookrightarrow \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}$ admits a left adjoint over $\widehat{\mathrm{Mon}}^{\mathbb{O},\mathrm{lax}}_{\infty}$.

Proof. The result follows from Proposition 2.1.8.13; to apply this we must show that if $\phi\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ is a lax monoidal functor, then ϕ_* preserves fully faithful and essentially surjective functors. It is clear that ϕ_* preserves fully faithful functors. To see that it preserves essentially surjective ones we note that if two points of $\iota_0\mathcal{C}$ are equivalent as objects of $\mathcal{C}$ then they are also equivalent as objects of $\phi_*\mathcal{C}$, since the map $I_{\mathcal{W}} \to \phi(I_{\mathcal{V}})$ induces a functor $E^1_{\mathcal{W}} \to \phi_* E^1_{\mathcal{V}}$. □

Lemma 4.2.7.2. Suppose $\mathcal{V}^{\otimes}$ and $\mathcal{W}^{\otimes}$ are monoidal ∞-categories compatible with small colimits, and $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a strong monoidal functor such that $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Then the induced functor $F_*\colon \mathrm{Cat}^{\mathcal{V}}_{\infty} \to \mathrm{Cat}^{\mathcal{W}}_{\infty}$ preserves colimits.

Proof. The functor F_* is the composite

$$\mathrm{Cat}^{\mathcal{V}}_{\infty} \hookrightarrow \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes}) \xrightarrow{F^{\mathrm{Alg}}_*} \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{W}^{\otimes}) \xrightarrow{L_{\mathcal{W}}} \mathrm{Cat}^{\mathcal{W}}_{\infty},$$

where $L_{\mathcal{W}}$ is the completion functor for $\mathcal{W}$, and we write F^{Alg}_* for the functor on $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}$ induced by composition with F for clarity. We know that F^{Alg}_* preserves local equivalences, so $F^{\mathrm{Alg}}_* L_{\mathcal{V}}\mathcal{C}$ and $F^{\mathrm{Alg}}_*\mathcal{C}$ are locally equivalent for all $\mathcal{C}$; it follows that $L_{\mathcal{W}} \circ F^{\mathrm{Alg}}_* \circ L_{\mathcal{V}} \simeq L_{\mathcal{W}} \circ F^{\mathrm{Alg}}_*$. If $\alpha \mapsto \mathcal{C}_{\alpha}$ is a diagram in $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ then its colimit is $L_{\mathcal{V}}(\operatorname{colim} \mathcal{C}_{\alpha})$ where this colimit is computed in $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$. Thus we have

$$\begin{aligned} F_*(\operatorname{colim} \mathcal{C}_{\alpha}) &\simeq L_{\mathcal{W}} F^{\mathrm{Alg}}_* L_{\mathcal{V}}(\operatorname{colim} \mathcal{C}_{\alpha}) \simeq L_{\mathcal{W}} F^{\mathrm{Alg}}_*(\operatorname{colim} \mathcal{C}_{\alpha}) \\ &\simeq \operatorname{colim} L_{\mathcal{W}} F^{\mathrm{Alg}}_*(\mathcal{C}_{\alpha}) \simeq \operatorname{colim} F_*\mathcal{C}_{\alpha}. \end{aligned}$$

□

Proposition 4.2.7.3. The restriction of the functor $\mathrm{Cat}^{(-)}_{\infty}$ to $\mathrm{Mon}^{\mathbb{O},\mathrm{Pr}}_{\infty}$ factors through Pr^{L}.

Proof. This follows from Lemma 4.2.7.2 and Lemma 4.2.3.5. □

Proposition 4.2.7.4. $\mathrm{Cat}^{(-)}_{\infty}$ is a lax monoidal functor with respect to the Cartesian product of monoidal ∞-categories.

Proof. Given a $\mathcal{V}$-∞-category $\mathcal{B}$, a $\mathcal{W}$-∞-category $\mathcal{C}$ and a $\mathcal{V}\times\mathcal{W}$-$\infty$-category $\mathcal{A}$ we clearly have

$$\mathrm{Map}(\mathcal{A},\mathcal{B}\boxtimes\mathcal{C})\simeq\mathrm{Map}(\pi_{1,*}\mathcal{A},\mathcal{B})\times\mathrm{Map}(\pi_{2,*}\mathcal{A},\mathcal{C}),$$

where π_1 and π_2 denote the projections from $\mathcal{V}\times\mathcal{W}$ to $\mathcal{V}$ and $\mathcal{W}$, respectively. Moreover $\pi_{i,*}E^X\simeq E^X$ for all X, so

$$\iota_\bullet(\mathcal{B}\boxtimes\mathcal{C})\simeq\iota_\bullet\mathcal{B}\times\iota_\bullet\mathcal{C}.$$

It follows that the complete enriched ∞-categories are closed under the exterior product in $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}$, and so the definition of the lax monoidal structure on the functor $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(-)$ implies that $\mathrm{Cat}^{(-)}_\infty$ is lax monoidal. □

Corollary 4.2.7.5. If $\mathcal{V}^\otimes$ is an $\mathbb{E}_n$-monoidal ∞-category, then the ∞-category $\mathrm{Cat}^{\mathcal{V}}_\infty$ inherits an $\mathbb{E}_{n-1}$-monoidal structure.

Proof. By Proposition 3.2.4.5 we can identify $\mathbb{E}_n$-monoidal ∞-categories with $\mathbb{E}_n$-algebras in Cat_∞. Since $\mathbb{E}_n\simeq(\mathbb{E}_1)^{\otimes n}$ by [Lur11, Theorem 5.1.2.2], we have

$$\mathrm{Mon}^{\mathbb{F},\mathbb{E}_n^\otimes}_\infty\simeq\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}^\otimes_{n-1}}(\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}^\otimes_1}(\mathrm{Cat}^\times_\infty))\simeq\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}^\otimes_{n-1}}(\mathrm{Alg}^{\mathbb{O}}_{\Delta^{\mathrm{op}}}(\mathrm{Cat}^\times_\infty))\simeq\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}^\otimes_{n-1}}(\mathrm{Mon}^{\mathbb{O}}_\infty).$$

Thus $\mathbb{E}_n$-monoidal ∞-categories are equivalent to $\mathbb{E}_{n-1}$-algebras in monoidal ∞-categories. Since $\mathrm{Cat}^{(-)}_\infty$ is lax monoidal, it takes $\mathbb{E}_{n-1}$-algebras in monoidal ∞-categories to $\mathbb{E}_{n-1}$-algebras in Cat_∞, i.e. $\mathbb{E}_{n-1}$-monoidal ∞-categories. □

Proposition 4.2.7.6. Suppose $\mathcal{V}$ is an ∞-category with finite products. Then the natural symmetric monoidal structure on $\mathrm{Cat}^{\mathcal{V}}_\infty$ is Cartesian.

Proof. This follows from Proposition 4.1.3.10, since the inclusion

$$\mathrm{Cat}^{\mathcal{V}}_\infty\hookrightarrow\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^\times)$$

preserves limits. □

Definition 4.2.7.7. If $\mathcal{V}^\otimes$ is an $\mathbb{E}_n$-monoidal ∞-category, we can iterate the enrichment functor k times for $k\leq n$ to obtain ∞-categories $\mathrm{Cat}^{\mathcal{V}}_{(\infty,k)}$ of *(∞,k)-categories enriched in $\mathcal{V}$*.

Proposition 4.2.7.8. When restricted to $\mathrm{Mon}^{\mathbb{O},\mathrm{Pr}}_\infty$, the functor $\mathrm{Cat}^{(-)}_\infty$ is lax monoidal with respect to the tensor product of presentable ∞-categories.

Proof. This follows because the complete enriched ∞-categories are closed under the exterior product, as in the proof of Proposition 4.2.7.4. □

4.3 Some Applications

In this section we describe two simple applications of our machinery: In §4.3.1 we show that enriching in a monoidal $(n,1)$-category gives an $(n+1,1)$-category, and use this to prove the Baez-Dolan stabilization hypothesis for k-tuply monoidal n-categories, and in §4.3.2 we prove that there is a fully faithful embedding of associative algebras in a monoidal ∞-category into pointed enriched ∞-categories.

4.3.1 The Baez-Dolan Stabilization Hypothesis

Recall that an $(n,1)$-*category* is an ∞-category where the mapping spaces are $(n-1)$-types, i.e. there are no non-trivial k-morphisms for $k > n$. Our first goal in this subsection is to prove that enriching in an $(n,1)$-category gives an $(n+1,1)$-category of enriched ∞-categories.

Remark 4.3.1.1. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category such that $\mathcal{V}^{\otimes}_{[1]}$ is an $(n,1)$-category. Then clearly $\mathcal{V}^{\otimes}$ is also an $(n,1)$-category. The phrase *monoidal* $(n,1)$-*category* is thus unambiguous.

Proposition 4.3.1.2. Suppose $\mathcal{V}^{\otimes}$ is a monoidal $(n,1)$-category and $\mathcal{C}$ is a $\mathcal{V}$-∞-category. Then the space $\iota\mathcal{C}$ is an n-type.

Proof. Let $s\colon \pi_0(\iota_0\mathcal{C}) \to \iota_0\mathcal{C}$ be a section of the projection $\iota_0\mathcal{C} \to \pi_0\iota_0\mathcal{C}$. Then the Cartesian morphism $s^*\mathcal{C} \to \mathcal{C}$ is fully faithful and essentially surjective, and so induces an equivalence $\iota(s^*\mathcal{C}) \to \iota\mathcal{C}$ by Proposition 4.2.2.5. Without loss of generality we may therefore assume that the space $\iota_0\mathcal{C}$ is discrete.

The simplicial space $\iota_\bullet\mathcal{C}$ is a groupoid object by Corollary 4.2.1.10. By [Lur09a, Corollary 6.1.3.20] this groupoid object is effective, and so we have a pullback diagram

$$\begin{tikzcd} \iota_1\mathcal{C} \arrow[r] \arrow[d] & \iota_0\mathcal{C} \arrow[d] \\ \iota_0\mathcal{C} \arrow[r] & \iota\mathcal{C}. \end{tikzcd}$$

If x is a point of $\iota_0\mathcal{C}$, we get a pullback diagram

$$\begin{tikzcd} \iota_1\mathcal{C}_{\{x\}} \arrow[r] \arrow[d] & \iota_0\mathcal{C} \arrow[d] \\ \{x\} \arrow[r] & \iota\mathcal{C}, \end{tikzcd}$$

where $\iota_1\mathcal{C}_{\{x\}}$ is the fibre of $\iota_1\mathcal{C} \to \iota_0\mathcal{C}$ at x. Since the map $\iota_0\mathcal{C} \to \iota\mathcal{C}$ is surjective on components, by considering the long exact sequence of homotopy groups associated to this fibre sequence we see that $\iota\mathcal{C}$ is an n-type provided the spaces $\iota_1\mathcal{C}_{\{x\}}$ are $(n-1)$-types for all $x \in \iota_0\mathcal{C}$.

The space $\iota_1\mathcal{C}_{\{x\}}$ is a union of components of $\iota_1\mathcal{C}$, so it suffices to show that $\iota_1\mathcal{C}$ is an $(n-1)$-type. Since $\iota_0\mathcal{C}$ is discrete, i.e. a 0-type, by [Lur09a, Lemma 5.5.6.14] this is equivalent to proving that the fibres of the map $\iota_1\mathcal{C} \to \iota_0\mathcal{C} \times \iota_0\mathcal{C}$ are $(n-1)$-types. But by Proposition 4.2.1.15 we can identify the fibre $\iota_1\mathcal{C}_{x,y}$ at $(x,y) \in \iota_0\mathcal{C}^{\times 2}$ with the space $\mathrm{Map}(I, \mathcal{C}(x,y))_{\mathrm{eq}}$ that is the union of the components of $\mathrm{Map}(I, \mathcal{C}(x,y))$ corresponding to equivalences. Since $\mathcal{V}$ is by assumption an n-category, the space $\mathrm{Map}(I, \mathcal{C}(x,y))$ is necessarily an $(n-1)$-type, hence so is the union of any subset of its components. $\square$

Theorem 4.3.1.3. Suppose $\mathcal{V}^{\otimes}$ is a monoidal $(n,1)$-category. Then $\mathrm{Cat}_\infty^{\mathcal{V}}$ is an $(n+1,1)$-category.

Proof. We need to show that if $\mathcal{C}$ and $\mathcal{D}$ are complete $\mathcal{V}$-∞-categories then the space

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)}(\mathcal{C}, \mathcal{D})$$

is an n-type. By Proposition 4.3.1.2, the space $\iota_0\mathcal{D} \simeq \iota\mathcal{D}$ is an n-type, hence so is the space $\mathrm{Map}_{\mathcal{S}}(\iota_0\mathcal{C}, \iota_0\mathcal{D})$. It then follows from [Lur09a, Lemma 5.5.6.14] that in order to prove that $\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)}(\mathcal{C}, \mathcal{D})$ is an n-type it suffices to show that the fibres of the map

$$\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes)}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathcal{S}}(\iota_0\mathcal{C}, \iota_0\mathcal{D})$$

induced by the projection $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes) \to \mathcal{S}$ are n-types.

Since the projection $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^\otimes) \to \mathcal{S}$ is a Cartesian fibration, by [Lur09a, Proposition 2.4.4.2] we can identify the fibre of this map at $f\colon \iota_0\mathcal{C} \to \iota_0\mathcal{D}$ with

$$\mathrm{Map}_{\mathrm{Alg}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}}(\mathcal{V}^\otimes)}(\mathcal{C}, f^*\mathcal{D}).$$

This space is the fibre of

$$\mathrm{Map}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}} \times \Delta^1, \mathcal{V}^\otimes) \to \mathrm{Map}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^\otimes) \times \mathrm{Map}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^\otimes)$$

at $(\mathcal{C}, f^*\mathcal{D})$. Since n-types are closed under all limits by [Lur09a, Proposition 5.5.6.5], it suffices to show that the spaces $\mathrm{Map}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^\otimes)$ and $\mathrm{Map}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}} \times \Delta^1, \mathcal{V}^\otimes)$ are n-types. Now these spaces are fibres of $\mathrm{Map}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^\otimes) \to \mathrm{Map}(\Delta^{\mathrm{op}}, \mathcal{V}^\otimes)$ and $\mathrm{Map}(\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}} \times \Delta^1, \mathcal{V}^\otimes) \to \mathrm{Map}(\Delta^{\mathrm{op}}, \mathcal{V}^\otimes)$, so by the same argument it's enough to show that these mapping spaces are n-types. But $\mathcal{V}^\otimes$ is by assumption an $(n,1)$-category, so this holds by [Lur09a, Proposition 2.3.4.18]. $\square$

It follows that if $\mathcal{V}^\otimes$ is a symmetric monoidal $(n,1)$-category, then $\mathbb{E}_k$-algebras in $\mathrm{Cat}_\infty^{\mathcal{V}}$ are equivalent to $\mathbb{E}_\infty$-algebras for k large:

Corollary 4.3.1.4. Let $\mathcal{V}^\otimes$ be a symmetric monoidal $(n,1)$-category. Then

(i) the map $\mathbb{E}_k^\otimes \to \Gamma^{\mathrm{op}}$ induces an equivalence

$$\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_k^\otimes}(\mathrm{Cat}_\infty^{\mathcal{V}}) \xrightarrow{\sim} \mathrm{Alg}^{\mathbb{F}}_{\Gamma^{\mathrm{op}}}(\mathrm{Cat}_\infty^{\mathcal{V}})$$

for $k \geq n+1$,

(ii) the stabilization map $i\colon \mathbb{E}_k^\otimes \to \mathbb{E}_{k+1}^\otimes$ (defined in [Lur11, §5.1.1]) induces an equivalence

$$i^*\colon \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{k+1}^\otimes}(\mathrm{Cat}_\infty^{\mathcal{V}}) \to \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_k^\otimes}(\mathrm{Cat}_\infty^{\mathcal{V}})$$

for $k \geq n+1$.

Proof. (i) is immediate from [Lur11, Corollary 5.1.1.7], and (ii) follows by the 2-out-of-3 property. $\square$

We end this subsection by observing that when $\mathcal{V}^\otimes$ is the monoidal ∞-category of n-categories, this yields the Baez-Dolan stabilization hypothesis, by the same proof as Lurie's version for $(n,1)$-categories [Lur09a, Example 5.1.2.3]. First we give the obvious definition of (weak) n-categories using our machinery:

Definition 4.3.1.5. The category Set of sets is a symmetric monoidal $(1,1)$-category. We can therefore define ∞-categories $\mathrm{Cat}_n := \mathrm{Cat}^{\mathrm{Set}}_{(\infty,n)}$ of Set-(∞,n)-categories, i.e. (weak) n-categories, as in Definition 4.2.7.7. Applying Theorem 4.3.1.3 inductively we see that Cat_n is an $(n+1,1)$-category. A *k-tuply monoidal n-category* is an $\mathbb{E}_k$-algebra in Cat_n, i.e. an $\mathbb{E}_k$-monoidal n-category.

Corollary 4.3.1.6 (Baez-Dolan Stabilization Hypothesis)**.** The stabilization map $i\colon \mathbb{E}_k^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ induces an equivalence

$$i^*\colon \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}^{\otimes}_{k+1}}(\mathrm{Cat}_n^{\times}) \to \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}^{\otimes}_{k}}(\mathrm{Cat}_n^{\times})$$

for $k \geq n+2$.

Proof. Apply Corollary 4.3.1.4 to Cat_n. $\square$

Remark 4.3.1.7. The Baez-Dolan stabilization hypothesis was originally stated by Baez and Dolan in [BD95]. A version of it was proved by Simpson [Sim98], who showed that for $k \geq n+2$ a k-tuply monoidal n-category can be "delooped" to a $(k+1)$-tuply monoidal n-category; the ∞-categorical version above extends this by showing that this construction gives an equivalence of ∞-categories.

4.3.2 $\mathbb{E}_n$-Algebras as Enriched (∞,n)-Categories

We now prove that the natural map from associative algebras in a monoidal ∞-category $\mathcal{V}^{\otimes}$ to pointed complete $\mathcal{V}$-∞-categories is fully faithful; we then show by induction that the same is true for the natural map from $\mathbb{E}_n$-algebras to pointed complete $\mathcal{V}$-(∞,n)-categories.

Definition 4.3.2.1. We have a fully faithful inclusion

$$\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^{\otimes}) \hookrightarrow \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$$

since $\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^{\otimes})$ is the fibre of $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ at $*$ in $\mathcal{S}$. The unit of $\mathcal{V}^{\otimes}$ is the initial object of $\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^{\otimes})$, so this functor factors through $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{E^0/}$. Composing this with the localization functor we get a functor $i\colon \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^{\otimes}) \to (\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}$.

Proposition 4.3.2.2. The functor $i\colon \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}}(\mathcal{V}^{\otimes}) \to (\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}$ is fully faithful.

Proof. Let A and B be two Δ^{op}-algebras in $\mathcal{V}^{\otimes}$. We have a fibre sequence

$$\mathrm{Map}_{(\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}}(i(A), i(B)) \to \mathrm{Map}_{\mathrm{Cat}^{\mathcal{V}}_{\infty}}(i(A), i(B)) \to \mathrm{Map}_{\mathrm{Cat}^{\mathcal{V}}_{\infty}}(E^0, i(B)).$$

Let $\widehat{B}$ be the completion of B, regarded as a $\mathcal{V}$-∞-category. Then we have equivalences

$$\mathrm{Map}_{\mathrm{Cat}^{\mathcal{V}}_{\infty}}(i(A), i(B)) \simeq \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})}(A, \widehat{B})$$

and

$$\mathrm{Map}_{\mathrm{Cat}^{\mathcal{V}}_{\infty}}(E^0, i(B)) \simeq \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})}(E^0, \widehat{B}).$$

The projection $\iota_0\colon \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes}) \to \mathcal{S}$ gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})}(A, \widehat{B}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})}(E^0, \widehat{B}) \arrow[d] \\
\mathrm{Map}_{\mathcal{S}}(*, \iota_0\widehat{B}) \arrow[r] & \mathrm{Map}_{\mathcal{S}}(*, \iota_0\widehat{B})
\end{tikzcd}$$

where the right vertical map is an equivalence by Lemma 4.2.1.5 and the bottom horizontal map is the identity, since $E^0 \to A$ is the identity on ι_0. Thus we can identify the fibre of the top horizontal map at the functor $E^0 \to \widehat{B}$ corresponding to a point $p\colon * \to \iota_0\widehat{B}$ with the corresponding fibre of the left vertical map, which is $\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{V}^{\otimes})}(A, p^*\widehat{B})$ by [Lur09a, Proposition 2.4.4.2].

Take p to be the underlying map of spaces of the completion functor $B \to \widehat{B}$; since this is fully faithful the induced map $B \to p^*\widehat{B}$ is an equivalence, and in particular

$$\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{V}^{\otimes})}(A, B) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{V}^{\otimes})}(A, p^*\widehat{B}).$$

Thus the map $\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{V}^{\otimes})}(A, B) \to \mathrm{Map}_{(\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}}(i(A), i(B))$ is also an equivalence, i.e. i is fully faithful. $\square$

Remark 4.3.2.3. A pointed $\mathcal{V}$-∞-category $\mathcal{C}$ is in the essential image of the functor i if and only if $\iota\mathcal{C}$ is connected, since then the functor $p^*\mathcal{C} \to \mathcal{C}$ induced by the chosen point $p : * \to \iota_0\mathcal{C}$ is fully faithful and essentially surjective, and $p^*\mathcal{C}$ is a $\mathbb{\Delta}^{\mathrm{op}}$-algebra. In other words, $\mathbb{\Delta}^{\mathrm{op}}$-algebras in $\mathcal{V}^{\otimes}$ are equivalent to $\mathcal{V}$-∞-categories with a single object.

Definition 4.3.2.4. By Proposition 3.2.4.7, monoidal ∞-categories are equivalent to $\mathbb{E}_1^{\otimes}$-monoidal ∞-categories, and $\mathbb{\Delta}^{\mathrm{op}}$-algebras in a monoidal ∞-category are equivalent to $\mathbb{E}_1^{\otimes}$-algebras in the associated $\mathbb{E}_1^{\otimes}$-monoidal ∞-category. Since $\mathbb{E}_n^{\otimes} \otimes \mathbb{E}_m^{\otimes} \simeq \mathbb{E}_{n+m}^{\otimes}$ for all n, m by [Lur11, Theorem 5.1.2.2], we get maps

$$\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_n^{\otimes}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}}(\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_1}(\mathcal{V}^{\otimes})^{\otimes}) \to \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes}_{E^0/}).$$

We can identify $(\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}$ with $\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_0^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes})$, so

$$\begin{aligned}
\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes}_{E^0/}) &\simeq \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}}(\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_0}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes})^{\otimes}) \\
&\simeq \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}\otimes\mathbb{E}_0^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes}) \\
&\simeq \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes}).
\end{aligned}$$

Thus we have maps

$$\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_n^{\otimes}}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_{n-1}^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{\infty})^{\otimes}) \to \cdots \to \mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_1^{\otimes}}((\mathrm{Cat}^{\mathcal{V}}_{(\infty,n-1)})^{\otimes}) \to (\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)})_{E^0/}.$$

Applying Proposition 4.3.2.2 inductively, we get the following:

Corollary 4.3.2.5. Suppose $\mathcal{V}$ is an $\mathbb{E}_n$-monoidal ∞-category. Then the composite functor

$$\mathrm{Alg}^{\mathbb{F}}_{\mathbb{E}_n^{\otimes}}(\mathcal{V}^{\otimes}) \to (\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)})_{E^0/}$$

is fully faithful.

4.4 Comparisons

Monoidal model categories are an important source of monoidal ∞-categories. One of our main goals in this section is to prove that if $\mathbf{V}$ is a nice monoidal model category, then the homotopy theory of $\mathbf{V}$-categories is equivalent to that of ∞-categories enriched in the monoidal ∞-category associated to $\mathbf{V}$. When the tensor product in $\mathbf{V}$ is the Cartesian product, we will use the same method to show that the latter is also equivalent to the homotopy theory of Segal categories enriched in $\mathbf{V}$. Our other main result is that $\mathcal{S}$-(∞, n)-categories are equivalent to n-fold Segal spaces.

In §4.4.1 we prove some technical results about ∞-categorical localizations of fibrations of categories, and in §4.4.2 we review some results on rectification of associative algebras in monoidal model categories. Then we carry out the comparison with enriched categories in §4.4.3 and the comparison with Segal categories in §4.4.4. Finally, in §4.4.5 we compare $\mathcal{S}$-(∞, n)-categories and n-fold Segal spaces.

Notation 4.4.0.6. In this section, if $\mathbf{V}$ is a model category we write $\mathbf{V}_\infty$ for the associated ∞-category. This can be constructed as the localization $\mathrm{N}\mathbf{V}^{\mathrm{cof}}[W^{-1}]$ where $\mathbf{V}^{\mathrm{cof}}$ is the full subcategory of $\mathbf{V}$ spanned by the cofibrant objects, and W is the class of weak equivalences in $\mathbf{V}$.

4.4.1 Fibrewise Localization

Suppose we have a functor of ordinary categories $F\colon \mathbf{C} \to \mathrm{Cat}$ together with a collection of weak equivalences in each category $F(c)$ that is preserved by the functors $F(f)$. Then we have two ways to construct an ∞-category over $\mathbf{C}$ where these weak equivalences are inverted: On the one hand we can invert the weak equivalences to get a functor $\mathbf{C} \to \mathrm{Cat}_\infty$, which corresponds to a coCartesian fibration $\mathcal{E} \to \mathbf{C}$. On the other hand, if $\mathbf{E} \to \mathbf{C}$ is a coGrothendieck fibration corresponding to F then there is a natural collection W of weak equivalences in $\mathbf{E}$ induced by those in the fibres, and we can invert these to get an ∞-category $\mathbf{E}[W^{-1}]$. Our main goal in this subsection is to prove that in this situation the natural map $\mathbf{E}[W^{-1}] \to \mathcal{E}$ is an equivalence of ∞-categories.

We will do this in two steps: first we show that the ∞-category $\mathcal{E}$ here is a fibrant replacement in the coCartesian model structure on $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$ for $\mathrm{N}\mathbf{E}$ equipped with a certain collection M of marked edges, and then we use an explicit model for $\mathbf{E}[W^{-1}]$ to show that this, equipped with a natural choice of marked edges, is also weakly equivalent to $(\mathrm{N}\mathbf{E}, M)$. In addition, we will prove that when the weak equivalences in each category $F(c)$ come from a (combinatorial) model structure, then there is a (combinatorial) model structure on $\mathbf{E}$ whose weak equivalences are the morphisms in W.

Let's explain the first step more precisely. Recall that a *relative category* is a category $\mathbf{C}$ equipped with a collection of "weak equivalences", i.e. a subcategory W containing all objects and isomorphisms. Write RelCat for the obvious category of relative categories; this has been studied as a model for the theory of $(\infty, 1)$-categories by Barwick and Kan [BK12]. The usual nerve functor from categories to simplicial sets extends to a functor $L\colon \mathrm{RelCat} \to \mathrm{Set}_\Delta^+$ that sends $(\mathbf{C}, W)$ to $(\mathrm{N}\mathbf{C}, \mathrm{N}W_1)$. In the model structure on Set_Δ^+, a fibrant replacement for $L(\mathbf{C}, W)$ is given by the ∞-categorical localization of $\mathbf{C}$ that inverts the morphisms in W (marked by the equivalences).

In [Lur09a, §3.5.2] Lurie describes a right Quillen equivalence $\mathrm{N}^+_{\mathbf{C}}$ from the projective model structure on $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$ to the coCartesian model structure on $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$. Given a functor $F\colon \mathbf{B} \to \mathrm{RelCat}$ we therefore have two reasonable ways to construct a fibrant object of $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$:

(i) Find a fibrant replacement $\bar{F}$ for the functor $LF\colon \mathbf{C} \to \mathrm{Set}^+_\Delta$, and then form $\mathrm{N}^+_{\mathbf{C}}\bar{F}$.

(ii) Construct a coGrothendieck fibration $\mathbf{E} \to \mathbf{C}$ associated to F, regarded as a functor to categories, and write S for the collection of 1-simplices in $\mathrm{N}\mathbf{E}$ that correspond to composites of (fibrewise) weak equivalences and coCartesian morphisms. Then find a fibrant replacement in $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ for $(\mathrm{N}\mathbf{E}, S) \to \mathrm{N}\mathbf{C}$.

The precise statement of our first goal in this subsection is to prove that these give weakly equivalent objects. We begin by reviewing the definition of the functor $\mathrm{N}^+_{\mathbf{C}}$:

Definition 4.4.1.1. Let $\mathbf{C}$ be a category. Given a functor $F\colon \mathbf{C} \to \mathrm{Set}_\Delta$, we define $\mathrm{N}_{\mathbf{C}}F$ to be the simplicial set characterized by the property that a morphism $\Delta^I \to \mathrm{N}_{\mathbf{C}}F$, where I is a partially ordered set, is determined by:

(1) a functor $\sigma\colon I \to \mathbf{C}$,

(2) for every non-empty subset $J \subseteq I$ with maximal element j, a map $\tau_J\colon \Delta^J \to F(\sigma(j))$,

such that for all subsets $K \subseteq J \subseteq I$ with maximal elements $k \in K$ and $j \in J$, the diagram

$$\begin{tikzcd} \Delta^K \arrow[r, "\tau_K"] \arrow[d] & F(\sigma(k)) \arrow[d] \\ \Delta^J \arrow[r, "\tau_J"'] & F(\sigma(j)) \end{tikzcd}$$

commutes. This defines a functor $\mathrm{N}_{\mathbf{C}}\colon \mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta) \to (\mathrm{Set}_\Delta)_{/\mathrm{N}\mathbf{C}}$.

The functor $\mathrm{N}_{\mathbf{C}}$ has a left adjoint, which we denote

$$\mathfrak{F}_{\mathbf{C}}\colon (\mathrm{Set}_\Delta)_{/\mathrm{N}\mathbf{C}} \to \mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta).$$

Proposition 4.4.1.2. Let $\pi\colon \mathbf{E} \to \mathbf{C}$ be a functor. Then $\mathfrak{F}_{\mathbf{C}}\mathrm{N}\mathbf{E}$ is isomorphic to the functor $O_\pi\colon \mathbf{C} \to \mathrm{Set}_\Delta$ defined by $c \mapsto \mathrm{N}\mathbf{E}_{/c}$.

Proof. We must show that there is a natural isomorphism $\mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}(-)) \cong \mathrm{Hom}(O_\pi, -)$; we will do this by defining explicit natural transformations

$$\phi\colon \mathrm{Hom}(O_\pi, -) \to \mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}(-))$$

and

$$\psi\colon \mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}(-)) \to \mathrm{Hom}(O_\pi, -)$$

that are inverse to each other.

Given $X\colon \mathbf{C} \to \mathrm{Set}_\Delta$ and a natural transformation $\eta\colon O_\pi \to X$, define $\phi(\eta)\colon \mathrm{N}\mathbf{E} \to \mathrm{N}_{\mathbf{C}}X$ to be the morphism that sends a simplex $\sigma\colon \Delta^I \to \mathrm{N}\mathbf{E}$ (which we can identify with a functor $I \to \mathbf{E}$) to the simplex of $\mathrm{N}_{\mathbf{C}}X$ determined by

- the composite functor $I \to \mathbf{E} \to \mathbf{C}$,

- for $J \subseteq I$ with maximal element j, the composite $\Delta^J \to \mathrm{N}E_{/\pi(\sigma(j))} \xrightarrow{\eta_{\pi(\sigma(j))}} X(\pi(\sigma(j)))$.

Conversely, given a map $G\colon \mathrm{N}\mathbf{E} \to \mathrm{N}_{\mathbf{C}}X$ of simplicial sets over $\mathrm{N}\mathbf{C}$, let $\psi(G)$ be the natural transformation $O_\pi \to X$ determined as follows: for $c \in \mathbf{C}$, the morphism $\psi(G)_c\colon \mathrm{N}E_{/c} \to X(c)$ sends a simplex $\sigma\colon \Delta^I \to \mathrm{N}E_{/c}$, where I has maximal element i, to the composite

$$\Delta^I \xrightarrow{\tau} X(\pi\sigma(i)) \xrightarrow{X(f)} X(c)$$

where

- τ is the I-simplex determined by the image under G of the I-simplex σ' of $\mathrm{N}\mathbf{E}$ underlying σ,

- f is the morphism $\pi(\sigma(i)) \to c$ in $\mathbf{C}$ from σ.

The remaining data in $G \circ \sigma'$ implies that this defines a map of simplicial sets $\mathrm{N}\mathbf{E}_{/c} \to X(c)$, and it is also easy to see that $\psi(G)$ is natural in c.

Both ϕ and ψ are obviously natural in X, and expanding out the definitions we see that $\phi\psi = \mathrm{id}$ and $\psi\phi = \mathrm{id}$, so we have the required natural isomorphism. □

Definition 4.4.1.3. Let $\mathbf{C}$ be a category. Given a functor $\bar{F}\colon \mathbf{C} \to \mathrm{Set}_\Delta^+$ we define $\mathrm{N}_{\mathbf{C}}^+\bar{F}$ to be the marked simplicial set $(\mathrm{N}_{\mathbf{C}}F, M)$ where F is the underlying functor $\mathbf{C} \to \mathrm{Set}_\Delta$ of $\bar{F}$, and M is the set of edges $\Delta^1 \to \mathrm{N}_{\mathbf{C}}F$ determined by

- a morphism $f\colon c \to c'$ in $\mathbf{C}$,

- a vertex $x \in F(c)$,

- a vertex $x' \in F(c')$ and an edge $F(f)(x) \to x'$ that is marked in $\bar{F}(c')$.

This determines a functor $\mathrm{N}_{\mathbf{C}}^+\colon \mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta^+) \to (\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$.

The functor $\mathrm{N}_{\mathbf{C}}^+$ has a left adjoint, which we denote $\mathfrak{F}_{\mathbf{C}}^+$.

Corollary 4.4.1.4. Let $\pi\colon \mathbf{E} \to \mathbf{C}$ be a functor, and let M be a set of edges of $\mathrm{N}\mathbf{E}$ that contains the degenerate edges. Then $\mathfrak{F}_{\mathbf{C}}^+(\mathrm{N}\mathbf{E}, M)$ is isomorphic to the functor $\bar{O}_\pi$ defined by $c \mapsto (\mathrm{N}\mathbf{E}_{/c}, M_c)$, where M_c is the collection of edges determined by $e \to e'$ in $\mathbf{E}$ and $\pi(e) \to \pi(e') \to c$ in $\mathbf{C}$ such that $\pi(e') = c$ and $e \to e'$ is in M.

Proof. We must show that there is a natural isomorphism

$$\mathrm{Hom}((\mathrm{N}\mathbf{E}, M)\mathrm{N}_{\mathbf{C}}^+(-)) \cong \mathrm{Hom}(\bar{O}_\pi, -).$$

Given $\bar{X}\colon \mathbf{C} \to \mathrm{Set}_\Delta^+$, with underlying functor $X\colon \mathbf{C} \to \mathrm{Set}_\Delta$, and a morphism $G\colon \mathrm{N}\mathbf{E} \to \mathrm{N}_{\mathbf{C}}X$, it is immediate from the definitions that G takes an edge $\sigma\colon e \to e'$ of $\mathrm{N}\mathbf{E}$ lying over $c \to c'$ in $\mathbf{C}$ to a marked edge of $\mathrm{N}_{\mathbf{C}}^+\bar{X}$ if and only if $\phi(G)_{c'}$ takes σ, regarded as an edge of $\mathrm{N}\mathbf{E}_{/c'}$, to a marked edge of $\bar{X}(c')$. Thus the natural isomorphism $\mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}X) \cong \mathrm{Hom}(O_\pi, X)$ of Proposition 4.4.1.2 identifies $\mathrm{Hom}((\mathrm{N}\mathbf{E}, M), \mathrm{N}_{\mathbf{C}}^+\bar{X})$, regarded as a subset of $\mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}X)$, with $\mathrm{Hom}(\bar{O}_\pi, \bar{X})$, regarded as a subset of $\mathrm{Hom}(O_\pi, X)$. □

Theorem 4.4.1.5 (Lurie, [Lur09a, Proposition 3.2.5.18])**.**

(i) The adjunction $\mathfrak{F}_{\mathbf{C}} \dashv \mathrm{N}_{\mathbf{C}}$ is a Quillen equivalence between $(\mathrm{Set}_\Delta)_{/\mathrm{N}\mathbf{C}}$ equipped with the covariant model structure and $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta)$ equipped with the projective model structure.

(ii) The adjunction $\mathfrak{F}^+_{\mathbf{C}} \dashv \mathrm{N}^+_{\mathbf{C}}$ is a Quillen equivalence between $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ equipped with the coCartesian model structure and $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$ equipped with the projective model structure.

Recall that if $\mathcal{C}$ is an ∞-category we write $\mathcal{C}^\natural$ for the marked simplicial set given by $\mathcal{C}$ marked by the equivalences, and that if $\mathcal{E} \to \mathrm{N}\mathbf{C}$ is a coCartesian fibration we write $\mathcal{E}^\natural$ for the object of $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ given by $\mathcal{E}$ marked by the coCartesian morphisms.

Lemma 4.4.1.6. Let $F\colon \mathbf{C} \to \mathrm{Cat}$ be a functor. Write $\pi\colon \mathbf{E} \to \mathbf{C}$ for the coGrothendieck fibration associated to F, so that $\mathbf{E}$ has objects pairs $(c \in \mathbf{C}, x \in F(c))$ and a morphism $(c, x) \to (d, y)$ in $\mathbf{E}$ is given by a morphism $f\colon c \to d$ in $\mathbf{C}$ and a morphism $F(f)(x) \to y$ in $F(d)$. Then:

(i) $\mathrm{N}_{\mathbf{C}}(\mathrm{N}F) \to \mathrm{N}\mathbf{C}$ is isomorphic to $\mathrm{N}\pi$.

(ii) $\mathrm{N}^+_{\mathbf{C}}(\mathrm{N}F^\natural) \to \mathrm{N}\mathbf{C}$ is isomorphic to $(\mathrm{N}\mathbf{E})^\natural \to \mathrm{N}\mathbf{C}$.

Proof. It is clear from the definition of $\mathrm{N}_{\mathbf{C}}$ that there is a natural isomorphism between n-simplices of $\mathrm{N}_{\mathbf{C}}(\mathrm{N}F)$ and n-simplices of $\mathrm{N}\mathbf{E}$, which proves (i). From Corollary 4.4.1.4, an edge of $\mathrm{N}^+_{\mathbf{C}}(\mathrm{N}F^\natural)$ is marked if it is given by $f\colon c \to c'$ in $\mathbf{C}$, $x \in F(c)$ and $F(f)(x) \to x'$ an isomorphism in $F(c')$. Under the identification with edges of $\mathrm{N}\mathbf{E}$, such edges precisely correspond to the coCartesian edges. This proves (ii). $\square$

Proposition 4.4.1.7. Given $F\colon \mathbf{C} \to \mathrm{RelCat}$, the counit map $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF \to LF$ is a weak equivalence in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$.

Proof. Since $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$ is equipped with the projective model structure, it suffices to show that for all $c \in \mathbf{C}$ the morphism $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF(c) \to LF(c)$ is a weak equivalence in Set^+_Δ. Let F_0 be the underlying functor $\mathbf{C} \to \mathrm{Cat}$, and let $\mathbf{E} \to \mathbf{C}$ be the canonical coGrothendieck fibration associated to F_0. Then by Lemma 4.4.1.6 we can identify $\mathrm{N}^+_{\mathbf{C}}\mathrm{N}F_0^\natural$ with $\mathrm{N}\mathbf{E}^\natural$, and so by Corollary 4.4.1.4 we can identify $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}\mathrm{N}F_0^\natural(c)$ with $\mathrm{N}\mathbf{E}_{/c}$, marked by the set M_c of coCartesian morphisms $e \to e'$ such that $\pi(e') = c$.

The adjunction $\mathfrak{F}^+_{\mathbf{C}} \dashv \mathrm{N}^+_{\mathbf{C}}$ is a Quillen equivalence, so since $\mathrm{N}F_0^\natural$ is fibrant and every object of $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ is cofibrant, the counit $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}\mathrm{N}F_0^\natural \to \mathrm{N}F_0^\natural$ is a weak equivalence in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$. In particular, $(\mathrm{N}\mathbf{E}_{/c}, M_c) \to \mathrm{N}F_0(c)^\natural$ is a weak equivalence.

Let M'_c be the set of edges of $\mathrm{N}\mathbf{E}_{/c}$ corresponding to weak equivalences in $F(c)$. Then we have a pushout diagram

$$\begin{tikzcd}
(\mathrm{N}\mathbf{E}_{/c}, M_c) \arrow[r] \arrow[d] & \mathrm{N}F_0(c)^\natural \arrow[d] \\
(\mathrm{N}\mathbf{E}_{/c}, M_c \cup M'_c) \arrow[r] & LF(c),
\end{tikzcd}$$

since both vertical maps are pushouts along $\coprod_{f \in M'_c} \Delta^1 \hookrightarrow \coprod_{f \in M'_c} (\Delta^1)^\sharp$. As the model structure on Set^+_Δ is left proper, it follows that $(\mathrm{N}\mathbf{E}_{/c}, M_c \cup M'_c) \to LF(c)$ is a weak equivalence.

By Corollary 4.4.1.4 we can identify $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF(c)$ with the simplicial set $\mathrm{N}\mathbf{E}_{/c}$, marked by the set M''_c of morphisms $e \to e'$ with $\pi(e') = c$ such that given a coCartesian factorization $e \to \bar{e} \to e'$ the morphism $\bar{e} \to e'$ is a weak equivalence in $LF(c)$. The obvious map $(\mathrm{N}\mathbf{E}_{/c}, M_c \cup M'_c) \to \mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF(c)$ is therefore marked anodyne, since the edges in M''_c are precisely the composites of edges in M_c and M'_c. In particular this is also a weak equivalence, and so by the 2-out-of-3 property the map $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF(c) \to LF(c)$ is a weak equivalence, as required. □

Corollary 4.4.1.8. Given $F\colon \mathbf{C} \to \mathrm{RelCat}$, let $LF \to \bar{F}$ be a fibrant replacement in the projective model structure on $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$. Then $\mathrm{N}^+_{\mathbf{C}}LF \to \mathrm{N}^+_{\mathbf{C}}\bar{F}$ is a coCartesian equivalence in $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$.

Proof. The adjunction $\mathfrak{F}^+_{\mathbf{C}} \dashv \mathrm{N}^+_{\mathbf{C}}$ is a Quillen equivalence, so since $\bar{F}$ is fibrant and every object of $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ is cofibrant, the morphism $\mathrm{N}^+_{\mathbf{C}}LF \to \mathrm{N}^+_{\mathbf{C}}\bar{F}$ is a weak equivalence if and only if the adjunct morphism $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF \to \bar{F}$ is a weak equivalence. This follows by the 2-out-of-3 property, since in the commutative diagram

$$\begin{tikzcd} \mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF \arrow[rr] \arrow[dr] & & LF \arrow[dl] \\ & \bar{F} & \end{tikzcd}$$

the morphism $LF \to \bar{F}$ is a weak equivalence by assumption, and $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF \to LF$ is a weak equivalence by Proposition 4.4.1.7. □

Using Lemma 4.4.1.6 we can equivalently state this as:

Corollary 4.4.1.9. Given $F\colon \mathbf{C} \to \mathrm{RelCat}$, suppose $\pi\colon \mathbf{E} \to \mathbf{C}$ is a coGrothendieck fibration corresponding to the underlying functor $\mathbf{C} \to \mathrm{Cat}$. Let M be the set of morphisms $f\colon e \to e'$ in $\mathbf{E}$ such that given a coCartesian factorization $e \to \pi(f)_!e \to e'$, the morphism $\pi(f)_!e \to e'$ is a weak equivalence in $F(\pi(e'))$. Then if $LF \to \bar{F}$ is a fibrant replacement in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$, there is a coCartesian equivalence $(\mathrm{N}\mathbf{E}, M) \to \mathrm{N}^+_{\mathbf{C}}\bar{F}$.

Our next goal is to prove that, with $F\colon \mathbf{C} \to \mathrm{RelCat}$ and $\pi\colon \mathbf{E} \to \mathbf{C}$ as above, inverting the collection W of fibrewise weak equivalences in $\mathbf{E}$ gives a coCartesian fibration $\mathbf{E}[W^{-1}] \to \mathbf{C}$. As a corollary, we will also see that $\mathbf{E}[W^{-1}]$ is the total space of the coCartesian fibration associated to the functor obtained from F by inverting the weak equivalences in the relative categories $F(c)$. We will prove this result by analyzing an explicit model for $\mathbf{E}[W^{-1}]$ as a simplicial category, namely the *hammock localization*. We now recall the definition of this, specifically the version defined in [DHKS04, §35], and its basic properties:

Definition 4.4.1.10. A *zig-zag type* $Z = (Z_+, Z_-)$ consists of a decomposition $\{1, \ldots, n\} = Z_+ \amalg Z_-$. The *zig-zag category* $\mathbf{ZZ}$ is the category with objects zig-zag types and morphisms $Z \to Z'$ given by order-preserving morphisms $f\colon \{1, \ldots, n\} \to \{1, \ldots, n'\}$ such that $f(Z_+) \subseteq Z'_+$ and $f(Z_-) \subseteq Z'_-$. If Z is a zig-zag type, the associated zig-zag category $|Z|$ is the category with objects $0, \ldots, n$ and

$$|Z|(i,j) = \begin{cases} *, & i \leq j, k \in Z_+ \text{ for } k = i+1, \ldots, j, \\ *, & i \geq j, k \in Z_- \text{ for } k = j+1, \ldots, i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

This clearly gives a functor $|{-}|\colon \mathbf{ZZ} \to \mathrm{Cat}$. If n is an odd integer, we abbreviate

$$\langle n\rangle := (\{2,4,\dots,n-1\},\{1,3,\dots,n\})$$

and if n is an even integer we abbreviate

$$\langle n\rangle := (\{1,3,\dots,n-1\},\{2,4,\dots,n\}).$$

Definition 4.4.1.11. Suppose $(\mathbf{C},W)$ is a relative category. For $x,y \in \mathbf{C}$ and $Z \in \mathbf{ZZ}$ we define $\mathbf{L}_W\mathbf{C}_Z(x,y)$ to be the subcategory of $\mathrm{Fun}(|Z|,\mathbf{C})$ whose objects are the functors $F\colon |Z| \to \mathbf{C}$ such that $F(0) = x$, $F(n) = y$, and $F(i \to (i-1))$ is in W for all $i \in Z_-$, and whose morphisms are the natural transformations $\eta\colon F \to G$ such that $\eta_0 = \mathrm{id}_x$, $\eta_n = \mathrm{id}_y$, and η_i is in W for all i. We write $\mathcal{L}_W\mathbf{C}_Z(x,y) := \mathrm{N}\mathbf{L}_W\mathbf{C}_Z(x,y)$.

This construction gives a functor $\mathbf{ZZ}^{\mathrm{op}} \to \mathrm{Cat}$; we let $\mathbf{L}_W\mathbf{C}(x,y) \to \mathbf{ZZ}$ be the fibration associated to it by the Grothendieck construction. Using concatenation of zig-zags we get a strict 2-category $\mathbf{L}_W\mathbf{C}$ with the same objects as $\mathbf{C}$ and with mapping categories $\mathbf{L}_W\mathbf{C}(x,y)$; taking nerves, this gives a simplicial category $\mathcal{L}_W\mathbf{C}$ whose mapping spaces are $\mathcal{L}_W\mathbf{C}(x,y) := \mathrm{N}\mathbf{L}_W\mathbf{C}(x,y)$. This simplicial category is the *hammock localization* of $(\mathbf{C},W)$.

Theorem 4.4.1.12 (Dwyer-Kan)**.** Let $(\mathbf{C},W)$ be a relative category. Then:

(i) The diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathcal{L}_W W \arrow[d] \\ \mathbf{C} \arrow[r] & \mathcal{L}_W\mathbf{C} \end{tikzcd}$$

is a homotopy pushout square in simplicial categories.

(ii) If $\mathcal{L}_W W \to \bar{\mathcal{L}}_W W$ is a fibrant replacement in simplicial categories, then $\mathrm{N}\bar{\mathcal{L}}_W W$ is a Kan complex and $\mathrm{N}W \to \mathrm{N}\bar{\mathcal{L}}_W W$ is a weak equivalence of simplicial sets.

Proof.

(i) This follows by combining [DHKS04, Proposition 35.7], [DK80b, Proposition 2.2], and [DK80a, §4.5] (observe that a cofibration in the model structure on simplicial categories with a fixed set of objects described in [DK80a, §7] is also a cofibration in the model structure on simplicial categories).

(ii) It follows from [DK80a, §9.1] that $\mathcal{L}_W W$ is a simplicial groupoid. If $\mathcal{L}_W W \to \bar{\mathcal{L}}_W W$ is a fibrant replacement in simplicial categories, then $\mathrm{N}\bar{\mathcal{L}}_W W$ is the nerve of a fibrant simplicial groupoid, hence a Kan complex by [DK84, Theorem 3.3]. Let $\mathfrak{G}$ denote the left adjoint to the nerve of simplicial groupoids, as defined in [DK84, §3.1]; by [DK84, Theorem 3.3] the morphism $\mathrm{N}W \to \mathrm{N}\bar{\mathcal{L}}_W W$ is a weak equivalence if and only if the adjunct $\mathfrak{G}\mathrm{N}W \to \bar{\mathcal{L}}_W W$ is a weak equivalence of simplicial groupoids. This follows from [DK80a, §5.5], since this implies that the mapping spaces in both are the appropriate loop spaces of $\mathrm{N}W$. □

Corollary 4.4.1.13. Let $(\mathbf{C},W)$ be a relative category. Suppose $\mathcal{L}_W\mathbf{C} \to \bar{\mathcal{L}}_W\mathbf{C}$ is a fibrant replacement in the model category of simplicial categories. Then

$$L(\mathbf{C},W) \to \mathrm{N}\bar{\mathcal{L}}_W\mathbf{C}^{\natural}$$

is a weak equivalence in Set_Δ^+.

Proof. We must show that for every ∞-category $\mathcal{D}$, the induced map

$$\mathrm{Map}_{\mathrm{Set}_\Delta^+}(\mathcal{C}^\natural, \mathcal{D}^\natural) \to \mathrm{Map}_{\mathrm{Set}_\Delta^+}(L(\mathbf{C}, W), \mathcal{D}^\natural)$$

is a weak equivalence of simplicial sets. Observe that

$$\mathrm{Map}_{\mathrm{Set}_\Delta^+}(L(\mathbf{C}, W), \mathcal{D}^\natural) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}\mathbf{C}, \mathcal{D}) \times_{\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}W,\mathcal{D})} \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}W, \iota\mathcal{D})$$

and $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}W, \iota\mathcal{D}) \simeq \mathrm{Map}_{\mathcal{S}}(\overline{\mathrm{N}W}, \iota\mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\overline{\mathrm{N}W}, \mathcal{D})$, where $\mathrm{N}W \to \overline{\mathrm{N}W}$ denotes a fibrant replacement in the usual model structure on simplicial sets, so this is equivalent to requiring

$$\begin{tikzcd}
\mathrm{N}W \arrow[r] \arrow[d] & \overline{\mathrm{N}W} \arrow[d] \\
\mathrm{N}\mathbf{C} \arrow[r] & \mathrm{N}\bar{\mathcal{L}}_W\mathbf{C}
\end{tikzcd}$$

to be a homotopy pushout square. Theorem 4.4.1.12(i) implies that

$$\begin{tikzcd}
\mathrm{N}W \arrow[r] \arrow[d] & \mathrm{N}\bar{\mathcal{L}}_W W \arrow[d] \\
\mathrm{N}\mathbf{C} \arrow[r] & \mathrm{N}\bar{\mathcal{L}}_W\mathbf{C}
\end{tikzcd}$$

is a homotopy pushout square, since N is a right Quillen equivalence and all the objects are fibrant. By Theorem 4.4.1.12(ii) we also have that $\mathrm{N}W \to \mathrm{N}\bar{\mathcal{L}}_W W$ is a fibrant replacement in the usual model structure on simplicial sets, so the result follows. □

We now fix a functor $F\colon \mathbf{C} \to \mathrm{RelCat}$, and let $\pi\colon \mathbf{E} \to \mathbf{C}$ be a coGrothendieck fibration associated to the underlying functor $\mathbf{C} \to \mathrm{Cat}$. We say a morphism $\bar{f}\colon x \to y$ in $\mathbf{E}$ lying over $f\colon a \to b$ in $\mathbf{C}$ is a *weak equivalence* if f is an isomorphism and $f_! x \to y$ is a weak equivalence in $F(b)$; write W for the subcategory of $\mathbf{E}$ whose morphisms are the weak equivalences. Our goal is to show that the nerve of $\mathcal{L}_W\mathbf{E} \to \mathbf{C}$ is (equivalent to) a coCartesian fibration. To prove this we need a technical hypothesis on the relative categories $F(c)$:

Definition 4.4.1.14. A relative category $(\mathbf{C}, W)$ satisfies the *two-out-of-three property* if given morphisms $r\colon A \to B$ and $s\colon B \to C$ such that two out of $r, s, s \circ r$ are in W, then so is the third.

Definition 4.4.1.15. We say that a relative category $\overline{\mathbf{C}} = (\mathbf{C}, W)$ is a *partial model category* if $\overline{\mathbf{C}}$ satisfies the two-out-of-three property and $\overline{\mathbf{C}}$ admits a *three-arrow calculus*, i.e. there exist subcategories $U, V \subseteq W$ such that

(i) for every zig-zag $A' \xleftarrow{u} A \xrightarrow{f} B$ in $\mathbf{C}$ with $u \in U$, there exists a functorial zig-zag $A' \xrightarrow{f'} B' \xleftarrow{u'} B$ with $u' \in U$ such that $u'f = f'u$ and u' is an isomorphism if u is,

(ii) for every zig-zag $X \xrightarrow{g} Y' \xleftarrow{v} Y$ in $\mathbf{C}$ with $v \in V$, there exists a functorial zig-zag $X \xleftarrow{v'} X' \xrightarrow{g'} Y$ with $v' \in V$ such that $gv' = vg'$ and v' is an isomorphism if v is,

(iii) every map $w \in W$ admits a functorial factorization $w = vu$ with $u \in U$ and $v \in V$.

Remark 4.4.1.16. If $\mathbf{M}$ is a model category (with functorial factorizations), then the relative category obtained by equipping $\mathbf{M}$ with the weak equivalences in the model structure is a partial model category. Similarly, the relative categories obtained from the full subcategories $\mathbf{M}^{\mathrm{cof}}$ of cofibrant objects, $\mathbf{M}^{\mathrm{fib}}$ of fibrant objects, and $\mathbf{M}^{\circ}$ of fibrant-cofibrant objects together with the weak equivalences between these objects are all partial model categories. The term "partial model category" is taken from [BK], but we use the more general definition of [DHKS04, 36.1] since the more restrictive definition of Barwick and Kan does not include what is for us the key example, namely $\mathbf{M}^{\mathrm{cof}}$ for $\mathbf{M}$ a model category.

Theorem 4.4.1.17 (Dwyer-Kan)**.** Suppose $(\mathbf{C}, W)$ is a partial model category. Then for every pair of objects $X, Y \in \mathbf{C}$, the morphism $\mathcal{L}_W\mathbf{C}_{\langle n \rangle}(X,Y) \to \mathcal{L}_W\mathbf{C}(X,Y)$ is a weak equivalence of simplicial sets for all $n \geq 3$.

Proof. For $n = 3$ this is [DK80b, Proposition 6.2(i)]; the general case follows similarly. □

Proposition 4.4.1.18. Suppose $F\colon \mathbf{C} \to \mathrm{RelCat}$ is a functor such that $F(C)$ is a partial model category for each $C \in \mathbf{C}$. Let $\phi\colon A \to B$ be a morphism in $\mathbf{C}$, and let X and Y be objects of $\mathbf{E}_A$ and $\mathbf{E}_B$, respectively. Write $\mathcal{L}_W\mathbf{E}(X,Y)_\phi$ for the subspace of $\mathcal{L}_W\mathbf{E}(X,Y)$ over ϕ. The morphism

$$\bar{\phi}^*\colon \mathcal{L}_W\mathbf{E}_B(\phi_!X, Y) \to \mathcal{L}_W\mathbf{E}(X,Y)_\phi$$

given by composition with a coCartesian morphism $\bar{\phi}\colon X \to \phi_!X$ is a weak equivalence of simplicial sets.

Proof. It is easy to see that $\mathbf{E}$ is also a partial model category. The maps $\mathcal{L}_W\mathbf{E}_{\langle 4 \rangle}(X,Y)_\phi \to \mathcal{L}_W\mathbf{E}(X,Y)_\phi$ and $\mathcal{L}_W(\mathbf{E}_B)_{\langle 4 \rangle}(\phi_!X, Y) \to \mathcal{L}_W\mathbf{E}_B(\phi_!X,Y)$ are therefore weak equivalences by Theorem 4.4.1.17. Since composition with $\bar{\phi}$ gives a functor $\bar{\phi}^*\colon \mathbf{L}_B := \mathbf{L}_W(\mathbf{E}_B)_{\langle 4 \rangle}(\phi_!X,Y) \to \mathbf{L}_W\mathbf{E}_{\langle 4 \rangle}(X,Y)_\phi =: \mathbf{L}$ it therefore suffices to prove that this gives a weak equivalence upon taking nerves.

We will prove this in two steps. Let $\mathbf{L}^1$ denote the full subcategory of $\mathbf{L}$ spanned by objects

$$X = X_0 \xrightarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xleftarrow{f_4} X_4 = Y$$

such that $X_i \in \mathbf{E}_B$ for $i \geq 1$ and f_i lies over id_B in $\mathbf{C}$ for $i \geq 2$. Then $\bar{\phi}^*$ factors as

$$\mathbf{L}_B \xrightarrow{f} \mathbf{L}^1 \xrightarrow{i} \mathbf{L};$$

we will show that each of these functors gives a weak equivalence of nerves.

First we consider $f\colon \mathbf{L}_B \to \mathbf{L}^1$, given by composition with $\bar{\phi}$. Define $q\colon \mathbf{L}^1 \to \mathbf{L}_B$ by sending a zig-zag

$$X \xrightarrow{g} Z \leftarrow Z' \to Y' \leftarrow Y$$

in $\mathbf{L}^1$ to

$$\phi_!X \xrightarrow{g'} Z \leftarrow Z' \to Y' \leftarrow Y$$

where $X \xrightarrow{\bar{\phi}} \phi_! X \xrightarrow{g'} Z$ is the coCartesian factorization of g (which exists since the other maps lie over id_B). Then it is clear that $qf \simeq \mathrm{id}$ and $fq \simeq \mathrm{id}$, so f is an equivalence of categories.

Next we want to define a functor $p\colon \mathbf{L} \to \mathbf{L}^1$. Given a zig-zag

$$X \xrightarrow{g} Z' \leftarrow Z \xrightarrow{h} Y' \leftarrow Y$$

in $\mathbf{L}$, this lies over

$$A \to C' \xleftarrow{\gamma} C \to B' \xleftarrow{\beta} B$$

where γ and β are isomorphisms, since weak equivalences in $\mathbf{E}$ map to isomorphisms in $\mathbf{C}$. Thus the coCartesian maps $Z' \to \gamma_!^{-1} Z'$ and $B' \to \beta_!^{-1} B'$ are isomorphisms, and our zig-zag is isomorphic to the zig-zag

$$X \to \gamma_!^{-1} Z' \leftarrow Z \to \beta_!^{-1} Y' \leftarrow Y.$$

To define p we may therefore assume that β and γ are identities, in which case p sends

$$X \xrightarrow{f} Z' \leftarrow Z \xrightarrow{g} Y' \leftarrow Y$$

lying over

$$A \xrightarrow{\alpha} C \xleftarrow{\mathrm{id}} C \xrightarrow{\psi} B \xleftarrow{\mathrm{id}} B$$

to

$$X \to \psi_! Z' \leftarrow \psi_! Z \to Y' \leftarrow Y$$

in $\mathbf{L}^1$; this is clearly functorial.

We wish to prove that p gives an inverse to i after taking nerves. It is obvious that $p \circ i \simeq \mathrm{id}$, so it suffices to show that $i \circ p$ is homotopic to the identity after taking nerves. To see this we consider the natural transformation $\eta\colon \mathbf{L} \to \mathrm{Fun}([1], \mathbf{L}_W\mathbf{E}_{\langle 6 \rangle}(x,y)_\phi)$ that sends our zig-zag to the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathrm{id}"] & Z' \arrow[d, "\mathrm{id}"] & Z \arrow[l] \arrow[r] \arrow[d] & \psi_! Z \arrow[d] & \psi_! Z \arrow[l, "\mathrm{id}"'] \arrow[r] \arrow[d, "\mathrm{id}"] & Y' \arrow[d, "\mathrm{id}"] & Y \arrow[l] \arrow[d, "\mathrm{id}"] \\
X \arrow[r] & Z' & Z' \arrow[l, "\mathrm{id}"] \arrow[r] & \psi_! Z' & \psi_! Z \arrow[l] \arrow[r] & Y' & Y, \arrow[l]
\end{tikzcd}$$

After composing with the inclusion $\mathbf{L}_W\mathbf{E}_{\langle 6 \rangle}(x,y)_\phi \to \mathbf{L}_W\mathbf{E}(x,y)_\phi$ the functor η_0 is clearly linked to the inclusion $\mathbf{L} \to \mathbf{L}_W\mathbf{E}(x,y)_\phi$ by a sequence of natural transformations, and similarly η_1 is linked to the composite of $i \circ p$ with this inclusion. Since natural transformations give homotopies of the induced maps between nerves it follows from Theorem 4.4.1.17 that the morphism on nerves induced by $i \circ p$ is homotopic to the identity. This completes the proof. $\square$

Corollary 4.4.1.19. Suppose $F\colon \mathbf{C} \to \mathrm{RelCat}$ is a functor such that $F(C)$ is a partial model category for each $C \in \mathbf{C}$. There is an ∞-category $\mathbf{E}[W^{-1}]$ such that $L(\mathbf{E}, W) \to \mathbf{E}[W^{-1}]^\natural$ is a weak equivalence in Set_Δ^+, and $\mathbf{E}[W^{-1}] \to \mathrm{N}\mathbf{C}$ is a coCartesian fibration.

Proof. Let $\mathcal{L}_W\mathbf{E} \to \bar{\mathcal{L}}_W\mathbf{E} \to \mathbf{C}$ denote a factorization of $\mathcal{L}_W\mathbf{E} \to \mathbf{C}$ as a trivial cofibration followed by a fibration in the model category of simplicial categories. Then $(\mathrm{N}\bar{\mathcal{L}}_W\mathbf{E})^\natural$ is a fibrant replacement for $L(\mathbf{E}, W)$ in Set_Δ^+. By [Lur09a, Proposition 2.4.4.3] to prove that $\mathrm{N}\bar{\mathcal{L}}_W\mathbf{E} \to \mathrm{N}\mathbf{C}$ is a coCartesian fibration it suffices to show that for each morphism $f\colon c \to d$ in $\mathbf{C}$ and each x in $\mathbf{E}_c$ we have a homotopy pullback square of simplicial sets

$$\begin{tikzcd}
\mathcal{L}_W\mathbf{E}(f_!x, y) \arrow[r] \arrow[d] & \mathcal{L}_W\mathbf{E}(x, y) \arrow[d] \\
\mathbf{C}(d, e) \arrow[r] & \mathbf{C}(c, e)
\end{tikzcd}$$

for all $e \in \mathbf{C}$ and $y \in \mathbf{E}_e$, where $\bar{f}\colon x \to f_!x$ denotes a coCartesian morphism in $\mathbf{E}$ over f.

Since the inclusion of a point in a discrete simplicial set is a Kan fibration and the model structure on simplicial sets is right proper, given $g\colon d \to e$ the fibres at $\{g\}$ and $\{g \circ f\}$ in this diagram are homotopy fibres. To see that the diagram is a homotopy pullback square it thus suffices to show that composition with $\bar{f}$ induces a weak equivalence

$$\mathcal{L}_W\mathbf{E}(f_!x, y)_g \to \mathcal{L}_W\mathbf{E}(x, y)_{gf}$$

for all $g\colon d \to e$. But by Proposition 4.4.1.18, in the commutative diagram

$$\begin{tikzcd}
 & \mathcal{L}_W\mathbf{E}_e((gf)_!x, y) \arrow[dl] \arrow[dr] & \\
\mathcal{L}_W\mathbf{E}(f_!x, y)_g \arrow[rr] & & \mathcal{L}_W\mathbf{E}(x, y)_{gf}
\end{tikzcd}$$

the diagonal morphisms are both weak equivalences, hence by the 2-out-of-3 property so is the horizontal morphism. $\square$

Corollary 4.4.1.20. Suppose $F\colon \mathbf{C} \to \mathrm{RelCat}$ is a functor such that $F(C)$ is a partial model category for each $C \in \mathbf{C}$. Let $LF \to \bar{F}$ be a fibrant replacement in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta^+)$. Then there is a weak equivalence $L(\mathbf{E}, W) \to (\mathrm{N}_\mathbf{C}\bar{F})^\natural$ in Set_Δ^+.

Proof. The obvious map of categorical patterns $p\colon \mathfrak{P}^{\mathrm{eq}}_{\mathrm{N}\mathbf{C}} \to \mathfrak{P}^{\mathrm{coCart}}_{\mathrm{N}\mathbf{C}}$ induces a Quillen adjunction

$$p_!\colon (\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}^\natural} \rightleftarrows (\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}^\sharp} : p^*$$

where $p_!$ is the identity on the underlying marked simplicial sets, and p^* forgets the marked edges that do not lie over isomorphisms in $\mathbf{C}$. Since all objects are cofibrant, $p_!$ preserves weak equivalences.

By Proposition 4.4.1.19, there exists a coCartesian fibration $\mathbf{E}[W^{-1}] \to \mathrm{N}\mathbf{C}$ with a map $\phi\colon L(\mathbf{E}, W) \to \mathbf{E}[W^{-1}]^\natural$ that is a weak equivalence in Set_Δ^+. The map ϕ is also a weak equivalence when regarded as a morphism in $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}^\natural}$, and since $p_!$ preserves weak equivalences it is a weak equivalence in $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}^\sharp}$ as well.

Let M' be the set of edges of $\mathrm{N}\mathbf{E}$ corresponding to coCartesian morphisms in $\mathbf{E}$, and let $\mathbf{E}[W^{-1}]^+$ denote the marked simplicial set obtained from $\mathbf{E}[W^{-1}]^\natural$ by also marking the morphisms in the image of M'. We have a pushout diagram

$$\begin{tikzcd} L(\mathbf{E}, W) \arrow[r] \arrow[d] & \mathbf{E}[W^{-1}]^{\natural} \arrow[d] \\ (\mathrm{N}\mathbf{E}, \mathrm{N}W_1 \cup M') \arrow[r] & \mathbf{E}[W^{-1}]^{+}, \end{tikzcd}$$

as both vertical maps are pushouts along $\coprod_{f \in M'} \Delta^1 \hookrightarrow \coprod_{f \in M'} (\Delta^1)^{\sharp}$. Since the model structure on $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}}$ is left proper, it follows that $(\mathrm{N}\mathbf{E}, \mathrm{N}W_1 \cup M') \to \mathbf{E}[W^{-1}]^{+}$ is a weak equivalence.

Let $\mathbf{E}[W^{-1}]^{*}$ denote $\mathbf{E}[W^{-1}]$, marked by the coCartesian morphisms. These are composites of equivalences and morphisms in the image of M', so $\mathbf{E}[W^{-1}]^{+} \to \mathbf{E}[W^{-1}]^{*}$ is marked anodyne. Moreover, $\mathrm{N}\mathbf{E}$ marked by the composites of morphisms in $\mathrm{N}W_1$ and M' is precisely $\mathrm{N}^{+}_{\mathbf{C}}F$, so $(\mathrm{N}\mathbf{E}, \mathrm{N}W_1 \cup M') \to \mathrm{N}^{+}_{\mathbf{C}}F$ is also marked anodyne. By the 2-out-of-3 property we therefore have a weak equivalence $\mathrm{N}^{+}_{\mathbf{C}}LF \to \mathbf{E}[W^{-1}]^{*}$. Thus $\mathbf{E}[W^{-1}]^{*}$ and $\mathrm{N}^{+}_{\mathbf{C}}\bar{F}$ are both fibrant replacements for $\mathrm{N}^{+}_{\mathbf{C}}LF$, and so are linked by a zig-zag of weak equivalences between fibrant objects.

This implies that the underlying ∞-categories $\mathbf{E}[W^{-1}]$ and $\mathrm{N}_{\mathbf{C}}LF$ are equivalent, and so by the 2-out-of-3 property the map $(\mathrm{N}\mathbf{E}, W) \to (\mathrm{N}_{\mathbf{C}}\bar{F})^{\natural}$ is a weak equivalence in $\mathrm{Set}_{\Delta}^{+}$, as required. □

Although not strictly necessary for the applications we are interested in below, we will now show that if the functor $F\colon \mathbf{C} \to \mathrm{RelCat}$ is obtained from a suitable functor from $\mathbf{C}$ to combinatorial model categories, then the relative category structure on $\mathbf{E}$ considered above also comes from a combinatorial model category.

Definition 4.4.1.21. Let $\mathrm{ModCat}^{\mathrm{R}}$ be the category of model categories and right Quillen functors. A *right Quillen presheaf* on a category $\mathbf{C}$ is a functor $\mathbf{C}^{\mathrm{op}} \to \mathrm{ModCat}^{\mathrm{R}}$. A right Quillen presheaf is *combinatorial* if it factors through the full subcategory of combinatorial model categories.

Definition 4.4.1.22. Suppose $\mathbf{C}$ is a κ-accessible category. A right Quillen presheaf on $\mathbf{C}$ is *κ-accessible* if for each κ-filtered diagram $i\colon \mathbf{I} \to \mathbf{C}$ with colimit x, the category $F(x)$ is the limit of the categories $F(i(\alpha))$, and the model structure on $F(x)$ is induced by those on $F(i(\alpha))$ in the sense that a map $f\colon a \to b$ in $F(x)$ is a (trivial) fibration if and only if $F(g_{\alpha})(f)$ is a (trivial) fibration in $F(i(\alpha))$ for all $\alpha \in \mathbf{I}$, where g_{α} is the canonical morphism $i(\alpha) \to x$. We say a right Quillen presheaf F on an accessible category $\mathbf{C}$ is *accessible* if there exists a cardinal κ such that $\mathbf{C}$ and F are κ-accessible.

Proposition 4.4.1.23. Suppose $\mathbf{C}$ is a complete and cocomplete category and F is a right Quillen presheaf on $\mathbf{C}$. Let $\pi\colon \mathbf{E} \to \mathbf{C}$ be the Grothendieck fibration corresponding to F. Then there exists a model structure on $\mathbf{E}$ such that a morphism $\phi\colon x \to y$ with image $f\colon a \to b$ in $\mathbf{C}$ is

(W) a weak equivalence if and only if f is an isomorphism in $\mathbf{C}$ and the morphism $f_!x \to y$ is a weak equivalence in $F(b)$.

(F) a fibration if and only if $x \to f^*y$ is a fibration in $F(a)$.

(C) a cofibration if and only if $f_!x \to y$ is a cofibration in $F(b)$.

Moreover, if **C** is a presentable category and F is an accessible and combinatorial right Quillen presheaf, then this model structure on **E** is combinatorial.

Remark 4.4.1.24. If $f\colon a \to b$ is an isomorphism in **C**, then $f^* = F(f)$ is an isomorphism of model categories with inverse $f_!$. Thus if $\phi\colon x \to y$ is a morphism in **E** such that $f = \pi(\phi)$ is an isomorphism in **C**, then $f_!x \to y$ is a weak equivalence in $\mathbf{E}_b$ if and only if $x \to f^*y$ is a weak equivalence in $\mathbf{E}_a$.

Remark 4.4.1.25. This model category structure is a particular case of that constructed by Roig [Roi94] (though he does not consider the combinatorial case), but we include a proof for completeness.

Proof. Limits in **E** are computed by first taking Cartesian pullbacks to the fibre over the limit of the projection of the diagram to **C**, and then taking the limit in that fibre. Since all the fibres $\mathbf{E}_x$ have limits, it is therefore clear that **E** has limits. Similarly, since each functor ϕ^* for ϕ in **C** has a left adjoint, and each of the fibres $\mathbf{E}_x$ has all colimits, it is clear that **E** has colimits.

To show that **E** is a model category we must now prove that the weak equivalences satisfy the 2-out-of-3 property, and the cofibrations and trivial fibrations, as well as the trivial cofibrations and fibrations, form weak factorization systems. We check the 2-out-of-3 property first: Suppose we have morphisms $\bar{f}\colon x \to y$ and $\bar{g}\colon y \to z$ in **E** lying over $f\colon a \to b$ and $g\colon b \to c$ in **C**. If two out of the three morphisms $\bar{f}$, $\bar{g}$ and $\bar{g}\bar{f}$ are weak equivalences, it is clear that f and g must be isomorphisms. Thus $g_!$ is an isomorphism of model categories, and $g_!f_!x \to g_!y$ is a weak equivalence in $\mathbf{E}_c$ if and only if $f_!x \to y$ is a weak equivalence in $\mathbf{E}_b$. Combining this with the 2-out-of-3 property for weak equivalences in $\mathbf{E}_c$ gives the 2-out-of-3 property for **E**.

We now prove that the cofibrations and trivial fibrations form a weak factorization system:

(1) Any morphism has a factorization as a cofibration followed by a trivial fibration: Given $\bar{f}\colon x \to y$ in **E** lying over $f\colon a \to b$ in **C**, choose a factorization $f_!x \to z \to y$ of $f_!x \to y$ as a cofibration followed by a trivial fibration in $\mathbf{E}_b$. Then by definition $x \to z$ is a cofibration and $z \to y$ is a trivial fibration in **E**.

(2) A morphism that has the left lifting property with respect to all trivial fibrations is a cofibration: Suppose $\bar{f}\colon x \to y$, lying over $f\colon a \to b$ in **C**, has the left lifting property with respect to all trivial fibrations. Then in particular there exists a lift in all diagrams

$$\begin{tikzcd} x \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & y' \end{tikzcd}$$

where $x' \to y'$ is a trivial fibration in $\mathbf{E}_b$. By the universal property of coCartesian morphisms, this clearly implies that $f_!x \to y$ has the left lifting property with respect to trivial fibrations in $\mathbf{E}_b$, and so is a cofibration in $\mathbf{E}_b$. Thus $\bar{f}$ is a cofibration.

(3) Cofibrations have the left lifting property with respect to trivial fibrations: Suppose $\bar{f}\colon x \to y$, lying over $f\colon a \to b$ in $\mathbf{C}$, is a cofibration, and $\bar{g}\colon x' \to y'$, lying over $g\colon a' \to b'$, is a trivial fibration. Given a commutative diagram

$$\begin{tikzcd} x \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{f}"'] & x' \arrow[d, "\bar{g}"] \\ y \arrow[r, "\bar{\beta}"'] & y' \end{tikzcd}$$

lying over

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "f"'] & a' \arrow[d, "g"] \\ b \arrow[r, "\beta"'] & b' \end{tikzcd}$$

we must show there exists a lift $y \to x'$. Since $\bar{g}$ is a trivial fibration, g is an isomorphism. Pulling back along g^{-1} and pushing forward along $g\alpha = \beta f$ and β gives a diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & \beta_! f_! x \arrow[r] \arrow[d] & (g^{-1})^* x' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & \beta_! y \arrow[r] & y' \arrow[r] & y' \end{tikzcd}$$

Here $\beta_! f_! x \to \beta_! y$ is a cofibration in $\mathbf{E}_{b'}$ since $f_! x \to y$ is a cofibration in $\mathbf{E}_b$ and $\beta_!$ is a left Quillen functor, and $(g^{-1})^* x' \to (g^{-1})^* g^* y = y$ is a trivial fibration in $\mathbf{E}_{b'}$ since $x \to g^* y$ is a trivial fibration in $\mathbf{E}_{a'}$ and $(g^{-1})^*$ is a right Quillen functor. Thus there exists a lift $\beta_! y \to (g^{-1})^* x'$ which gives the desired lift $y \to x'$.

(4) A morphism that has the right lifting property with respect to all cofibrations is a trivial fibration: Suppose $\bar{g}\colon x' \to y'$, lying over $g\colon a' \to b'$ in $\mathbf{C}$, has the right lifting property with respect to all cofibrations. Then in particular there exists a lift in all diagrams

$$\begin{tikzcd} x \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & y' \end{tikzcd}$$

where $x \to y$ is a cofibration in $\mathbf{E}_{a'}$. By the universal property of Cartesian morphisms, this clearly implies that $x' \to g^* y'$ has the right lifting property with respect to cofibrations in $\mathbf{E}_{a'}$, and so is a trivial fibration in $\mathbf{E}_{a'}$. On the other hand, there exists a lift in the diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x' \arrow[d] \\ g_!x' \arrow[r] & y', \end{tikzcd}$$

and projecting this down to **C** we see that g must be an isomorphism. Thus $\bar{g}$ is a trivial fibration in **E**.

The proof that trivial cofibrations and fibrations form a weak factorization system is dual to that for cofibrations and trivial fibrations, so we omit the details.

This completes the proof that **E** is a model category. Now suppose the right Quillen presheaf F is combinatorial and accessible. It follows from [MP89, Theorem 5.3.4] that the category **E** is accessible, and the functor π is accessible, thus **E** is a presentable category since we already proved that it has small colimits.

Let κ be a cardinal such that **C** is κ-accessible and $\mathbf{E}_x$ is κ-accessible for each κ-compact object x in **C**. For $x \in \mathbf{C}$, let I_x and J_x be sets of generating cofibrations and trivial cofibrations for $\mathbf{E}_x$. Let I and J be the unions of I_x and J_x, respectively, over all κ-compact objects $x \in \mathbf{C}$; then I and J are sets.

Suppose a morphism $\bar{f}: x \to y$, lying over $f: a \to b$ in **C**, has the right lifting property with respect to the morphisms in J; then $x \to f^*y$ is a fibration in $\mathbf{E}_a$: To see this let $\mathbf{K} \to \mathbf{C}$, $\alpha \mapsto a_\alpha$, be a κ-filtered diagram of κ-compact objects with colimit a, and let $\gamma_\alpha : a_\alpha \to a$ be the canonical morphism. Then $\gamma_\alpha^* x \to \gamma_\alpha^* f^* y$ has the right lifting property with respect to a set of generating trivial cofibrations in $\mathbf{E}_{a_\alpha}$, and hence this is a fibration in $\mathbf{E}_{a_\alpha}$. Since the right Quillen presheaf F is κ-accessible, this implies that $x \to f^*y$ is a fibration in $\mathbf{E}_a$. This means $\bar{f}$ is a fibration in **E**, so J is a set of generating trivial cofibrations.

Similarly, if $\bar{f}$ has the right lifting property with respect to the morphisms in I, then $x \to f^*y$ is a trivial fibration in $\mathbf{E}_a$. To find a set of generating cofibrations we consider also the set I' of morphisms $\emptyset_\emptyset \to \emptyset_c$ and $\emptyset_{c \amalg c} \to \emptyset_c$ where c is a κ-compact object of **C** and $\emptyset_c$ denotes the initial object of $\mathbf{E}_c$. We claim that if $\bar{f}: x \to y$ in **E**, with image $f: a \to b$ in **C**, has the right lifting property with respect to the morphisms in I', then f is an isomorphism in **C**. To prove this it suffices to show that for every object $c \in \mathbf{C}$ the map $f_*: \mathrm{Hom}_{\mathbf{C}}(c, a') \to \mathrm{Hom}_{\mathbf{C}}(c, b')$ induced by composition with f is a bijection; since **C** is κ-presentable it is enough to prove this for c a κ-compact object. Since $\bar{f}$ has the right lifting property with respect to $\emptyset_\emptyset \to \emptyset_c$ and every morphism $c \to b$ induces a morphism $\emptyset_c \to y$, there exists a lift in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & a \arrow[d, "f"] \\ c \arrow[r] \arrow[ur, dashed] & b \end{tikzcd}$$

for every map $c \to b$; this shows that f_* is surjective. Moreover, given two morphisms

$c \to a$ such that the composites $c \to b$ are equal, we get a lift in the diagram

$$\begin{tikzcd} c \amalg c \arrow[r] \arrow[d] & a \arrow[d, "f"] \\ c \arrow[r] \arrow[ur, dashed] & b \end{tikzcd}$$

since $\bar{f}$ has the right lifting property with respect to $\emptyset_{c \amalg c} \to \emptyset_c$; thus the two morphisms $c \to a$ must be equal and so f_* is injective. It follows that if a morphism in $\mathbf{E}$ has the right lifting property with respect to the union $I \amalg I'$ then it is a trivial fibration, so $I \amalg I'$ is a set of generating cofibrations for $\mathbf{E}$. Hence $\mathbf{E}$ is a combinatorial model category. $\square$

Remark 4.4.1.26. Let F be a right Quillen presheaf on a category $\mathbf{C}$, and let $\mathbf{E} \to \mathbf{C}$ be a coGrothendieck fibration associated to the underlying functor to categories. Write G for the associated "left Quillen presheaf" obtained by passing to left adjoints, and let $G^c \colon \mathbf{C} \to \mathrm{RelCat}$ be the functor to relative categories obtained by restricting to cofibrant objects. Then the full subcategory $\mathbf{E}^{\mathrm{cof}}$ of cofibrant objects in $\mathbf{E}$, with the model structure defined above, is the total space of the coGrothendieck fibration associated to G^{cof}, and the weak equivalences in $\mathbf{E}^{\mathrm{cof}}$ are precisely those considered above.

4.4.2 Rectifying Associative Algebras

In [Lur11, §4.1.4] Lurie proves a rectification result for associative algebras: if $\mathbf{V}$ is a nice symmetric monoidal model category, then the ∞-category of ∞-categorical associative algebras in $\mathbf{V}_\infty$, i.e. the ∞-category of algebras for the non-symmetric ∞-operad $\mathbb{\Delta}^{\mathrm{op}}$, is equivalent to that associated to the model category of (strictly) associative algebras in $\mathbf{V}$ (constructed by Schwede and Shipley [SS00]). This is proved by showing that both are equivalent to the ∞-category of algebras for the free associative algebra monad on $\mathbf{V}_\infty$. We would like to use the same idea to show that the ∞-category associated to the model category $\mathrm{Cat}_X(\mathbf{V})$ of $\mathbf{V}$-categories with a fixed set X of objects is equivalent to $\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}_X}(\mathbf{V}_\infty)$; to do this we need a generalization of Schwede and Shipley's results to the case of non-symmetric monoidal model categories. Luckily this generalization has been carried out by Muro [Mur11] as part of his work on model structures for algebras over non-symmetric operads. We will now review this case of Muro's work, and then observe that they allow the technical parts of Lurie's proof to work exactly as in [Lur11].

First we recall an observation of Schwede and Shipley on model structures for algebras over monads:

Definition 4.4.2.1. Let T be a monad on a model category $\mathbf{C}$. We say that T is an *admissible* monad if there exists a model structure on the category $\mathrm{Alg}(T)$ of T-algebras where a morphism is a weak equivalence or fibration if and only if the underlying morphism in $\mathbf{C}$ is a weak equivalence or fibration.

Write $F_T \colon \mathbf{C} \rightleftarrows \mathrm{Alg}(T) \colon U_T$ for the associated adjunction. If $\mathbf{C}$ is a combinatorial model category with sets I and J of generating cofibrations and trivial cofibrations, we say that T is *combinatorially admissible* if it is admissible and the model structure on $\mathrm{Alg}(T)$ is combinatorial with $F_T(I)$ and $F_T(J)$ as sets of generating cofibrations and trivial cofibrations.

Lemma 4.4.2.2 (Schwede-Shipley, [SS00, Lemma 2.3])**.** Suppose **C** is a combinatorial model category and T is a filtered-colimit-preserving monad on **C**, and let J be a set of generating trivial cofibrations for **C**. If every morphism in the weakly saturated class generated by $F_T(J)$ is a weak equivalence in **C** then T is combinatorially admissible.

Remark 4.4.2.3. Since weak equivalences in **C** are closed under retracts and transfinite composites, the weakly saturated class generated by $F_T(J)$ will be contained in the weak equivalences provided the pushout of any morphism in $F_T(J)$ along any morphism in $\mathrm{Alg}(T)$ is a weak equivalence.

Definition 4.4.2.4. Let **C** be a biclosed monoidal category. If $f\colon A \to B$ and $g\colon A' \to B'$ are morphisms in **C**, let $f \Box g$ be the induced morphism

$$A \otimes B' \amalg_{A \otimes A'} B \otimes A' \to B \otimes B'.$$

Definition 4.4.2.5. Let **C** be a model category equipped with a biclosed monoidal structure. We say that **C** is a *monoidal model category* if $f \Box g$ is a cofibration whenever f and g are both cofibrations, and a trivial cofibration if either f or g is also a weak equivalence.

Definition 4.4.2.6. Suppose **C** is a monoidal model category. Let U be the set of morphisms in **C** of the form $f_1 \Box \cdots \Box f_n$ where each f_i is either a trivial cofibration or of the form $\emptyset \to X_i$ for some $X_i \in \mathbf{C}$, with at least one f_i being a trivial cofibration. We say that **C** satisfies the *monoid axiom* if the weakly saturated class $\bar{U}$ generated by U is contained in the weak equivalences in **C**.

Remark 4.4.2.7. If **C** is symmetric monoidal, then this is equivalent to the corresponding statement where U consists of morphisms of the form $f \otimes \mathrm{id}_X$ with f a trivial cofibration. This is the original form of the monoid axiom, due to Schwede and Shipley.

We can now state the special case of Muro's results on algebras over non-symmetric operads that we will make use of:

Theorem 4.4.2.8 (Muro [Mur11, Theorem 8.6])**.** Suppose **C** is a combinatorial biclosed monoidal model category satisfying the monoid axiom. Write $\mathrm{Alg}(\mathbf{C})$ for the category of associative algebra objects of **C** and $F\colon \mathbf{C} \rightleftarrows \mathrm{Alg}(\mathbf{C}) : U$ for the free algebra functor and forgetful functor. Let $f\colon X \to Y$ be a morphism in **C** and $g\colon F(X) \to A$ be a morphism in $\mathrm{Alg}(\mathbf{C})$. If

$$\begin{tikzcd}
F(X) \arrow[r, "f"] \arrow[d, "g"'] & F(Y) \arrow[d, "g'"] \\
A \arrow[r, "f'"'] & B
\end{tikzcd}$$

is a pushout diagram in $\mathrm{Alg}(\mathbf{C})$, then there is a sequence of morphisms in **C**

$$A = B_0 \xrightarrow{\phi_1} B_1 \xrightarrow{\phi_2} B_2 \cdots$$

such that $B = \mathrm{colim}_t\, B_t$ and ϕ_t is a pushout of

$$\coprod_{n \geq 1} \coprod_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = t}} k_1^S \Box \cdots \Box k_n^S$$

where

$$k_i^S = \begin{cases} f, & i \in S \\ \varnothing \to A, & i \notin S. \end{cases}$$

Corollary 4.4.2.9. Suppose $\mathbf{C}$ is a combinatorial biclosed monoidal model category satisfying the monoid axiom. Then the free associative algebra monad on $\mathbf{C}$ is combinatorially admissible.

Proof. By Remark 4.4.2.3 it suffices to show that if $f\colon X \to Y$ is a trivial cofibration in $\mathbf{C}$ and $g\colon F(X) \to A$ a morphism in $\mathrm{Alg}(\mathbf{C})$, and

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[d, "g"'] & F(Y) \arrow[d, "g'"] \\ A \arrow[r, "f'"'] & B \end{tikzcd}$$

is a pushout diagram in $\mathrm{Alg}(\mathbf{C})$, then f' is a weak equivalence in $\mathbf{C}$. By Theorem 4.4.2.8, the morphism f' is a transfinite composite of pushouts of morphisms ϕ_t that are clearly contained in the class U from Definition 4.4.2.6, so f' is contained in the weakly saturated closure $\bar{U}$. Since $\mathbf{C}$ satisfies the monoid axiom, this implies that f' is a weak equivalence in $\mathbf{C}$. □

This allows us to generalize the key technical result [Lur11, Lemma 4.1.4.13] to non-symmetric monoidal categories:

Definition 4.4.2.10. A model category is *tractable* if it is combinatorial and there exists a set of generating cofibrations that consists of morphisms between cofibrant objects.

Lemma 4.4.2.11. Suppose $\mathbf{C}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom and $\mathbf{I}$ is a small category such that $\mathrm{N}\mathbf{I}$ is sifted. Then the forgetful functor $U_\infty\colon \mathrm{Alg}(\mathbf{C})_\infty \to \mathbf{C}_\infty$ preserves $\mathrm{N}\mathbf{I}$-indexed colimits.

The proof is almost the same as that of [Lur11, Lemma 4.1.4.13], but we include it for completeness:

Proof. By [Lur11, Proposition 1.3.3.11, Proposition 1.3.3.12] it suffices to show that the forgetful functor U preserves homotopy colimits indexed by $\mathbf{I}$. Regard the categories $\mathrm{Fun}(\mathbf{I}, \mathrm{Alg}_{\mathbf{M}}(\mathbf{C}))$ and $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ as model categories equipped with the projective model structures, let $C\colon \mathrm{Fun}(\mathbf{I}, \mathbf{C}) \to \mathbf{C}$ and $C_{\mathrm{Alg}}\colon \mathrm{Fun}(\mathbf{I}, \mathrm{Alg}(\mathbf{C})) \to \mathrm{Alg}(\mathbf{C})$ be colimit functors, and let $U^{\mathbf{I}}\colon \mathrm{Fun}(\mathbf{I}, \mathrm{Alg}(\mathbf{C})) \to \mathrm{Fun}(\mathbf{I}, \mathbf{C})$ be given by composition with U. Since $\mathrm{N}\mathbf{I}$ is sifted, there is a canonical isomorphism of functors $\alpha\colon C \circ U^{\mathbf{I}} \xrightarrow{\sim} U \circ C_{\mathrm{Alg}}$. We need to prove that this isomorphism persists after deriving all the relevant functors. Let $\mathbb{L}C$ and $\mathbb{L}C_{\mathrm{Alg}}$ be left derived functors of C and C_{Alg}; then α induces a natural transformation $\bar{\alpha}\colon \mathbb{L}C \circ U^{\mathbf{I}} \to U \circ \mathbb{L}C_{\mathrm{Alg}}$; we wish to prove that $\bar{\alpha}$ is a natural weak equivalence. Let $A\colon \mathbf{I} \to \mathrm{Alg}(\mathbf{C})$ be a projectively cofibrant functor; we must show that the natural map

$$\mathbb{L}CU^{\mathbf{I}}(A) \to U(\mathbb{L}C_{\mathrm{Alg}}(A)) \cong U(C_{\mathrm{Alg}}(A)) \cong C(U^{\mathbf{I}}A)$$

is a weak equivalence in $\mathbf{C}$.

Let's call an object $X \in \mathrm{Fun}(\mathbf{I}, \mathbf{C})$ *good* if

(i) the object $X(i)$ is cofibrant in $\mathbf{C}$ for all $i \in \mathbf{I}$,

(ii) the colimit $C(X)$ is cofibrant in $\mathbf{C}$,

(iii) the natural map $\mathbb{L}C(X) \to C(X)$ is a weak equivalence in $\mathbf{C}$, i.e. the colimit of X is also a homotopy colimit.

To complete the proof it suffices to show that $U^{\mathbf{I}}A$ is good whenever A is a projectively cofibrant object of $\mathrm{Fun}(\mathbf{I}, \mathrm{Alg}(\mathbf{C}))$.

Let's say a morphism $f\colon X \to Y$ in $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ is *good* if

(i) the objects X and Y are good,

(ii) the map $X(i) \to Y(i)$ is a cofibration for all $i \in \mathbf{I}$,

(iii) the map $C(f)\colon C(X) \to C(Y)$ is a cofibration in $\mathbf{C}$.

We now make the following observations:

(1) Good morphisms are stable under transfinite composition: Given an ordinal α and a direct system of objects $\{X^\beta\}_{\beta<\alpha}$ of $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ such that for every $0 < \beta < \alpha$ the map $\mathrm{colim}\{X^\gamma\}_{\gamma<\beta} \to X^\beta$ is good, then the induced map $X^0 \to X := \mathrm{colim}\{X^\beta\}_{\beta<\alpha}$ is good. The only non-obvious point is to show that the object X is good. For this we observe that X is a homotopy colimit of the system $\{X^\beta\}$ by (ii) and $C(X)$ is a homotopy colimit of $\{C(X^\beta)\}$ by (iii), and recall that homotopy colimit diagrams are stable under homotopy colimits.

(2) Suppose

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

is a pushout diagram in $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ such that f is good and the object X' is good. Then f' is also good: Again the only non-obvious point is to show the object Y' is good. The hypotheses imply that the diagram is a homotopy pushout square, and similarly $C(Y')$ is a homotopy pushout of $C(Y)$ with $C(X')$ over $C(X)$, so it follows that Y' is good since homotopy colimit diagrams are stable under homotopy colimits.

(3) Let $G\colon \mathbf{I} \to \mathbf{C}$ be a constant functor whose value is a cofibrant object of $\mathbf{C}$. Then G is good, since $\mathrm{N}\mathbf{I}$ is weakly contractible, using [Lur11, Proposition 1.3.3.11] and [Lur09a, Proposition 5.5.8.7].

(4) Every projectively cofibrant object of $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ is good, and every projective cofibration between projectively cofibrant objects is good.

(5) If X and Y are good, then so is $X \otimes Y$: The cofibrant objects of $\mathbf{C}$ are closed under tensor products, and because $\mathrm{N}\mathbf{I}$ is sifted, [Lur11, Proposition 1.3.3.11] gives a chain of isomorphisms in $\mathrm{h}\mathbf{C}$

$$\mathbb{L}C(X \otimes Y) \cong \mathbb{L}C(X) \otimes \mathbb{L}C(Y) \cong C(X) \otimes C(Y) \cong C(X \otimes Y).$$

(6) Let $f\colon X \to X'$ be a good morphism, and let Y be a good object. Then $f \otimes \mathrm{id}_Y$ is good: Condition (i) follows from (5), condition (ii) follows because tensoring with each $Y(c)$ preserves cofibrations, since $Y(c)$ is cofibrant, and condition (iii) holds by the same argument applied to $C(Y)$, since C commutes with tensor products.

(7) Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be good morphisms. Then $f\Box g$ is good. Condition (ii) holds since $\mathbf{C}$ is a monoidal model category, as does (iii) since C commutes with pushouts and tensor products. Then (i) holds by combining (5), (6), and (2).

(8) Every retract of a good object is good: this follows since cofibrations and weak equivalences are closed under retracts.

By assumption the model category $\mathbf{C}$ is left proper and tractable, which implies that the projective model structure on $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ is also tractable. Using the small object argument this implies that for every projectively cofibrant object $A \in \mathrm{Fun}(\mathbf{I}, \mathrm{Alg}(\mathbf{C}))$ there exists a transfinite sequence $\{A^\beta\}_{\beta \leq \alpha}$ such that

(a) A^0 is an initial object,

(b) A is a retract of A^α,

(c) if $\lambda \leq \alpha$ is a limit ordinal, then $A^\lambda \cong \mathrm{colim}\{A^\beta\}_{\beta<\lambda}$,

(d) for each $\beta < \alpha$ there is a pushout diagram

$$\begin{tikzcd} F(X') \arrow[r, "F(f)"] \arrow[d] & F(X) \arrow[d] \\ A^\beta \arrow[r] & A^{\beta+1} \end{tikzcd}$$

where f is a projective cofibration between projectively cofibrant objects of $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$.

By (b) and (8) to prove that $U^{\mathbf{I}}(A)$ is good it suffices to prove that $U^{\mathbf{I}}(A^\alpha)$ is good. We will show by transfinite induction that for each $\gamma \leq \beta \leq \alpha$ the induced morphism

$$u_{\gamma,\beta}\colon U^{\mathbf{I}}(A^\gamma) \to U^{\mathbf{I}}(A^\beta)$$

is good. If $\beta = 0$ this holds since $U^{\mathbf{I}}(A^0)$ is good by (a) and (3). If β is a non-zero limit ordinal, this follows from (c) and (1). It therefore suffices to consider the case where $\beta = \beta' + 1$ is a successor ordinal. Moreover, we may suppose $\gamma = \beta'$: if $\gamma < \beta'$ then $u_{\gamma,\beta} = u_{\beta',\beta} \circ u_{\gamma,\beta'}$ and composites of good morphisms are good by (1), while if $\gamma > \beta'$ then we must have $\gamma = \beta$ and we are reduced to proving that $U^{\mathbf{I}}(A^\beta)$ is good, which will follow from $u_{\beta',\beta}$ being good. Invoking (d) we thus need to prove that if

$$\begin{tikzcd} F(X') \arrow[r, "F(f)"] \arrow[d] & F(X) \arrow[d] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

is a pushout diagram where $f\colon X' \to X$ is a projective cofibration between projectively cofibrant objects of $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ and $U^{\mathbf{I}}B'$ is good, then $U^{\mathbf{I}}(v)$ is good. By Theorem 4.4.2.8 the morphism $U^{\mathbf{I}}(v)$ can be identified with a transfinite composite of morphisms $\phi_t\colon B_{t-1} \to B_t$; by (1) it suffices to show that each ϕ_t is good. But ϕ_t is a pushout of

$$\psi_t := \coprod_{n\geq 1} \coprod_{\substack{S\subseteq\{1,\ldots,n\} \\ |S|=t}} k_1^S \square \cdots \square k_n^S,$$

and since $B_0 = U(B')$ is good applying (2) inductively it suffices to prove that ψ_t is good. It is clear that an arbitrary coproduct of good morphisms is good, so by (7) to see this it suffices to show that each morphism k_i^S is good, which is true since this is either f, which is good by (4), or $\emptyset \to B'$, which is good since B' is good. $\square$

Remark 4.4.2.12. Applying the more general version of Theorem 4.4.2.8 actually proved in [Mur11], the same proof clearly implies, for example, that if $\mathbf{C}$ is a left proper tractable simplicial biclosed monoidal model category satisfying the monoid axiom, $\mathbf{O}$ is a small simplicial non-symmetric operad, and $\mathbf{I}$ is a small category such that $\mathrm{N}\mathbf{I}$ is sifted, then the forgetful functor $\mathrm{Alg}_{\mathbf{M}}(\mathbf{C})_\infty \to \mathbf{C}_\infty$ preserves $\mathrm{N}\mathbf{I}$-indexed colimits.

Proposition 4.4.2.13. Suppose $\mathbf{C}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom. Then the natural map

$$\mathrm{Alg}(\mathbf{C})_\infty \to \mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathbf{C}^{\otimes}_\infty)$$

is an equivalence.

Proof. We apply [Lur11, Corollary 6.2.2.14] as in the proof of [Lur11, Theorem 4.1.4.4]: We have a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}(\mathbf{C})_\infty \arrow[rr] \arrow[dr, "U_\infty"'] & & \mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathbf{C}^{\otimes}_\infty) \arrow[dl, "U'"] \\ & \mathbf{C}_\infty. & \end{tikzcd}$$

Then we observe:

(a) The ∞-category $\mathrm{Alg}(\mathbf{C})_\infty$ is presentable by [Lur11, Proposition 1.3.3.9], and the ∞-category $\mathrm{Alg}^{\mathbb{O}}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathbf{C}^{\otimes}_\infty)$ is presentable by Corollary 3.3.5.5 since $\mathbf{C}_\infty$ is presentable by [Lur11, Proposition 1.3.3.9] and the induced tensor product on $\mathbf{C}_\infty$ preserves colimits in each variable by [Lur11, Lemma 4.1.4.8].

(b) The functor U' admits a left adjoint F' by Theorem 3.3.4.6, and U_∞ admits a left adjoint F_∞ since it arises from a right Quillen functor.

(c) The functor U' is conservative by Lemma 3.3.5.3 and preserves sifted colimits by Proposition 3.3.5.2.

(d) The functor U_∞ is conservative by the definition of the weak equivalences in $\mathrm{Alg}(\mathbf{C})$, and preserves sifted colimits by Lemma 4.4.2.11.

(e) The canonical map $U' \circ F' \to U_\infty \circ F_\infty$ is an equivalence since both induce, on the level of homotopy categories, the free associative algebra functor $A \mapsto \coprod_{n\geq 0} C^{\otimes n}$ by Proposition 3.3.4.9.

The hypotheses of [Lur11, Corollary 6.2.2.14] thus hold, which implies that the morphism in question is an equivalence. $\square$

4.4.3 Comparison with Enriched Categories

Our goal in this subsection is to show that the homotopy theory of categories enriched in a nice monoidal model category $\mathbf{V}$ is equivalent to the homotopy theory of ∞-categories enriched in the monoidal ∞-category associated to $\mathbf{V}$. More precisely, we will prove that the ∞-category obtained from the category $\mathrm{Cat}(\mathbf{V})$ of small $\mathbf{V}$-categories by inverting the homotopically appropriate version of fully faithful and essentially surjective functors is equivalent to the ∞-category $\mathrm{Cat}_\infty^{\mathbf{V}_\infty}$ of small $\mathbf{V}_\infty$-∞-categories. We will do this in three steps: first we apply the results of §4.4.2 to get an equivalence between the ∞-category associated to a model structure on the category $\mathrm{Cat}_X(\mathbf{V})$ of $\mathbf{V}$-categories with a fixed set of objects X and the ∞-category $\mathrm{Alg}^{\mathcal{O}}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}_\infty)$ of Δ_X^{op}-algebras. Next, using the results of §4.4.1, we see that this induces an equivalence between the ∞-category associated to a certain model structure on $\mathrm{Cat}(\mathbf{V})$ and the ∞-category $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}_\infty^{\otimes})_{\mathrm{Set}}$ of categorical algebras in $\mathbf{V}_\infty$ whose spaces of objects are sets. Finally, we complete the comparison by showing that inverting the fully faithful and essentially surjective functors in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}_\infty^{\otimes})_{\mathrm{Set}}$ is equivalent to inverting them in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}_\infty^{\otimes})$.

If $\mathbf{V}$ is a biclosed monoidal category and X is a set then it is well-known that there is a monoidal structure on $\mathrm{Fun}(X \times X, \mathbf{V})$, given by

$$(F \otimes G)(x,y) = \coprod_{z\in X} F(x,z) \otimes G(z,y),$$

such that an associative algebra object in $\mathrm{Fun}(X \times X, \mathbf{V})$ is precisely a $\mathbf{V}$-category with objects X. This monoidal structure is well-behaved:

Proposition 4.4.3.1 (Muro [Mur11, Proposition 10.3])**.** If $\mathbf{V}$ is a monoidal model category satisfying the monoid axiom, then so is $\mathrm{Fun}(X \times X, \mathbf{V})$ for all sets X.

We can thus get a model structure on the category $\mathrm{Cat}_X(\mathbf{V})$ of $\mathbf{V}$-categories with fixed set of objects X:

Corollary 4.4.3.2. If $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom, then there is a combinatorial model category structure on the category $\mathrm{Cat}_X(\mathbf{V})$ such that a morphism is a fibration or weak equivalence if and only if its image in $\mathrm{Fun}(X \times X, \mathbf{V})$ is. Moreover, if $\mathbf{I}$ is a small category such that $\mathrm{N}\mathbf{I}$ is sifted then the forgetful functor $\mathrm{Cat}_X(\mathbf{V})_\infty \to \mathrm{Fun}(X \times X, \mathbf{V})_\infty$ preserves $\mathrm{N}\mathbf{I}$-indexed colimits.

Proof. Apply Corollary 4.4.2.9 and Lemma 4.4.2.11 to $\mathrm{Fun}(X \times X, \mathbf{V})$ equipped with the monoidal structure described above, so that associative algebras are $\mathbf{V}$-categories with set of objects X. $\square$

The ∞-category associated to this model category is equivalent to the ∞-category of Δ_X^{op}-algebras in $\mathbf{V}_\infty$:

Proposition 4.4.3.3. Suppose $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom, and let X be a set. The natural map $\eta_X : \mathrm{Cat}_X(\mathbf{V})_\infty \to \mathrm{Alg}^{\mathbb{O}}_{\Delta^{\mathrm{op}}_X}(\mathbf{V}^{\otimes}_\infty)$ is an equivalence.

Proof. This follows by exactly the same argument as that in the proof of Proposition 4.4.2.13, since the free associative algebra monad on $\mathrm{Fun}(X \times X, \mathbf{V})$ is the same as the free Δ^{op}_X-algebra monad by Proposition 3.3.4.9. $\square$

Using Proposition 4.4.1.23 we can combine these fibrewise model structures to get a model structure on the category $\mathrm{Cat}(\mathbf{V})$ of small $\mathbf{V}$-categories:

Proposition 4.4.3.4. Suppose $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom. Then there is a model structure on the category $\mathrm{Cat}(\mathbf{V})$ of small $\mathbf{V}$-categories such that a morphism $F : \mathbf{C} \to \mathbf{D}$ is a weak equivalence if and only if F is a bijection on objects and the induced morphism $\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is a weak equivalence in $\mathbf{V}$ for all $x, y \in \mathrm{ob}\,\mathbf{C}$, and a fibration if and only if $\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is a fibration in $\mathbf{V}$ for all $x, y \in \mathrm{ob}\,\mathbf{C}$.

We say that a functor $F : \mathbf{C} \to \mathbf{D}$ of $\mathbf{V}$-categories is *weakly fully faithful* if for all objects $x, y \in \mathbf{C}$ the morphism $\mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is a weak equivalence in $\mathbf{C}$; the weak equivalences in this model structure on $\mathrm{Cat}(\mathbf{V})$ are thus the weakly fully faithful functors that are bijective on objects. We therefore write $\mathrm{Cat}(\mathbf{V})^{\mathrm{FFB}}$ for $\mathrm{Cat}(\mathbf{V})$ equipped with this model structure.

The map $\eta_X : \mathrm{Cat}_X(\mathbf{V})_\infty \to \mathrm{Alg}^{\mathbb{O}}_{\Delta^{\mathrm{op}}_X}(\mathbf{V}^{\otimes}_\infty)$ is natural in X, so it induces a natural transformation of functors $\mathrm{Set} \to \mathrm{Set}^+_\Delta$. Applying Corollary 4.4.1.20 we get the following comparison of "algebraic" homotopy theories:

Theorem 4.4.3.5. The natural transformation η induces a functor

$$\mathrm{Cat}(\mathbf{V})^{\mathrm{FFB}}_\infty \to \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_\infty)_{\mathrm{Set}}$$

and this is an equivalence.

The weakly fully faithful functors that are bijective on objects are clearly not the right weak equivalences between $\mathbf{V}$-categories — just as for ordinary categories the equivalences are the functors that are fully faithful and essentially surjective, here they should be the functors that are weakly fully faithful and essentially surjective up to homotopy, in the following sense:

Definition 4.4.3.6. Let $\mathbf{V}$ be a monoidal model category. Then the projection $\mathbf{V} \to \mathrm{h}\mathbf{V}$ to the homotopy category is a monoidal functor; this therefore induces a functor $\mathrm{Cat}(\mathbf{V}) \to \mathrm{Cat}(\mathrm{h}\mathbf{V})$. We say a functor of $\mathbf{V}$-categories is *homotopically essentially surjective* if its image in $\mathrm{Cat}(\mathrm{h}\mathbf{V})$ is essentially surjective.

Weakly fully faithful and homotopically essentially surjective functors are often called *DK-equivalences*; they can also be described as the functors of $\mathbf{V}$-categories that induce equivalences of $\mathrm{h}\mathbf{V}$-categories.

Our next goal is to show that the ∞-category obtained by inverting the weakly fully faithful and homotopically essentially surjective functors in $\mathrm{Cat}(\mathbf{V})$, which we will denote by $\mathrm{Cat}(\mathbf{V})[\mathrm{FFES}^{-1}]$, is equivalent to the ∞-category $\mathrm{Cat}^{\mathbf{V}_\infty}_\infty$ of $\mathbf{V}_\infty$-enriched ∞-categories.

Remark 4.4.3.7. In many cases there is a model structure on $\mathrm{Cat}(\mathbf{V})$ where the weak equivalences are the weakly fully faithful and homotopically essentially surjective functors; see [Lur09a, BM12, Sta12] for general results on such model structures (and see [BM12, §1] for a historical discussion). If this model structure exists, then $\mathrm{Cat}(\mathbf{V})[\mathrm{FFES}^{-1}]$ is equivalent to the ∞-category associated to this model category.

The weakly fully faithful and homotopically essentially surjective functors in $\mathrm{Cat}(\mathbf{V})$ clearly correspond to the fully faithful and essentially surjective functors in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_{\infty})_{\mathrm{Set}}$. Theorem 4.4.3.5 therefore implies the following:

Proposition 4.4.3.8. Suppose $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom. Then $\mathrm{Cat}(\mathbf{V})[\mathrm{FFES}^{-1}]$ is equivalent to the localization $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_{\infty})_{\mathrm{Set}}[\mathrm{FFES}^{-1}]$ of $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_{\infty})_{\mathrm{Set}}$ with respect to the fully faithful and essentially surjective functors.

To prove our desired comparison result it therefore suffices to show that inverting the fully faithful and essentially surjective functors in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_{\infty})_{\mathrm{Set}}$ is equivalent to inverting them in the ∞-category $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_{\infty})$ of all categorical algebras. This is true for all monoidal ∞-categories:

Proposition 4.4.3.9. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category. The inclusion

$$i\colon \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\mathrm{Set}} \to \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$$

induces an equivalence $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\mathrm{Set}}[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Cat}^{\mathcal{V}}_{\infty}$ after inverting the fully faithful and essentially surjective functors.

Proof. Considering $\mathcal{S}$ as the ∞-category associated to the usual model structure on simplicial sets, we get a functor $j\colon \mathrm{Set}_{\Delta} \to \mathcal{S}$ that exhibits $\mathcal{S}$ as the localization of Set_{Δ} with respect to the weak equivalences. Let $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\Delta}$ be the ∞-category defined by the pullback square

$$\begin{tikzcd}
\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\Delta} \arrow[r, "j'"] \arrow[d] & \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes}) \arrow[d] \\
\mathrm{Set}_{\Delta} \arrow[r, "j"'] & \mathcal{S}.
\end{tikzcd}$$

Then $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\mathrm{Set}}$ is the pullback of $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\Delta}$ along the inclusion $\mathrm{Set} \to \mathrm{Set}_{\Delta}$ of the constant simplicial sets. This has a right adjoint $(-)_0\colon \mathrm{Set}_{\Delta} \to \mathrm{Set}$ that sends a simplicial set to its set of 0-simplices. The inclusion $i'\colon \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\mathrm{Set}} \hookrightarrow \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\Delta}$ therefore has a right adjoint $s\colon \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\Delta} \to \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\mathrm{Set}}$ that sends an object $(X \in \mathrm{Set}_{\Delta}, \mathcal{C} \in \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes}))$ to the pullback of $\mathcal{C}$ along the morphism $X_0 \to X \to \iota_0\mathcal{C}$. It is clear that i' preserves fully faithful and essentially surjective functors, as does s by the 2-out-of-3 property. Moreover, $si \simeq \mathrm{id}$ and the counit $is\colon \mathcal{C} \to \mathcal{C}$ is fully faithful and essentially surjective for all $\mathcal{C}$. It then follows from Lemma 2.1.8.4 that i' induces an equivalence

$$\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\mathrm{Set}}[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_{\Delta}[\mathrm{FFES}^{-1}]$$

after inverting the fully faithful and essentially surjective functors. Moreover, $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V})$ is the localization of $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V})_{\Delta}$ with respect to the morphisms that induce weak equivalences in Set_Δ and project to equivalences in $\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V})$. These are obviously among the fully faithful and essentially surjective functors, and so j' induces an equivalence

$$\mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})_\Delta[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Cat}^{\mathcal{V}}_\infty.$$

Composing these two equivalences gives the result. □

Corollary 4.4.3.10. Suppose $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom. The functor $\eta\colon \mathrm{Cat}(\mathbf{V})_\infty \to \mathrm{Alg}^{\mathbb{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_\infty)_{\mathrm{Set}}$ induces an equivalence $\mathrm{Cat}(\mathbf{V})[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Cat}^{\mathbf{V}_\infty}_\infty$.

4.4.4 Comparison with Segal Categories

Segal categories are a model for the theory of $(\infty,1)$-categories where composition is only associative up to coherent homotopy, inspired by Segal's model of A_∞-spaces. They were introduced by Hirschowitz and Simpson [HS98]. A generalization to Segal categories enriched in a monoidal model category where the tensor product is the Cartesian product was first given by Lurie [Lur09b], and later extensively studied by Simpson [Sim12].

Our goal in this subsection is to show that the homotopy theory of Segal categories enriched in $\mathbf{V}$ is equivalent to that of ∞-categories enriched in $\mathbf{V}_\infty$. Segal categories are usually regarded as fibrant objects in a certain model structure on *precategories*; we first review the definitions of Segal categories and precategories, and show that for our purposes we may equivalently consider Segal categories as objects in a larger category of functors. Then we prove the comparison result, using the same strategy as for the comparison with enriched categories.

We begin by recalling the definition of enriched Segal categories:

Definition 4.4.4.1. A model category is *Cartesian* if it is a monoidal model category with respect to the Cartesian product. If $\mathbf{V}$ is a Cartesian model category, a $\mathbf{V}$*-enriched Segal category* (or *Segal* $\mathbf{V}$*-category*) with set of objects S is a functor $\mathbf{C}\colon \mathbb{\Delta}^{\mathrm{op}}_S \to \mathbf{V}$ such that $\mathbf{C}(x,y)$ is fibrant for all $x,y \in S$ and for every object $(x_0,\ldots,x_n)$ of $\mathbb{\Delta}^{\mathrm{op}}_S$ the *Segal morphism* $\mathbf{C}(x_0,\ldots,x_n) \to \mathbf{C}(x_0,x_1) \times \cdots \mathbf{C}(x_{n-1},x_n)$ induced by the projections $(x_0,\ldots,x_n) \to (x_i,x_{i+1})$ is a weak equivalence.

Remark 4.4.4.2. We can regard (fibrant) $\mathbf{V}$-categories as the Segal categories where the Segal morphisms are *isomorphisms*, rather than just weak equivalences.

Now we construct a model category whose fibrant objects are Segal categories with a fixed set S of objects; for this we first need some notation:

Definition 4.4.4.3. If X is an object of $\mathbb{\Delta}^{\mathrm{op}}_S$, let $i_X\colon * \to \mathbb{\Delta}^{\mathrm{op}}_X$ denote the functor with image X, write $i_X^*\colon \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_S,\mathbf{V}) \to \mathbf{V}$ for the functor given by composition with i_X, and let $i_{X,!}\colon \mathbf{V} \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_S,\mathbf{V})$ be its left adjoint, given by left Kan extension along i_X. Then $i_{X,!}$ is a left Quillen functor with respect to the projective model structure on $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_S,\mathbf{V})$.

Observe that if $\mathbf{V}$ is a left proper combinatorial simplicial Cartesian model category, then a functor $\mathbf{C}\colon \mathbb{\Delta}^{\mathrm{op}}_S \to \mathbf{V}$ is a Segal category if and only if it is projectively fibrant and local with respect to the morphisms $i_{(x_0,x_1),!}A \coprod \cdots \coprod i_{(x_{n-1},x_n),!}A \to i_{(x_0,\ldots,x_n),!}A$ for all $(x_0,\ldots,x_n)$ in S and all A in a set of objects that generates $\mathbf{V}$ under colimits. Thus we can define a model structure whose fibrant objects are Segal categories as a left Bousfield localization of the projective model structure:

Definition 4.4.4.4. The *Segal category model structure on functors* is the left Bousfield localization of the projective model structure on $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})$ with respect to these morphisms. We write $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ for the category $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})$ equipped with this model structure.

Enriched Segal categories are more commonly considered as objects in a category of *precategories*:

Definition 4.4.4.5. A $\mathbf{V}$-*precategory* with set of objects S is a functor $\mathbf{C}\colon \triangle_S^{\mathrm{op}} \to \mathbf{V}$ such that $\mathbf{C}(x, \dots, x)$ is a final object for all constant sequences $(x, \dots, x)$ with $x \in S$. Write $\mathrm{Precat}_S(\mathbf{V})$ for the full subcategory of $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})$ spanned by the $\mathbf{V}$-precategories, and $u^*\colon \mathrm{Precat}_S(\mathbf{V}) \to \mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})$ for the inclusion. Then u^* has a left adjoint, which we denote $u_!$.

There is a model structure on $\mathrm{Precat}_S(\mathbf{V})$ analogous to that for $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})$ we described above:

Proposition 4.4.4.6 (Simpson [Sim12, Propostion 13.4.3])**.** There exists a model structure on $\mathrm{Precat}_S(\mathbf{V})$ where a morphism is a weak equivalence or fibration if it levelwise is one in $\mathbf{V}$. The functor $u^*\colon \mathrm{Precat}_S(\mathbf{V}) \to \mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})$ is a right Quillen functor.

Definition 4.4.4.7. The *Segal category model structure on precategories* is the left Bousfield localization of the projective model structure on $\mathrm{Precat}_S(\mathbf{V})$ with respect to the morphisms $u_!(i_{(x_0,x_1),!}A \coprod \cdots \coprod i_{(x_{n-1},x_n),!A}) \to u_! i_{(x_0,\dots,x_n),!}A$ for all $(x_0, \dots, x_n)$ in S and all A in a set of objects that generates $\mathbf{V}$ under colimits. We write $\mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$ for the category $\mathrm{Precat}_S(\mathbf{V})$ equipped with this model structure.

We now prove that these two model categories in the fixed-object case are equivalent:

Proposition 4.4.4.8. The adjunction $u_! \dashv u^*$ gives a Quillen equivalence

$$\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}} \rightleftarrows \mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}.$$

Proof. Since u^* is fully faithful, the counit $u_! u^* F \to F$ is an isomorphism in $\mathrm{Precat}_S(\mathbf{V})$ for all F. It thus remains to show that if X is a cofibrant object of $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ and X' is a fibrant replacement for $u_! X$ in $\mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$, then the composite $X \to u^* u_! X \to u^* X'$ is a weak equivalence in $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$.

By [Sim12, Lemma 14.2.1] the functor $u_!$ only changes the values of a functor at the constant sequences $(x, \dots, x)$ for $x \in S$, and so preserves fibrant objects. Moreover, if F is a fibrant object of $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$, and so in particular $F(x, \dots, x)$ is weakly equivalent to the final object, then the unit map $F \to u^* u_! F$ is a levelwise weak equivalence. Thus if X is a cofibrant object in $\mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ and $X \to F$ is a fibrant-cofibrant replacement for X, then $u_! X \to u_! F$ is a weak equivalence and $u_! F$ is fibrant in $\mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$, i.e. $u_! F$ is a fibrant replacement for $u_! X$. Since $X \to F$ and $F \to u^* u_! F$ are weak equivalences, it follows that the composite $X \to u^* u_! F$ is also a weak equivalence, as required. □

Using Proposition 4.4.1.23 we can combine these model structures as the set S varies:

Definition 4.4.4.9. Let $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ denote the total space of the right Quillen presheaf given by $S \mapsto \mathrm{Fun}(\triangle_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ and let $\mathrm{Precat}(\mathbf{V})$ denote the total space of the right Quillen presheaf given by $S \mapsto \mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$. The adjunction $u_! \dashv u^*$ is natural and so gives a natural transformation between these right Quillen presheaves.

Proposition 4.4.4.10. Let $\mathbf{V}$ be a left proper combinatorial simplicial Cartesian model category. There exist combinatorial model structures on the categories $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ and $\mathrm{Precat}(\mathbf{V})$ where a morphism $F\colon \mathbf{C} \to \mathbf{D}$ is a weak equivalence if and only if the induced morphism f on objects is a bijection and $\mathbf{C} \to f^*\mathbf{D}$ is a weak equivalence in $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{ob}\,\mathbf{C}}, \mathbf{V})_{\mathrm{Seg}}$ or $\mathrm{Precat}_{\mathrm{ob}\,\mathbf{C}}(\mathbf{V})_{\mathrm{Seg}}$ and a fibration if and only if $\mathbf{C} \to f^*\mathbf{D}$ is a fibration in $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{ob}\,\mathbf{C}}, \mathbf{V})_{\mathrm{Seg}}$ or $\mathrm{Precat}_{\mathrm{ob}\,\mathbf{C}}(\mathbf{V})_{\mathrm{Seg}}$. The adjunction

$$u_!\colon \mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V}) \leftrightarrows \mathrm{Precat}(\mathbf{V}) : u^*$$

induced by the natural transformations $u_!$ and u^* is a Quillen equivalence.

Our goal is now to prove that inverting an appropriate collection of weak equivalences in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ give an ∞-category equivalent to $\mathrm{Cat}_\infty^{\mathbf{V}_\infty}$. As in the case of enriched categories we begin by considering the fixed-object case, i.e. comparing the ∞-category associated to $\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V})_{\mathrm{Seg}}$ to $\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_X}(\mathbf{V}_\infty^\times)$.

We know the ∞-category associated to the projective model structure on $\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V})$ is equivalent to the ∞-categorical functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V}_\infty)$. The Bousfield-localized model category $\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V})_{\mathrm{Seg}}$ can therefore be identified with the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V}_\infty)$ spanned by the objects that are local with respect to certain maps. We can identify this with the ∞-category of Δ^{op}_X-monoids:

Definition 4.4.4.11. Suppose $\mathcal{V}$ is a presentable ∞-category and $\mathcal{M}$ is a generalized non-symmetric ∞-operad. For $m \in \mathcal{M}$, write $i_m\colon * \to \mathcal{M}$ for the inclusion of this object, and let $i_{m,!}$ denote left Kan extension along i_m. Then for any functor $F\colon \mathcal{M} \to \mathcal{V}$ and $X \in \mathcal{V}$ we have $\mathrm{Map}(i_{m,!}c_X, F) \simeq \mathrm{Map}(c_X, i_m^*F) \simeq \mathrm{Map}_{\mathcal{V}}(X, F(m))$, where c_X is the functor $* \to \mathcal{V}$ with image X.

Lemma 4.4.4.12. Suppose $\mathcal{V}$ is a presentable ∞-category such that the Cartesian product preserves colimits separately in each variable, and $\mathcal{M}$ is a small generalized non-symmetric ∞-operad. Then the ∞-category $\mathrm{Mnd}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{V})$ is the localization of $\mathrm{Fun}(\mathcal{M}, \mathcal{V})$ with respect to the morphisms $i_{m_1,!}X \amalg \cdots \amalg i_{m_n,!}X \to i_{m,!}X$ for all $m \in \mathcal{M}$ with X ranging over a set of objects that generates $\mathcal{V}$ under colimits.

Proof. A functor $F\colon \mathcal{M} \to \mathcal{V}$ is a monoid if and only if it is local with respect to these morphisms. $\square$

Since $\mathrm{Mnd}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{V})$ is equivalent to $\mathrm{Alg}^{\mathcal{O}}_{\mathcal{M}}(\mathcal{V}^\times)$, we have proved the following

Proposition 4.4.4.13. Suppose $\mathbf{V}$ is a left proper simplicial combinatorial Cartesian model category. Then the natural map $\alpha_X\colon (\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V})_{\mathrm{Seg}})_\infty \to \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_X}(\mathbf{V}_\infty^\times)$ is an equivalence.

The map $\alpha_X\colon (\mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathbf{V})_{\mathrm{Seg}})_\infty \to \mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_X}(\mathbf{V}_\infty^\times)$ is natural in X, so applying Corollary 4.4.1.20 and Proposition 4.4.1.23 we get the following comparison of "algebraic" homotopy theories:

Theorem 4.4.4.14. Suppose $\mathbf{V}$ is a left proper simplicial combinatorial Cartesian model category. The natural transformation α induces a functor $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})_\infty \to \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}_\infty^\otimes)_{\mathrm{Set}}$ and this is an equivalence.

The weak equivalences in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ are difficult to describe in general; however, a morphism $f\colon \mathbf{C} \to \mathbf{D}$ between fibrant objects, i.e. Segal categories, is a weak equivalence if and only if it is bijective on objects and a levelwise weak equivalence — given the Segal conditions, it suffices for f to give a weak equivalence $\mathbf{C}(x,y) \to \mathbf{D}(fx, fy)$ for all objects x, y in $\mathbf{C}$. To obtain the correct homotopy theory we clearly also need to invert the morphisms that are fully faithful and essentially surjective in the appropriate sense:

Definition 4.4.4.15. Composition with the projection $\mathbf{V} \to \mathrm{h}\mathbf{V}$ induces a functor

$$\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V}) \to \mathrm{Seg}_{\mathrm{Fun}}(\mathrm{h}\mathbf{V}).$$

This takes Segal categories to categories enriched in $\mathrm{h}\mathbf{V}$. We say a morphism between Segal categories in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ is *weakly fully faithful and homotopically essentially surjective* if its image in $\mathrm{Seg}_{\mathrm{Fun}}(\mathrm{h}\mathbf{V})$ corresponds to a fully faithful and essentially surjective functor of $\mathrm{h}\mathbf{V}$-categories.

This definition extends to give a notion of weak equivalence in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$, and it is possible to construct a model structure with these weak equivalences, cf. [Lur09b, Sim12]. For our purposes, however, it suffices to regard the ∞-category $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})_\infty$ as obtained by inverting the weak equivalences in the full subcategory of fibrant objects (i.e. Segal categories). Then we can construct an ∞-category $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})[\mathrm{FFES}^{-1}]$ by further inverting the weakly fully faithful and homotopically essentially surjective functors between Segal categories; this is equivalent to the ∞-category associated to the above-mentioned model categories.

The weakly fully faithful and homotopically essentially surjective functors between Segal categories clearly correspond to the fully faithful and essentially surjective functors between categorical algebras, so we get the following:

Proposition 4.4.4.16. Suppose $\mathbf{V}$ is a left proper simplicial combinatorial Cartesian model category. There is an equivalence

$$\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathbf{V}^{\otimes}_{\infty})_{\mathrm{Set}}[\mathrm{FFES}^{-1}].$$

Combining this with Proposition 4.4.3.9 gives our comparison result:

Corollary 4.4.4.17. Suppose $\mathbf{V}$ is a left proper simplicial combinatorial Cartesian model category. There is an equivalence

$$\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathbf{V}_\infty}.$$

4.4.5 Comparison with Iterated Segal Spaces

It follows from the results of the previous subsection that the ∞-category $\mathrm{Cat}^{\mathcal{S}}_{(\infty,n)}$ of $\mathcal{S}$-(∞,n)-categories, obtained by iterated enrichment in spaces, is equivalent to that associated to the model category of iterated Segal categories. Our goal in this subsection is to directly compare $\mathrm{Cat}^{\mathcal{S}}_{(\infty,n)}$ to another established model of (∞,n)-categories, namely the iterated Segal spaces of Barwick. We will deduce this comparison from a slightly more general result: we will prove that if $\mathcal{X}$ is an *absolute distributor*, in the sense of [Lur09b], then categorical algebras in $\mathcal{X}$ are equivalent to Segal spaces in $\mathcal{X}$, and complete categorical algebras are equivalent to complete Segal spaces. We begin with a brief review of the notion of distributor:

Definition 4.4.5.1. A *distributor* consists of an ∞-category $\mathcal{X}$ together with a full subcategory $\mathcal{Y}$ such that:

(1) The ∞-categories $\mathcal{X}$ and $\mathcal{Y}$ are presentable.

(2) The full subcategory $\mathcal{Y}$ is closed under small limits and colimits in $\mathcal{X}$.

(3) If $X \to Y$ is a morphism in $\mathcal{X}$ such that $Y \in \mathcal{Y}$, then the pullback functor $\mathcal{Y}_{/Y} \to \mathcal{X}_{/X}$ preserves colimits.

(4) Let $\mathcal{O}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ spanned by those morphisms $f\colon X \to Y$ such that $Y \in \mathcal{Y}$, and let $\pi\colon \mathcal{O} \to \mathcal{Y}$ be the functor given by evaluation at $1 \in \Delta^1$. Since $\mathcal{X}$ admits pullbacks, the evaluation-at-1 functor $\mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$ is a Cartesian fibration, hence so is π. Let $\chi\colon \mathcal{Y} \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{op}}$ be a functor that classifies π. Then χ preserves small limits.

Definition 4.4.5.2. An *absolute distributor* is a presentable ∞-category $\mathcal{X}$ such that the unique colimit-preserving functor $\mathcal{S} \to \mathcal{X}$ that sends $*$ to the final object is fully faithful, and $\mathcal{S} \subseteq \mathcal{X}$ is a distributor.

Proposition 4.4.5.3 ([Lur09b, Corollary 1.2.5])**.** Suppose $\mathcal{Y} \subseteq \mathcal{X}$ is a distributor. Let K be a small simplicial set, and let $\bar{\alpha}\colon \bar{p} \to \bar{q}$ be a natural transformation between functors $\bar{p}, \bar{q}\colon K^{\triangleright} \to \mathcal{X}$. If $\bar{q}$ is a colimit diagram in $\mathcal{Y}$ and $\alpha = \bar{\alpha}|_K$ is Cartesian, then $\bar{\alpha}$ is Cartesian if and only if $\bar{p}$ is a colimit diagram.

Lemma 4.4.5.4. Suppose $\mathcal{X}$ is an absolute distributor. Then for every space $X \in \mathcal{S}$, the map $\gamma_X\colon \mathrm{Fun}(X, \mathcal{X}) \to \mathcal{X}_{/X}$ that sends a functor $F\colon X \to \mathcal{X}$ to its colimit is an equivalence of ∞-categories.

Proof. Let $\xi\colon X \to \mathcal{X}$ be the constant functor at the final object $* \in \mathcal{S} \subseteq \mathcal{X}$. Since X is a space, a functor $F\colon X \to \mathcal{X}$ sends every morphism in X to an equivalence in $\mathcal{X}$, and so the unique natural transformation $F \to \xi$ is Cartesian.

Write $\bar{\xi}\colon X^{\triangleright} \to \mathcal{X}$ for a colimit diagram extending ξ. Then γ_X factors as

$$\mathrm{Fun}(X, \mathcal{X}) \simeq \mathrm{Fun}(X, \mathcal{X})_{/\xi} \xrightarrow{\phi_1} \mathrm{Fun}(X^{\triangleright}, \mathcal{X})_{/\bar{\xi}} \xrightarrow{\phi_2} \mathcal{X}_{/X},$$

where ϕ_2 is given by evaluation at the cone point. The functor ϕ_1 gives an equivalence between $\mathrm{Fun}(X, \mathcal{X})_{/\xi}$ and the full subcategory $\mathcal{E}_1$ of $\mathrm{Fun}(X^{\triangleright}, \mathcal{X})_{/\bar{\xi}}$ spanned by the colimit diagrams. On the other hand, the restriction of ϕ_2 to the full subcategory $\mathcal{E}_2$ spanned by the Cartesian natural transformations to $\bar{\xi}$ is also clearly an equivalence. By Proposition 4.4.5.3 $\mathcal{E}_1 = \mathcal{E}_2$, and so the composite γ_X is indeed an equivalence. □

Proposition 4.4.5.5. Let $\mathcal{O}$ be an ∞-category, and let $F\colon \mathcal{O} \to \mathcal{S}$ be a functor; write $\pi\colon \mathcal{O}_F \to \mathcal{O}$ for the left fibration associated to F. Suppose $\mathcal{X}$ is an absolute distributor. Then left Kan extension along π gives an equivalence $\mathrm{Fun}(\mathcal{O}_F, \mathcal{X}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}, \mathcal{X})_{/F}$.

Proof. By Proposition 2.1.5.13 the ∞-category $\mathrm{Fun}(\mathcal{O}_F, \mathcal{X})$ is equivalent to the ∞-category of sections of the Cartesian fibration $\mathcal{E} \to \mathcal{O}$ whose fibre at $x \in \mathcal{O}$ is $\mathrm{Fun}(F(x), \mathcal{X})$. Since $\mathcal{X}$ is an absolute distributor, by Lemma 4.4.5.4 the ∞-category $\mathcal{E}$ is equivalent over $\mathcal{O}$ to the total space $\mathcal{E}'$ of the Cartesian fibration associated to the functor sending x to $\mathcal{X}_{/F(x)}$. Then $\mathcal{E}'$ is the pullback along F of the Cartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$ given by evaluation at 1, so we have an equivalence between the ∞-category $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{E}')$ of sections and the fibre of $\mathrm{Fun}(\mathcal{O} \times \Delta^1, \mathcal{X}) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{O}, \mathcal{X})) \to \mathrm{Fun}(\mathcal{O}, \mathcal{X})$ at F. This is clearly equivalent to $\mathrm{Fun}(\mathcal{O}, \mathcal{X})_{/F}$, which completes the proof. $\square$

Remark 4.4.5.6. In the cases we are most interested in, where $\mathcal{X}$ is the distributor of n-fold iterated complete Segal spaces in $\mathcal{S}$, we can also prove this without using Proposition 2.1.5.13, by instead rewriting everything in terms of left fibrations over products of Δ^{op}.

Definition 4.4.5.7. Let $\mathcal{X}$ be an absolute distributor. A *Segal space* in $\mathcal{X}$ is a category object $F \colon \Delta^{\mathrm{op}} \to \mathcal{X}$ such that $F([0])$ is in $\mathcal{S} \subseteq \mathcal{X}$.

Proposition 4.4.5.8. Under the equivalence $\pi_! \colon \mathrm{Fun}(\Delta^{\mathrm{op}}_X, \mathcal{X}) \xrightarrow{\sim} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})_{/j_*X}$, the full subcategory $\mathrm{Mnd}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_X}(\mathcal{X})$ of Δ^{op}_X-monoids corresponds to $\mathrm{Seg}(\mathcal{X})_X$, the ∞-category of Segal spaces with 0th space X.

Proof. It is clear that $\pi_!$ takes $\mathrm{Mon}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_X}(\mathcal{X})$ into the ∞-category of functors $\Delta^{\mathrm{op}} \to \mathcal{X}$ that sends $[0]$ to X. Since $\mathrm{Seg}(\mathcal{X})_X$ is a full subcategory of this, it suffices to show that $F \colon \Delta^{\mathrm{op}}_X \to \mathcal{X}$ is a Δ^{op}_X-monoid if and only if $\pi_! F$ is a Segal space in $\mathcal{X}$.

We must show that the Segal morphism

$$\pi_! F([n]) \to \pi_! F([1]) \times_X \cdots \times_X \pi_! F([1]) =: (\pi_! F)^{\mathrm{Seg}}_{[n]}$$

is an equivalence for all n if and only if F is a Δ^{op}_X-monoid. Since π is a coCartesian fibration, $\pi_! F([n]) \simeq \mathrm{colim}_{\xi \in X^{\times(n+1)}} F(\xi)$. It thus suffices to show that $(\pi_! F)^{\mathrm{Seg}}_{[n]}$ is also a colimit of this diagram if and only if F is a Δ^{op}_X-monoid. Using Proposition 4.4.5.3 we see that this condition is equivalent to the natural transformation of functors $(X^{\times(n+1)})^{\triangleright} \to \mathcal{X}$ given by

$$\begin{tikzcd} F(\xi) \arrow[r] \arrow[d] & (\pi_! F)^{\mathrm{Seg}}_{[n]} \arrow[d] \\ \{\xi\} \arrow[r] & X^{\times(n+1)} \end{tikzcd}$$

being Cartesian. Since X is a space, it suffices to check that this square is a pullback. In other words, we must show that the fibre of $(\pi_! F)^{\mathrm{Seg}}_{[n]} \to X^{\times(n+1)}$ at $\xi = (x_0, \ldots, x_n)$ is $F(\xi)$ if and only if F is a Δ^{op}_X-monoid. Since limits commute, it is clear that this fibre is the fibre product

$$(\pi_! F[1])_{(x_0, x_1)} \times_{(\pi_! F[0])_{(x_1)}} \cdots \times_{(\pi_! F[0])_{(x_{n-1})}} (\pi_! F[1])_{(x_{n-1}, x_n)}.$$

But by Proposition 4.4.5.3 again, the natural maps $F(x, y) \to (\pi_! F[1])_{(x,y)}$ and $* = F(x) \to (\pi_! F[0])_x$ are equivalences. Thus the map $F(\xi) \to (\pi_! F[n])_\xi$ is equivalent to the natural map

$$F(\xi) \to F(x_0, x_1) \times \cdots \times F(x_{n-1}, x_n),$$

which is an equivalence if and only if F is a Δ^{op}_X-monoid. $\square$

Corollary 4.4.5.9. Suppose $\mathcal{X}$ is an absolute distributor. The map $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{X}^{\times}) \to \mathrm{Seg}(\mathcal{X})$

given by left Kan extension of the corresponding monoids along the maps $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$ is an equivalence.

Proof. The projection $j^*\colon \mathrm{Seg}(\mathcal{X}) \to \mathcal{S}$ given by composition with $j\colon \{[0]\} \to \Delta^{\mathrm{op}}$ has a right adjoint j_* given by right Kan extension. It follows from Lemma 2.1.6.4 that j^* is a Cartesian fibration. The functor $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{X}^\times) \to \mathrm{Seg}(\mathcal{X})$ clearly preserves Cartesian morphisms, so it suffices to show that this functor induces an equivalence fibrewise, which we proved in Proposition 4.4.5.8. □

Definition 4.4.5.10. Let $\mathcal{X}$ be an absolute distributor, and let $\Lambda\colon \mathcal{X} \to \mathcal{S}$ denote the right adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathcal{X}$. The inclusion $\mathrm{Gpd}(\mathcal{S}) \hookrightarrow \mathrm{Seg}(\mathcal{S}) \hookrightarrow \mathrm{Seg}(\mathcal{X})$ admits a right adjoint $\iota\colon \mathrm{Seg}(\mathcal{X}) \to \mathrm{Gpd}(\mathcal{S})$, which is the composite of the functor $\Lambda\colon \mathrm{Seg}(\mathcal{X}) \to \mathrm{Seg}(\mathcal{S})$ induced by Λ, and $\iota\colon \mathrm{Seg}(\mathcal{S}) \to \mathrm{Gpd}(\mathcal{S})$. We say a Segal space $F\colon \Delta^{\mathrm{op}} \to \mathcal{X}$ is *complete* if the groupoid object ιF is constant.

Remark 4.4.5.11. By Lemma 2.1.10.4, a Segal space F is complete if and only if the map $\iota F(s^0)\colon \iota F[0] \to \iota F[1]$ is an equivalence.

Definition 4.4.5.12. Let E^n denote the Segal space $j_*\{0,\ldots,n\}$. If $\mathcal{X}$ is an absolute distributor we also write E^n for E^n regarded as a Segal space in $\mathcal{X}$ via the inclusion $\mathcal{S} \hookrightarrow \mathcal{X}$.

Proposition 4.4.5.13. Suppose $\mathcal{X}$ is an absolute distributor. Then a Segal space F in $\mathcal{X}$ is complete if and only if it is local with respect to the morphism $E^1 \to E^0$.

Proof. It is clear that F is local with respect to $E^1 \to E^0$, considered as a morphism in $\mathrm{Seg}(\mathcal{X})$, if and only if the Segal space ΛF in $\mathcal{S}$ is local with respect to $E^1 \to E^0$, considered as a morphism in $\mathrm{Seg}(\mathcal{S})$. On the other hand, F is complete if and only if ΛF is complete, so it suffices to prove this for Segal spaces in $\mathcal{S}$. This case is part of [Rez01, Proposition 6.4]. □

Definition 4.4.5.14. Let $\mathrm{CSS}(\mathcal{X})$ denote the full subcategory of $\mathrm{Seg}(\mathcal{X})$ spanned by the complete Segal spaces; by Proposition 4.4.5.13 this is the localization of $\mathrm{Seg}(\mathcal{X})$ with respect to the morphism $E^1 \to E^0$.

Corollary 4.4.5.15. Let $\mathcal{X}$ be an absolute distributor. The equivalence $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{X}^\times) \xrightarrow{\sim} \mathrm{Seg}(\mathcal{X})$ induces an equivalence $\mathrm{Cat}^{\mathcal{X}}_\infty \xrightarrow{\sim} \mathrm{CSS}(\mathcal{X})$.

Proof. It is clear that $E^n_{\mathcal{X}} \in \mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{X}^\times)$ corresponds to $E^n \in \mathrm{Seg}(\mathcal{X})$ under this equivalence. Both sides are therefore the localization with respect to $E^1 \to E^0$. □

Definition 4.4.5.16. By [Lur09b, Corollary 1.3.4], if $\mathcal{X}$ is an absolute distributor, then $\mathrm{CSS}(\mathcal{X})$ is also an absolute distributor. We therefore have absolute distributors $\mathrm{CSS}^n(\mathcal{X})$ of *n-fold iterated complete Segal spaces* in $\mathcal{X}$.

Applying Corollary 4.4.5.15 inductively, we get:

Corollary 4.4.5.17. Let $\mathcal{X}$ be an absolute distributor. Then $\mathrm{Cat}^{\mathcal{X}}_{(\infty,n)} \simeq \mathrm{CSS}^n(\mathcal{X})$.

In particular, taking $\mathcal{X}$ to be the ∞-category $\mathcal{S}$ of spaces, we obtain the desired comparison with iterated Segal spaces:

Corollary 4.4.5.18. There is an equivalence $\mathrm{Cat}^{\mathcal{S}}_{(\infty,n)} \simeq \mathrm{CSS}^n(\mathcal{S})$.

4.5 Natural Transformations and Functor Categories

In this section we consider two approaches to defining *natural transformations* in an enriched ∞-category: In §4.5.1 we consider an internal definition; this is probably the clearer definition, and leads to a functor ∞-category that is easily seen to be an ∞-category. Then in §4.5.2 we consider an external definition, and show the resulting functor ∞-category is equivalent to the internal one; in §4.5.3 we use this definition to construct an $(\infty, 2)$-category of enriched ∞-categories, functors, and natural transformations.

4.5.1 Internal Natural Transformations

In this subsection we introduce an *internal* definition of natural transformations between functors between enriched ∞-categories. We then use this to construct ∞-categories of functors between enriched ∞-categories and show that this is the underlying ∞-category of the internal hom when this exists.

Definition 4.5.1.1. Let $\mathcal{G}_n$ denote the $\mathcal{S}$-graph with objects $\{0, \ldots, n\}$ and

$$\mathcal{G}_n(i,j) = \begin{cases} *, & i < j \\ \emptyset, & j \geq i. \end{cases}$$

We write $[n]_{\mathcal{S}}$ for the free $\mathcal{S}$-∞-category on the graph $\mathcal{G}_n$. If $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category, we write $[n]_{\mathcal{V}}$ for $E^0_{\mathcal{V}} \otimes [n]_{\mathcal{S}}$.

Remark 4.5.1.2. Let $[1]^{\times}$ denote the full subcategory of $\mathcal{S}^{\times}$ on $\emptyset$ and $*$. Then the graph $\mathcal{G}_n$ is obviously defined over $[1]^{\times}$, and the $\mathcal{V}$-∞-category $[n]_{\mathcal{V}}$ exists provided $\mathcal{V}$ has an initial object $\emptyset$ and $x \otimes \emptyset \simeq \emptyset$ for all $x \in \mathcal{V}$.

Remark 4.5.1.3. The inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$ induces an inclusion $\mathrm{Set}_{\Delta} \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$. Let $\delta[n]$ denote the simplicial space associated to the nerve $\mathrm{N}[n]$ under this functor. This is a Segal space, and using our description of free enriched ∞-categories it is easy to see that under the equivalence $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{S}^{\times}) \simeq \mathrm{Seg}^{\mathcal{O}}_{\infty}$ the $\mathcal{S}$-∞-category $[n]_{\mathcal{S}}$ is equivalent to the Segal space $\delta[n]$.

Definition 4.5.1.4. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose F_0 and F_1 are functors $\mathcal{C} \to \mathcal{D}$ of $\mathcal{V}$-∞-categories. A *natural transformation* from F_1 to F_0 is a functor $\phi\colon \mathcal{C} \otimes [1]_{\mathcal{V}} \to \mathcal{D}$ such that $\phi \circ (\mathrm{id}_{\mathcal{C}} \otimes d^i) \simeq F_i$.

Proposition 4.5.1.5. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category. The simplicial $\mathcal{V}$-∞-category $[\bullet]_{\mathcal{V}}$ is a coSegal object in $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$.

Proof. We must show that the natural maps $[1]_{\mathcal{V}} \amalg_{[0]_{\mathcal{V}}} \cdots \amalg_{[0]_{\mathcal{V}}} [1]_{\mathcal{V}} \to [n]_{\mathcal{V}}$ are equivalences. Since $- \otimes E^0_{\mathcal{V}}$ preserves colimits, it suffices to prove this in $\mathcal{S}$. By definition, $[n]_{\mathcal{S}}$ is the free $\mathcal{S}$-∞-category on the graph $\mathcal{G}_n$, and it is obvious that the map $\mathcal{G}_1 \amalg_{\mathcal{G}_0} \cdots \amalg_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_n$ is an equivalence. Since the formation of free $\mathcal{S}$-∞-categories preserves colimits, this implies that $[\bullet]_{\mathcal{S}}$ is a coSegal object. $\square$

Definition 4.5.1.6. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. The (*internal*) *functor ∞-category* $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is the simplicial space $\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})}(\mathcal{C} \otimes [\bullet]_{\mathcal{S}}, \mathcal{D})$.

Corollary 4.5.1.7. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. Then $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is a Segal space.

Remark 4.5.1.8. Using the results of Joyal and Tierney [JT07] we can describe the quasicategory associated to this Segal space as the simplicial set $\mathrm{Hom}(\mathcal{C} \otimes [\bullet]_{\mathcal{V}}, \mathcal{D})$.

Proposition 4.5.1.9. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. For any Segal space X we have an equivalence

$$\mathrm{Map}_{\mathrm{Seg}_{\infty}^{\mathcal{O}}}(X, \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})}(\mathcal{C} \otimes X, \mathcal{D}),$$

where on the right we regard X as an $\mathcal{S}$-∞-category.

Proof. Every Segal space can be canonically written as a colimit of a diagram of the objects $\delta[n]$. Specifically, the Segal space X is the coend of

$$\bar{X} \colon \Delta \times \Delta^{\mathrm{op}} \to \mathrm{Seg}_{\infty}^{\mathcal{O}}, \qquad ([n], [m]) \mapsto \operatorname*{colim}_{X_m} \delta[n].$$

Since $\mathrm{Map}(\delta[n], \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Map}(\mathcal{C} \otimes [n]_{\mathcal{S}}, \mathcal{D})$ we then have

$$\begin{aligned} \mathrm{Map}(X, \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})) &\simeq \mathrm{Map}(\mathrm{coend}\, \bar{X}, \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})) \\ &\simeq \mathrm{end}\, \mathrm{Map}(\bar{X}, \mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})) \\ &\simeq \mathrm{end}\, \mathrm{Map}(\mathcal{C} \otimes \bar{X}, \mathcal{D}) \\ &\simeq \mathrm{Map}(\mathcal{C} \otimes X, \mathcal{D}). \end{aligned}$$

□

Corollary 4.5.1.10. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. The underlying space $\iota\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ of the Segal space $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is $|\mathrm{Map}(\mathcal{C} \otimes E^{\bullet}_{\mathcal{V}}, \mathcal{D})|$. In particular, if $\mathcal{D}$ is a complete $\mathcal{V}$-∞-category then $\iota\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is equivalent to $\mathrm{Map}_{\mathrm{Cat}_{\infty}^{\mathcal{V}}}(\mathcal{C}, \mathcal{D})$, so the Segal space $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is complete.

Proof. The underlying groupoid object of a Segal space X is $\mathrm{Map}(E^{\bullet}, X)$. By Proposition 4.5.1.9, the underlying groupoid object of $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is therefore $\mathrm{Map}(\mathcal{C} \otimes E^{\bullet}, \mathcal{D})$, and the underlying space is the colimit of this simplicial space. By Corollary 4.2.4.10 it follows that if $\mathcal{D}$ is complete then $\iota\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Map}(\mathcal{C}, \mathcal{D})$. □

Now suppose $\mathcal{V}^{\otimes}$ is a presentably symmetric monoidal ∞-category. Then $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$ and $\mathrm{Cat}_{\infty}^{\mathcal{V}}$ are also symmetric monoidal, and the induced tensor products preserve colimits in each variable. This implies that $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$ and $\mathrm{Cat}_{\infty}^{\mathcal{V}}$ have internal hom objects; we write $\mathcal{D}^{\mathcal{C}}$ for the internal hom object for maps $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})$.

Lemma 4.5.1.11. Let $\mathcal{V}$ be a presentably symmetric monoidal ∞-category, and suppose $\mathcal{D}$ is a complete $\mathcal{V}$-∞-category. Then $\mathcal{D}^{\mathcal{C}}$ is also a complete $\mathcal{V}$-∞-category, for all $\mathcal{V}$-∞-categories $\mathcal{C}$. Moreover, $\mathcal{D}^{\mathcal{C}}$ is also the internal hom in $\mathrm{Cat}_{\infty}^{\mathcal{V}}$.

Proof. We must show that $\mathrm{Map}(E^0, \mathcal{D}^{\mathcal{C}}) \to \mathrm{Map}(E^1, \mathcal{D}^{\mathcal{C}})$ is an equivalence. Passing to left adjoints this is $\mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}(E^1 \otimes \mathcal{C}, \mathcal{D})$, which is an equivalence since $\mathcal{C} \otimes E^1 \to \mathcal{C}$ is a local equivalence by Proposition 4.2.4.9.

Since $\mathcal{D}^{\mathcal{C}}$ is complete we have, for any complete $\mathcal{V}$-∞-category $\mathcal{A}$,

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cat}_{\infty}^{\mathcal{V}}}(\mathcal{A}, \mathcal{D}^{\mathcal{C}}) &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})}(\mathcal{A}, \mathcal{D}^{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}^{\mathcal{O}}(\mathcal{V}^{\otimes})}(\mathcal{A} \otimes \mathcal{C}, \mathcal{D}) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\infty}^{\mathcal{V}}}(\mathcal{A} \otimes \mathcal{C}, \mathcal{D}), \end{aligned}$$

hence $\mathcal{D}^{\mathcal{C}}$ is also the internal hom in $\mathrm{Cat}_\infty^{\mathcal{V}}$. $\square$

Proposition 4.5.1.12. Let $\mathcal{V}^\otimes$ be a presentably monoidal ∞-category. Write $t\colon \mathcal{S} \to \mathcal{V}$ for the unique colimit-preserving strong monoidal functor sending $*$ to the unit I, and let $u\colon \mathcal{V} \to \mathcal{S}$ be its lax monoidal right adjoint, given by $\mathrm{Map}(I,–)$. Then if $\mathcal{C}$ is a $\mathcal{V}$-∞-category the Segal space corresponding to the $\mathcal{S}$-∞-category $u_*\mathcal{C}$ is $\mathrm{Map}([\bullet]_{\mathcal{V}}, \mathcal{C})$.

Proof. Since $[n]_{\mathcal{S}}$ is the $\mathcal{S}$-∞-category corresponding to $\delta[n]$, the Segal space corresponding to $u_*\mathcal{C}$ is $\mathrm{Map}([n]_{\mathcal{S}}, u_*\mathcal{C}) \simeq \mathrm{Map}(t_*[n]_{\mathcal{S}}, \mathcal{C}) \simeq \mathrm{Map}([n]_{\mathcal{V}}, \mathcal{C})$. $\square$

Corollary 4.5.1.13. Let $\mathcal{V}$ be a presentably symmetric monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. The Segal space corresponding to the $\mathcal{S}$-∞-category $u_*\mathcal{D}^{\mathcal{C}}$ is $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D})$.

Proof. The Segal space associated to $u_*\mathcal{C}^{\mathcal{D}}$ is given by

$$\mathrm{Map}([\bullet]_{\mathcal{V}}, \mathcal{D}^{\mathcal{C}}) \simeq \mathrm{Map}(\mathcal{C} \otimes [\bullet]_{\mathcal{S}}, \mathcal{D}). \qquad \square$$

4.5.2 External Natural Transformations

In this section we give an *external* definition of natural transformations, and prove that this is equivalent to the internal definition. We first introduce some notation:

Definition 4.5.2.1. If $X\colon \Delta^{\mathrm{op}} \to \mathcal{S}$ is a Segal space, then the associated right fibration $\mathcal{X} \to \Delta^{\mathrm{op}}$ is a double ∞-category. We write $\Delta^{\mathrm{op}}[n] \to \Delta^{\mathrm{op}}$ for the double ∞-category associated in this way to the nerve of the category $[n]$, regarded as a Segal space via the inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$.

Remark 4.5.2.2. The ∞-category $\Delta^{\mathrm{op}}[n]$ can be identified with the category of simplices $\mathrm{Simp}(\mathrm{N}[n])$ or $\mathrm{Simp}(\Delta^n)$ of the nerve of $[n]$. Its objects can be described as sequences $(i_0, \ldots, i_m)$, where $0 \leq i_j \leq i_{j+1} \leq n$, and for every $\phi\colon [k] \to [m]$ in Δ there is a unique morphism $(i_0, \ldots, i_m) \to (i_{\phi(0)}, \ldots, i_{\phi(k)})$.

Definition 4.5.2.3. For X a space, let $\Delta^{\mathrm{op}}_X[n]$ denote the double ∞-category $\Delta^{\mathrm{op}}_X \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}[n]$.

Remark 4.5.2.4. Objects of $\Delta^{\mathrm{op}}_X[n]$ can be described as lists $((x_0, i_0), \ldots, (x_k, i_k))$ where $x_i \in X$ and $0 \leq i_0 \leq \cdots \leq i_k \leq n$.

Definition 4.5.2.5. Let $\mathcal{V}^\otimes$ be a monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. If F_0 and F_1 are functors $\mathcal{C} \to \mathcal{D}$, an (*external*) *natural transformation* from F_0 to F_1 is a morphism of $\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[1]$-algebras $\eta\colon s^{0,*}\mathcal{C} \to \phi^* s^{0,*}\mathcal{D}$, where $\phi\colon \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[1] \to \Delta^{\mathrm{op}}_{\iota_0\mathcal{D}}[1]$ is the morphism induced by $(\iota_0 F_0, \iota_0 F_1)\colon \iota_0\mathcal{C} \times [1] \to \iota_0\mathcal{D}$, such that η restricts to F_i when restricted to $\Delta^{\mathrm{op}}_X \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{\{i\}}$.

Remark 4.5.2.6. The natural transformation η thus determines morphisms

$$\mathcal{C}(x, y) \to \mathcal{D}(F_0 x, F_1 y)$$

in $\mathcal{V}$ for all $x, y \in \mathcal{C}$. These are compatible with composition, which implies that, as expected, they are determined by the images $I \to \mathcal{D}(F_0 x, F_1 x)$ of the identity morphisms $I \to \mathcal{C}(x, x)$ for $x \in \mathcal{C}$.

Definition 4.5.2.7. Let $\mathcal{V}^{\otimes}$ be a monoidal ∞-category. The objects $\Delta^{\mathrm{op}}[n]$ clearly form a cosimplicial object in generalized non-symmetric ∞-operads, hence they determine a natural transformation of simplicial ∞-categories $\mathrm{Alg}^{\mathcal{O}}_{/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes}) \to (\mathrm{Opd}^{\mathcal{O},\mathrm{gen}}_{\infty})_{/\Delta^{\mathrm{op}}[n]}$ (cf. Remark 3.2.8.2 for this notation). If $\mathcal{C}$ is a $\mathcal{V}$-∞-category, the $\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[n]$-algebras $\pi_n^*\mathcal{C}$, where $\pi_n\colon \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[n] \to \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}$ denotes the map of generalized non-symmetric ∞-operads induced by the unique morphism $\pi_n\colon [n] \to [0]$ in Δ, determine a section. Given $\mathcal{V}$-∞-categories $\mathcal{C}$ and $\mathcal{D}$ we therefore get a simplicial space $\mathrm{Fun}^{\mathcal{V}}_{\mathrm{ext}}(\mathcal{C}, \mathcal{D})$ with

$$\mathrm{Fun}^{\mathcal{V}}_{\mathrm{ext}}(\mathcal{C}, \mathcal{D})_n := \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes})}(\pi_n^*\mathcal{C}, \pi_n^*\mathcal{D}).$$

This is the (*external*) *functor ∞-category* from $\mathcal{C}$ to $\mathcal{D}$.

Lemma 4.5.2.8. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ is a $\mathcal{V}$-∞-category. Write I_n for the inclusion $\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[n] \to \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}\times[n]}$. Then the functor $\mathcal{C} \otimes [n] \to \mathcal{C}$ determines an equivalence $I_n^*(\mathcal{C} \otimes [n]) \xrightarrow{\sim} \pi_n^*\mathcal{C}$.

Proof. It suffices to observe that for $i \leq j$ and any $x, y \in \mathcal{C}$ the morphism

$$(\mathcal{C} \otimes [n])((x,i),(y,j)) \to \mathcal{C}(x,y)$$

is an equivalence. □

Proposition 4.5.2.9. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ is a $\mathcal{V}$-∞-category. The morphism $I_{n,!}(\pi_n^*\mathcal{C}) \to \mathcal{C} \otimes [n]$ adjunct to the inverse equivalence $\pi_n^*\mathcal{C} \xrightarrow{\sim} I_n^*(\mathcal{C} \otimes [n])$ is an equivalence.

Proof. This is immediate from the description of free algebras in terms of operadic colimits. □

Corollary 4.5.2.10. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. Using Lemma 4.5.2.8, the inclusions $I_n\colon \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[n] \to \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}\times[n]}$ induce a natural transformation

$$\eta_n\colon \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{/\Delta^{\mathrm{op}}_{[n]}}(\mathcal{V}^{\otimes})}(\mathcal{C} \otimes [n], p_n^*\mathcal{D}) \to \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes})}(\pi_n^*\mathcal{C}, \pi_n^*\mathcal{D}),$$

where p_n denotes the morphism induced by the projection $\Delta^{\mathrm{op}}_{[n]} \to \Delta^{\mathrm{op}}$, i.e. a morphism of simplicial spaces

$$\mathrm{Fun}^{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathcal{V}}_{\mathrm{ext}}(\mathcal{C}, \mathcal{D}).$$

This is an equivalence.

Proof. To show that η_n is an equivalence, it suffices to show that it gives an equivalence on the fibres over each map $\phi\colon \iota_0\mathcal{C} \times [n] \to \iota_0\mathcal{D}$. This can be identified with

$$\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}\times[n]}}(\mathcal{V}^{\otimes})}(\mathcal{C} \otimes [n], \phi^* p_n^*\mathcal{D}) \to \mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[n]}(\mathcal{V}^{\otimes})}(\pi_n^*\mathcal{C}, I_n^*\phi^* p_n^*\mathcal{D}).$$

This is an equivalence, since $\mathcal{C} \otimes [n]$ is $I_{n,!}\pi_n^*\mathcal{C}$ by Proposition 4.5.2.9. □

Conjecture 4.5.2.11. Let $\mathcal{V}^{\otimes}$ be a presentably symmetric monoidal ∞-category. A natural transformation $\eta\colon \mathcal{C} \otimes [1] \to \mathcal{D}$ is a natural equivalence (i.e. extends to a functor from

$\mathcal{C} \otimes E^1$) if and only if for each $c \in \mathcal{C}$ the morphism $[1] \to \mathcal{D}$ given by restricting to c is an equivalence in $\mathcal{D}$.

Sketch Proof. The "only if" direction is obvious. We may therefore assume given a natural transformation $\eta\colon \mathcal{C} \otimes [1] \to \mathcal{D}$ from F to G such that the induced maps $[1] \to \mathcal{D}$ are equivalences for all $c \in \mathcal{C}$. To show that this extends to a natural equivalence, we will show that the adjunct morphism $[1]_{\mathcal{V}} \to \mathcal{D}^{\mathcal{C}}$ is an equivalence. Since $[1]_{\mathcal{V}}$ is $t_*[1]_{\mathcal{S}}$, we may equivalently show that the associated functor $[1]_{\mathcal{S}} \to u_*\mathcal{D}^{\mathcal{C}}$ is an equivalence. By Proposition 4.2.1.16 it suffices to show that for any $H \in \mathcal{D}^{\mathcal{C}}$ the map $u_*\mathcal{D}^{\mathcal{C}}(H, F) \to u_*\mathcal{D}^{\mathcal{C}}(H, G)$ given by composition with η is an equivalence.

By Corollary 4.5.2.10 and Corollary 4.5.1.13 we may identify $u_*\mathcal{D}^{\mathcal{C}}$ with the external functor ∞-category $\mathrm{Fun}^{\mathcal{V}}_{\mathrm{ext}}(\mathcal{C}, \mathcal{D})$. For fixed $H\colon \mathcal{C} \to \mathcal{D}$, let $\phi\colon \iota_0\mathcal{C} \times [1] \to \mathcal{D}$ be the map determined by (H, F). Then we can identify $u_*\mathcal{D}^{\mathcal{C}}(H, F)$ with the fibre of

$$\mathrm{Map}_{\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[1]}(\mathcal{V}^{\otimes})}(\pi_1^*\mathcal{C}, \phi^*\pi_1^*\mathcal{D}) \to \mathrm{Map}(\mathcal{C}, H^*\mathcal{D}) \times \mathrm{Map}(\mathcal{C}, F^*\mathcal{D})$$

at (H, F). Now $\mathrm{Alg}^{\mathcal{O}}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[1]}(\mathcal{V}^{\otimes})$ is monadic over $\mathrm{Fun}(\iota_0\mathcal{C} \times \iota_0\mathcal{C} \times \{(0,0),(0,1),(1,1)\}, \mathcal{V})$. This means we can describe the mapping space as the limit of a diagram of spaces whose vertices are mapping spaces between the free $\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}[1]$-algebra monad applied some number of times to the underlying functors for $\pi_1^*\mathcal{C}$ and $\phi^*\pi_1^*\mathcal{D}$ in this functor category. After taking the appropriate fibres, we see that this means the map $u_*\mathcal{D}^{\mathcal{C}}(H, F) \to u_*\mathcal{D}^{\mathcal{C}}(H, G)$ given by composition with η is indeed an equivalence, since equivalences in functor categories are detected pointwise. □

Remark 4.5.2.12. This result should clearly also be true without the assumption that $\mathcal{V}$ is symmetric monoidal, but this proof seems to rely essentially on the existence of the internal hom $\mathcal{D}^{\mathcal{C}}$ to reduce the construction of the inverse from $\mathcal{V}$ to $\mathcal{S}$.

4.5.3 The $(\infty, 2)$-Category of $\mathcal{V}$-∞-Categories

In this section we use the external definition of natural transformations to define an $(\infty, 2)$-category of $\mathcal{V}$-∞-categories, functors, and natural transformations.

It is clear that the full subcategory of $(\mathrm{Opd}^{\mathcal{O},\mathrm{gen}}_{\infty})_{/\Delta^{\mathrm{op}}[n]}$ spanned by generalized non-symmetric ∞-operads of the form $\Delta^{\mathrm{op}}_X[n]$ for some space X is equivalent to the full subcategory $\mathrm{diag}_n\mathcal{S}$ of $\mathcal{S}^{\times n}$ spanned by objects of the form $(X, \ldots, X)$.

Definition 4.5.3.1. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category. Write $\mathcal{A}_n$ for the pullback

$$\begin{tikzcd}
\mathcal{A}_n \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathcal{O}}_{/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes}) \arrow[d] \\
\mathrm{diag}_n\mathcal{S} \arrow[r] & (\mathrm{Opd}^{\mathcal{O},\mathrm{gen}}_{\infty})_{/\Delta^{\mathrm{op}}[n]}.
\end{tikzcd}$$

Then we define $\mathrm{Cat}^{\mathcal{V}}_{\infty}[n]$ to be the full subcategory of $\mathcal{A}_n$ spanned by objects of the form $\pi_n^*\mathcal{C}$ where $\mathcal{C}$ is a complete $\mathcal{V}$-∞-category. This gives a simplicial ∞-category $\mathrm{CAT}^{\mathcal{V}}_{\infty} := \mathrm{Cat}^{\mathcal{V}}_{\infty}[\bullet]$.

Remark 4.5.3.2. We must restrict to complete $\mathcal{V}$-∞-categories to get the right mapping spaces: By Remark 4.2.5.6 if $\mathcal{D}$ is not complete then $\iota\mathrm{Fun}^{\mathcal{V}}(\mathcal{C},\mathcal{D}) \simeq |\mathrm{Map}(\mathcal{C}\otimes E^{\bullet},\mathcal{D})|$ is not in general equivalent to the space of maps from $\mathcal{C}$ to $\mathcal{D}$ in $\mathrm{Cat}_{\infty}^{\mathcal{V}}$.

Proposition 4.5.3.3. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category. The simplicial ∞-category $\mathrm{CAT}_{\infty}^{\mathcal{V}}$ is a double ∞-category.

Proof. We must show that for each n the Segal morphism

$$\mathrm{Cat}_{\infty}^{\mathcal{V}}[n] \to \mathrm{Cat}_{\infty}^{\mathcal{V}}[1] \times_{\mathrm{Cat}_{\infty}^{\mathcal{V}}} \cdots \times_{\mathrm{Cat}_{\infty}^{\mathcal{V}}} \mathrm{Cat}_{\infty}^{\mathcal{V}}[1]$$

is an equivalence. On both sides the objects are just complete $\mathcal{V}$-∞-categories, so this functor is clearly essentially surjective; it remains to show that it is fully faithful. Let $\mathcal{C}$ and $\mathcal{D}$ be two complete $\mathcal{V}$-∞-categories; we must show that the morphism

$$\mathrm{Map}(\pi_n^*\mathcal{C},\pi_n^*\mathcal{D}) \to \mathrm{Map}(\pi_1^*\mathcal{C},\pi_1^*\mathcal{D}) \times_{\mathrm{Map}(\mathcal{C},\mathcal{D})} \cdots \times_{\mathrm{Map}(\mathcal{C},\mathcal{D})} \mathrm{Map}(\pi_1^*\mathcal{C},\pi_1^*\mathcal{D})$$

is an equivalence. Using Corollary 4.5.2.10 we can identify the left-hand side here as $\mathrm{Map}(\mathcal{C}\otimes[n],\mathcal{D})$ and the right-hand side as

$$\mathrm{Map}(\mathcal{C}\otimes[1],\mathcal{D}) \times_{\mathrm{Map}(\mathcal{C},\mathcal{D})} \cdots \times_{\mathrm{Map}(\mathcal{C},\mathcal{D})} \mathrm{Map}(\mathcal{C}\otimes[1],\mathcal{D}).$$

This map is therefore an equivalence by Corollary 4.5.1.7. □

Lemma 4.5.3.4. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category. Then the Segal space $\mathrm{Map}(\Delta^1,\mathrm{Cat}_{\infty}^{\mathcal{V}}[\bullet])$ is complete.

Sketch Proof. We must show that $s^0\colon \mathrm{Map}(\Delta^1,\mathrm{Cat}_{\infty}^{\mathcal{V}}) \to \mathrm{Map}(E^1,\mathrm{Map}(\Delta^1,\mathrm{Cat}_{\infty}^{\mathcal{V}}[\bullet]))$ is an equivalence. To see this it suffices to show that the map induces an equivalence on fibres over all $(\mathcal{C},\mathcal{D})$ in $(\mathrm{Cat}_{\infty}^{\mathcal{V}})^{\times 2}$. It follows from Proposition 4.5.1.9 that over $(\mathcal{C},\mathcal{D})$ we get the map

$$s^0\colon \mathrm{Map}(\mathcal{C},\mathcal{D}) \to \mathrm{Map}(\mathcal{C}\otimes E^1,\mathcal{D}),$$

and this is an equivalence since $\mathcal{D}$ is by assumption complete. □

Proposition 4.5.3.5. Let $\mathcal{V}^{\otimes}$ be a presentably monoidal ∞-category. Then the simplicial space $\iota\mathrm{Cat}_{\infty}^{\mathcal{V}}[\bullet]$ is constant.

Proof. This follows by combining Lemma 4.5.3.4 and Proposition 2.2.2.13 since it is obvious from the definition that $s_0\colon \mathrm{Cat}_{\infty}^{\mathcal{V}}[0] \to \mathrm{Cat}_{\infty}^{\mathcal{V}}[1]$ is essentially surjective. □

The simplicial ∞-category $\mathrm{CAT}_{\infty}^{\mathcal{V}}$ is thus a Segal object in ∞-categories whose underlying simplicial space is constant. This means that we may consider it as an $(\infty,2)$-category — the $(\infty,2)$-category of $\mathcal{V}$-∞-categories, functors, and natural transformations. The ∞-category of morphisms from $\mathcal{C}$ to $\mathcal{D}$ in $\mathrm{CAT}_{\infty}^{\mathcal{V}}$ is precisely $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C},\mathcal{D})$, as it should be.

4.6 Correspondences

If $\mathbf{V}$ is a closed symmetric monoidal category and $\mathbf{C}$ and $\mathbf{D}$ are $\mathbf{V}$-categories, a *correspondence* or *profunctor* from $\mathbf{C}$ to $\mathbf{D}$ is a functor $\mathbf{C}\otimes\mathbf{D}^{\mathrm{op}} \to \overline{\mathbf{V}}$, where $\overline{\mathbf{V}}$ is $\mathbf{V}$ regarded as a $\mathbf{V}$-category via the internal hom. Our goal in this section is to introduce an ∞-categorical version of correspondences between enriched ∞-categories; our definition will be "external", using algebras for certain double ∞-categories, and is inspired by that given by Bacard [Bac10] in the context of 2-categories.

4.6.1 Correspondences between $\mathcal{V}$-∞-Categories

To give our definition of a correspondence, we first introduce some notation:

Definition 4.6.1.1. Given spaces X and Y, consider the functor $F_{X,Y}\colon \{0,1\} \to \mathcal{S}$ that sends 0 to X and 1 to Y. Let $j\colon \{0,1\} \hookrightarrow \Delta^{\mathrm{op}}[1]$ denote the inclusion of the fibre at $[0]$. The right Kan extension $j_*F_{X,Y}$ is clearly a $\Delta^{\mathrm{op}}[1]$-category object. We write $\Delta^{\mathrm{op}}_{X<Y} \to \Delta^{\mathrm{op}}[1]$ for the left fibration associated to $j_*F_{X,Y}$. Then $\Delta^{\mathrm{op}}_{X<Y}$ is a double ∞-category.

Remark 4.6.1.2. If $X = Y$, then $\Delta^{\mathrm{op}}_{X<X}$ is precisely $\Delta^{\mathrm{op}}_{X}[1]$ as defined above.

Example 4.6.1.3. If $X_0, \dots, X_n$ are sets, we can represent objects of $\Delta^{\mathrm{op}}_{X_0,\dots,X_n}$ as sequences $(x_0^1, \dots, x_0^{m_0} | x_1^1, \dots, x_1^{m_1} | \dots | x_n^1, \dots, x_n^{m_n})$ where $x_i^j \in X_i$.

Remark 4.6.1.4. Pulling back $\Delta^{\mathrm{op}}_{X<Y} \to \Delta^{\mathrm{op}}[1]$ along the two inclusions $d^0, d^1\colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}[1]$ clearly gives Δ^{op}_X and Δ^{op}_Y, respectively.

Definition 4.6.1.5. Let $\mathcal{V}^\otimes$ be a monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. A *correspondence* from $\mathcal{C}$ to $\mathcal{D}$ is a $\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}<\iota_0\mathcal{D}}$-algebra $M\colon \Delta^{\mathrm{op}}_{\iota_0\mathcal{C}<\iota_0\mathcal{D}} \to \mathcal{V}^\otimes$ such that the restrictions to $\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}$ and $\Delta^{\mathrm{op}}_{\iota_0\mathcal{D}}$ are $\mathcal{C}$ and $\mathcal{D}$, respectively. We will use the notation $M\colon \mathcal{C} \nrightarrow \mathcal{D}$ for a correspondence M from $\mathcal{C}$ to $\mathcal{D}$.

Definition 4.6.1.6. Let $\mathcal{V}^\otimes$ be a monoidal ∞-category, and suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-∞-categories. The *∞-category of $\mathcal{V}$-correspondences* $\mathrm{Corr}^{\mathcal{V}}(\mathcal{C},\mathcal{D})$ from $\mathcal{C}$ to $\mathcal{D}$ is

$$\{\mathcal{C}\} \times_{\mathrm{Alg}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}}}(\mathcal{V}^\otimes)} \mathrm{Alg}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{C}<\iota_0\mathcal{D}}}(\mathcal{V}^\otimes) \times_{\mathrm{Alg}_{\Delta^{\mathrm{op}}_{\iota_0\mathcal{D}}}(\mathcal{V}^\otimes)} \{\mathcal{D}\}.$$

Remark 4.6.1.7. There should of course be an inclusion $\mathrm{Fun}^{\mathcal{V}}(\mathcal{C},\mathcal{D}) \to \mathrm{Corr}^{\mathcal{V}}(\mathcal{C},\mathcal{D})$, but using our definitions it is not obvious how to construct this.

If the ∞-category $\mathcal{V}$ is presentably monoidal, then we can compose correspondences. To see this, we first need some more notation:

Definition 4.6.1.8. Given spaces $X_0, \dots, X_n$, consider the functor $F_{X_0,\dots,X_n}\colon \{0,\dots,n\} \to \mathcal{S}$ that sends i to X_i. Let $j\colon \{0,\dots,n\} \hookrightarrow \Delta^{\mathrm{op}}[n]$ denote the inclusion of the fibre at $[0]$. The right Kan extension $j_*F_{X_0,\dots,X_n}$ is clearly a $\Delta^{\mathrm{op}}[n]$-category object. We write $\Delta^{\mathrm{op}}_{X_0<\cdots<X_n} \to \Delta^{\mathrm{op}}[n]$ for the right fibration associated to $j_*F_{X_0,\cdots,X_n}$. Then $\Delta^{\mathrm{op}}_{X_0<\cdots<X_n}$ is a double ∞-category.

Definition 4.6.1.9. Given spaces $X_0, \dots, X_n$, let $\Delta^{\mathrm{op},\amalg}_{X_0<\cdots<X_n}$ denote the colimit of generalized non-symmetric ∞-operads

$$\Delta^{\mathrm{op}}_{X_0<X_1} \amalg_{\Delta^{\mathrm{op}}_{X_1}} \cdots \amalg_{\Delta^{\mathrm{op}}_{X_{n-1}}} \Delta^{\mathrm{op}}_{X_{n-1}<X_n}.$$

Let $\kappa^{X_0<\dots<X_n}$ (or just κ) denote the inclusion $\Delta^{\mathrm{op},\amalg}_{X_0<\cdots<X_n} \to \Delta^{\mathrm{op}}_{X_0<\cdots<X_n}$.

Definition 4.6.1.10. Given spaces $X_0, \ldots, X_n$, we say a $\triangle^{\mathrm{op}}_{X_0 < \cdots < X_n}$-algebra M in $\mathcal{V}^{\otimes}$ is a *composite* if it is the left operadic Kan extension of its restriction to $\triangle^{\mathrm{op},\amalg}_{X_0 < \cdots < X_n}$, i.e. if the adjunction morphism $\kappa_! \kappa^* M \to M$ is an equivalence.

Definition 4.6.1.11. Given $\mathcal{V}$-∞-categories $\mathcal{C}_0, \ldots, \mathcal{C}_n$, let $\mathrm{Corr}^{\mathcal{V}}(\mathcal{C}_0, \ldots, \mathcal{C}_n)$ denote the full subcategory of

$$\mathrm{Alg}_{\triangle^{\mathrm{op}}_{\iota_0\mathcal{C}_0 < \cdots < \iota_0\mathcal{C}_n}}(\mathcal{V}^{\otimes}) \times_{\mathrm{Alg}_{\triangle^{\mathrm{op}}_{\iota_0\mathcal{C}_0}}(\mathcal{V}^{\otimes}) \times \cdots \times \mathrm{Alg}_{\triangle^{\mathrm{op}}_{\iota_0\mathcal{C}_n}}(\mathcal{V}^{\otimes})} \{(\mathcal{C}_0, \ldots, \mathcal{C}_n)\}$$

spanned by those $\triangle^{\mathrm{op}}_{\iota_0\mathcal{C}_0 < \cdots < \iota_0\mathcal{C}_n}$-algebras that are composites.

Given $\mathcal{V}$-∞-categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, the projection

$$\mathrm{Corr}^{\mathcal{V}}(\mathcal{C}, \mathcal{D}, \mathcal{E}) \to \mathrm{Corr}^{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \times \mathrm{Corr}^{\mathcal{V}}(\mathcal{D}, \mathcal{E}),$$

given by restriction along the vertices $(0,1)$ and $(1,2)$ in $\triangle^{\mathrm{op}}[2]$ is an equivalence — this will follow from Corollary 4.6.2.6. Since there is also a map $\mathrm{Corr}^{\mathcal{V}}(\mathcal{C}, \mathcal{D}, \mathcal{E}) \to \mathrm{Corr}^{\mathcal{V}}(\mathcal{C}, \mathcal{E})$ coming from restriction to $(0,2)$, this means that given correspondences $\mathcal{C} \nrightarrow \mathcal{D}$ and $\mathcal{D} \nrightarrow \mathcal{E}$ we can compose them to get a correspondence $\mathcal{C} \nrightarrow \mathcal{E}$.

It is possible to define a Segal category using the ∞-categories $\mathrm{Corr}^{\mathcal{V}}(\mathcal{C}_0, \ldots, \mathcal{C}_n)$, giving a model for the $(\infty, 2)$-category of $\mathcal{V}$-∞-categories and correspondences. In §4.6.3 we will construct a different model for this $(\infty, 2)$-category as the subcategory of horizontal morphisms in a double ∞-category of $\mathcal{V}$-∞-categories, functors, and correspondences.

4.6.2 The Double ∞-Categories $\triangle^{\mathrm{op},\amalg}_{X_0 < \cdots < X_n}$

In this subsection we will give an explicit model for the pushout of generalized non-symmetric ∞-operads $\triangle^{\mathrm{op},\amalg}_{X_0 < \cdots < X_n}$, which will allow us to better understand the functors $\kappa_!$.

Definition 4.6.2.1. Let $\mathbb{\Lambda}^{\mathrm{op}}[n]$ be the full subcategory of $\triangle^{\mathrm{op}}[n]$ spanned by those objects $(i_0, \ldots, i_n)$ such that $|i_{k+1} - i_k| \leq 1$ (i.e. the i_j's can jump by at most 1 at each step).

Definition 4.6.2.2. Given spaces $X_0, \ldots, X_n$, let $\mathbb{\Lambda}^{\mathrm{op}}_{X_0 < \cdots < X_n}$ be defined by the pullback

$$\begin{tikzcd}
\mathbb{\Lambda}^{\mathrm{op}}_{X_0 < \cdots < X_n} \arrow[r] \arrow[d] & \triangle^{\mathrm{op}}_{X_0 < \cdots < X_n} \arrow[d] \\
\mathbb{\Lambda}^{\mathrm{op}}[n] \arrow[r] & \triangle^{\mathrm{op}}[n].
\end{tikzcd}$$

Lemma 4.6.2.3. Given spaces $X_0, \ldots, X_n$, the projection $\mathbb{\Lambda}^{\mathrm{op}}_{X_0 < \cdots < X_n} \to \triangle^{\mathrm{op}}$ is a generalized non-symmetric ∞-operad

Proof. $\mathbb{\Lambda}^{\mathrm{op}}_{X_0 < \cdots < X_n} \to \triangle^{\mathrm{op}}$ is the full subcategory on some of the objects in the fibre at $[1]$. $\square$

Theorem 4.6.2.4. For spaces $X_0, \ldots, X_n$, let $\triangle^{\mathrm{op},\amalg}_{X_0 < \cdots < X_n}$ denote the colimit in simplicial sets

$$\triangle^{\mathrm{op}}_{X_0 < X_1} \amalg_{\triangle^{\mathrm{op}}_{X_1}} \cdots \amalg_{\triangle^{\mathrm{op}}_{X_{n-1}}} \triangle^{\mathrm{op}}_{X_{n-1} < X_n}.$$

Then the inclusion $\mathbb{\Delta}^{\mathrm{op},\mathrm{II}}_{X_0<\cdots<X_n} \hookrightarrow \mathbb{\Lambda}^{\mathrm{op}}_{X_0<\cdots<X_n}$ is a trivial cofibration in the generalized non-symmetric ∞-operad model structure.

Lemma 4.6.2.5. Suppose $\phi\colon \mathcal{M} \to \mathcal{N}$ is a left fibration of generalized non-symmetric ∞-operads and $\bar{\sigma}\colon \Delta^n \to \mathcal{N}$ is a simplex lying over $\sigma\colon \Delta^n \to \mathbb{\Delta}^{\mathrm{op}}$. Let $\pi_{\bar{\sigma}}\colon \Delta^n * \mathcal{G}^{\mathrm{O}}_{[r]/} \to \mathcal{N}$ be an inert extension of $\bar{\sigma}$ (with $[r]$ the final vertex of σ), and let $\pi^{\partial}_{\bar{\sigma}}$ denote the restriction of $\pi_{\bar{\sigma}}$ to $\partial\Delta^n * \mathcal{G}^{\mathrm{O}}_{[r]/}$. Let $\mathcal{M}_{\bar{\sigma}}$ and $\mathcal{M}^{\partial}_{\bar{\sigma}}$ denote the pullbacks of $\mathcal{M}$ along $\pi_{\bar{\sigma}}$ and $\pi^{\partial}_{\bar{\sigma}}$, respectively. Then the inclusion $\mathcal{M}^{\partial}_{\bar{\sigma}} \hookrightarrow \mathcal{M}_{\bar{\sigma}}$ is a trivial cofibration of generalized non-symmetric ∞-operads.

Sketch Proof. Write $\mathcal{X} \subseteq \mathcal{M}_{\bar{\sigma}}$ for the subspace lying over $\mathcal{G}^{\mathrm{O}}_{[r]/} \subseteq \Delta^n * \mathcal{G}^{\mathrm{O}}_{[r]/}$. Let $\{\tau_b\}_{b\in B}$ be the set of non-degenerate simplices $\tau_b\colon \Delta^p \to \mathcal{M}_{\bar{\sigma}}$ such that $\phi\tau_b$ is a degeneracy of $\bar{\sigma}$. Choose a well-ordering of B such that the dimension of τ_b is (non-strictly) increasing in b. For each $b \in B$, define $\mathcal{M}^{\leq b}_{\bar{\sigma}}$ to be the subspace of $\mathcal{M}_{\bar{\sigma}}$ generated by $\mathcal{M}^{\partial}_{\bar{\sigma}}$ together with the simplices $\tau\colon \Delta^p \to \mathcal{M}_{\bar{\sigma}}$ such that for some $q \leq p$ the restriction $\tau|_{\Delta^{\{0,\ldots,q\}}}$ factors through $\tau_{b'}$ for some $b' \leq b$, the morphism $\tau(q) \to \tau(q')$ is inert, and $\tau|_{\Delta^{q+1,\ldots,p}}$ factors through $\mathcal{X}$. Define $\mathcal{M}^{<b}_{\bar{\sigma}}$ similarly. Since weak equivalences are closed under transfinite composition, it suffices to prove that the inclusions $\mathcal{M}^{<b}_{\bar{\sigma}} \to \mathcal{M}^{\leq b}_{\bar{\sigma}}$ are trivial cofibrations.

Now from [Lur11, Lemma 3.1.2.5] we conclude that there is a homotopy pushout diagram

$$\begin{tikzcd}
\partial\Delta^p * \mathcal{X}_{\tau_b/} \arrow[r] \arrow[d] & \mathcal{M}^{<b}_{\bar{\sigma}} \arrow[d] \\
\Delta^p * \mathcal{X}_{\tau_b/} \arrow[r] & \mathcal{M}^{\leq b}_{\bar{\sigma}},
\end{tikzcd}$$

where $\mathcal{X}_{\tau_b/}$ denotes $(\mathcal{M}^{\leq b}_{\bar{\sigma}})_{\tau_b/} \times_{\mathcal{M}^{\leq b}_{\bar{\sigma}}} \mathcal{X}$, so it suffices to prove that the inclusion $\partial\Delta^p * \mathcal{X}_{\tau_b/} \to \Delta^p * \mathcal{X}_{\tau_b/}$ is a trivial cofibration. Now choose inert morphisms extending τ_b to a diagram $\Delta^p * \mathcal{G}^{\mathrm{O}}_{[s]/} \to \mathcal{M}_{\bar{\sigma}}$. Then the resulting map $\mathcal{G}^{\mathrm{O}}_{[s]/} \to \mathcal{X}_{\tau_b/}$ is a categorical equivalence (since an object of $\mathcal{X}_{\tau_b/}$ must be given by an inert map), hence in the diagram

$$\begin{tikzcd}
\partial\Delta^p * \mathcal{G}^{\mathrm{O}}_{[s]/} \arrow[r] \arrow[d] & \partial\Delta^p * \mathcal{X}_{\tau_b/} \arrow[d] \\
\Delta^p * \mathcal{G}^{\mathrm{O}}_{[s]/} \arrow[r] & \Delta^p * \mathcal{X}_{\tau_b/}
\end{tikzcd}$$

the horizontal morphisms are weak equivalences, as is the left vertical morphism. By the 2-out-of-3 property, it follows that so is the right vertical morphism, which completes the proof. □

Proof of Theorem 4.6.2.4. To make the proof slightly easier to read, we will omit mention of the markings of the simplicial sets involved. Observe that an n-simplex of $\mathbb{\Delta}^{\mathrm{op}}[n]$ is uniquely described by

- an n-simplex $\sigma = (f_1, \ldots, f_n)$ of $\mathbb{\Delta}^{\mathrm{op}}$ where each $f_i\colon [r_i] \to [r_{i-1}]$ is a morphism of $\mathbb{\Delta}$,
- an object $J = (j_0, j_1, \ldots j_{r_0})$ of $\mathbb{\Delta}^{\mathrm{op}}[n]$.

Such an object lies in $\mathbb{\Lambda}^{\mathrm{op}}[n]$ if and only if J and all the objects $J_i = f_i^* \cdots f_1^* J$ are in $\mathbb{\Lambda}^{\mathrm{op}}[n]$.

We'll say that a simplex (σ, J) of $\mathbb{\Lambda}^{\mathrm{op}}[n]$ is

- *old* if (σ, J) is in $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{II}}[n]$, i.e. if $J \in \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{II}}[n]$, and *new* otherwise,
- *narrow* if $r_n = 1$ and *wide* if $r_n > 1$.

We say a morphism ϕ in $\mathbb{\Delta}^{\mathrm{op}}$ is *neutral* if it is neither active nor inert.

For an object J of $\mathbb{\Lambda}^{\mathrm{op}}[n]$ over $[r]$, write $\pi_J \colon \mathcal{G}^{\mathrm{O}}_{[r]/} \to \mathbb{\Lambda}^{\mathrm{op}}[n]$ for the diagram of inert morphisms from α.

More generally for (σ, J) an n-simplex of $\mathbb{\Lambda}^{\mathrm{op}}[n]$, write $\pi_{(\sigma,J)} \colon \Delta^n * \mathcal{G}^{\mathrm{O}}_{[r]/} \to \mathbb{\Lambda}^{\mathrm{op}}[n]$ for the corresponding diagram, and $\pi^{\partial}_{(\sigma,J)}$ for the restriction of $\pi_{(\sigma,J)}$ to $\partial\Delta^n * \mathcal{G}^{\mathrm{O}}_{[r]/}$.

We now divide the non-degenerate new simplices of $\mathbb{\Lambda}^{\mathrm{op}}[n]$ into various groups:

- Let $S_1[n]$ be the set of nondegenerate wide new n-simplices (σ, J) such that f_n^σ is inert.
- Let $S_1'[n]$ be the set of non-degenerate new $(n+1)$-simplices (σ, J) such that r_{n+1} is either 0 or 1 and f_{n+1}^σ and f_n^σ are both inert.
- Let $S_1''[n]$ be the set of non-degenerate new $(n+2)$-simplices (σ, J) such that $r_{n+1} = 1$, $r_{n+2} = 0$, and f_{n+1}^σ and f_n^σ are both inert.
- For $1 \leq k < r \leq n$, let $T_1[n](k,r)$ be the set of nondegenerate narrow new n-simplices (σ, J) such that f_k^σ is inert, f_r^σ is neutral, and f_i^σ is active for $k < i < r$ and $i > r$. Let $T_1[n](k)$ be the union of $T_1[n](k,r)$ for all $r > k$.
- For $1 \leq k < r \leq n$, let $T_1'[n](k,r)$ be the set of nondegenerate narrow new $(n+1)$-simplices (σ, J) such that f_k^σ is inert, f_r^σ is inert, and f_i^σ is active for $k < i < r$ and $i > r$. Let $T_1'[n](k)$ be the union of $T_1'[n](k,r)$ for all $r > k$.
- For $1 \leq k < r \leq n$, let $\bar{T}_1'[n](k,r)$ be the set of nondegenerate new $(n+1)$-simplices (σ, J) such that $r_n = 1$, $r_{n+1} = 0$, f_k^σ is inert, f_r^σ is neutral, and f_i^σ is active for $k < i < r$ and $r < i < n+1$. Let $\bar{T}_1'[n](k)$ be the union of $\bar{T}_1'[n](k,r)$ for all $r > k$.
- For $1 \leq k < r \leq n$, let $\bar{T}_1''[n](k,r)$ be the set of nondegenerate new $(n+2)$-simplices (σ, J) such that $r_{n+1} = 1$, $r_{n+2} = 0$, f_k^σ is inert, f_r^σ is inert, and f_i^σ is active for $k < i < r$ and $r < i < n+2$. Let $\bar{T}_1''[n](k)$ be the union of $\bar{T}_1''[n](k,r)$ for all $r > k$.
- For $1 \leq k < n$ let $T_2[n](k)$ be the set of nondegenerate wide new n-simplices (σ, J) such that f_k^σ is inert and f_i^σ is active for $i > k$.
- For $1 \leq k < n$ let $T_2'[n](k)$ be the set of nondegenerate new $(n+1)$-simplices (σ, J) such that $r_{n+1} = 1$ or 0, f_k^σ and f_{n+1}^σ are inert and f_i^σ is active for $k < i < n+1$.
- For $1 \leq k < n$ let $T_2''[n](k)$ be the set of nondegenerate new $(n+2)$-simplices (σ, J) such that $r_{n+1} = 1$, $r_{n+2} = 0$, f_k^σ and f_{n+1}^σ are inert, and f_i^σ is active for $k < i < n+1$.
- Let $S_2[n](k)$ be the union of $T_1[n](k)$ and $T_2[n](k)$, let $S_2'[n](k)$ be the union of $T_1'[n](k)$, $\bar{T}_1'[n](k)$, and $T_2'[n](k)$, and let $S_2''[n](k)$ be the union of $\bar{T}_1''[n](k)$ and $T_2''[n](k)$. Let $S_2[n]$, $S_2'[n]$ and $S_2''[n]$ be the unions of $S_2[n](k)$, $S_2'[n](k)$, and $S_2''[n](k)$, respectively, over all k.

- For $1 \leq k \leq n$ let $S_3[n](k)$ be the set of nondegenerate narrow new n-simplices (σ, J) such that f_k^σ is neutral and f_i^σ is active for $i > k$, and (σ, J) is not contained in $T_1(l)$ for any $l < k$.
- For $1 \leq k \leq n$ let $S'_3[n](k)$ be the set of nondegenerate narrow new $(n+1)$-simplices (σ, J) such that f_k^σ is inert and f_i^σ is active for $i > k$, and (σ, X) is not contained in $T'_1(l)$ for any $l < k$.
- For $1 \leq k \leq n$ let $\bar{S}'_3[n](k)$ be the set of nondegenerate new $(n+1)$-simplices (σ, J) such that $r_n = 1$, $r_{n+1} = 0$, f_k^σ is neutral, and f_i^σ is active for $k < i < n+1$, and (σ, J) is not contained in $\bar{T}'_1(l)$ for any $l < k$.
- For $1 \leq k \leq n$ let $\bar{S}''_3[n](k)$ be the set of nondegenerate new $(n+2)$-simplices (σ, J) such that $r_{n+1} = 1$, $r_{n+2} = 0$, f_k^σ is inert and f_i^σ is active for $k < i < n+2$, and (σ, X) is not contained in $\bar{T}''_1(l)$ for any $l < k$.
- Let $S_3[n]$ be the union of $S_3[n](k)$ for all k, let $S'_3[n]$ be the union of $S'_3[n](k)$ and $\bar{S}'_3[n](k)$ for all k, and let $S''_3[n]$ be the union of $\bar{S}''_3[n](k)$ for all k.
- Let $S_4[n]$ be the set of nondegenerate wide new n-simplices (σ, J) that are not contained in $S_1[n]$ or $S_2[n]$.
- Let $S'_4[n]$ be the set of nondegenerate new $(n+1)$-simplices (σ, J) such that $r_{n+1} = 0$ or 1 and f_{n+1}^σ is inert that are not contained in $S'_1[n]$ or $S'_2[n]$.
- Let $S''_4[n]$ be the set of nondegenerate new $(n+2)$-simplices (σ, J) such that $r_{n+1} = 1$, $r_{n+2} = 0$, and f_{n+1}^σ is inert that are not contained in $S''_1[n]$ or $S''_2[n]$.

Observe that if (σ, J) is an n-simplex such that all f_i^σ are active and $r_n = 1$ or 0, then (σ, J) must be old.

Now let $\mathcal{F}(n)$ be the subset of $\mathbb{\Lambda}^{\mathrm{op}}[n]$ containing the old simplices together with the non-degenerate new n-simplices, the $(n+1)$-simplices in $S'_i[n]$ and the $(n+2)$-simplices in $S''_i[n]$ for all i, and let $\overline{\mathcal{F}}(n)$ be the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0 < \cdots < X_n}$ over $\mathcal{F}(n)$. It then suffices to prove that the inclusions $\mathbb{\Delta}^{\mathrm{op,II}}_{X_0 < \cdots < X_n} = \overline{\mathcal{F}}(-1) \subseteq \overline{\mathcal{F}}(0) \subseteq \overline{\mathcal{F}}(1) \subseteq \ldots$ are trivial cofibrations.

For $k = 0, \ldots, 3$ define $\mathcal{F}_{n,k} \subseteq \mathcal{F}(n)$ to be the subset containing the simplices in $\mathcal{F}(n-1)$ together with those in $S_i[n]$, $S'_i[n]$, and $S''_i[n]$ for $i \leq k$, and let $\overline{\mathcal{F}}_{n,k}$ be the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0 < \cdots < X_n}$ over $\mathcal{F}_{n,k}$. Then it suffices to prove that the inclusions

$$\overline{\mathcal{F}}(n-1) = \overline{\mathcal{F}}_{n,0} \subseteq \overline{\mathcal{F}}_{n,1} \subseteq \overline{\mathcal{F}}_{n,2} \subseteq \overline{\mathcal{F}}_{n,3} \subseteq \overline{\mathcal{F}}_{n,4} = \overline{\mathcal{F}}(n)$$

are trivial cofibrations.

$k = 1$: For (σ, J) in $S_1[n]$, observe that since any narrow new n-simplex whose final map is inert is contained in $\mathcal{F}(n-1) = \mathcal{F}_{n,0}$, as is any new $(n+1)$-simplex whose final map is $[1] \to [0]$ and whose penultimate map is inert, the map $\pi^{\partial}_{(\sigma,J)}$ factors through $\mathcal{F}(n-1)$. Thus we have a pushout diagram

$$\begin{tikzcd}
\coprod_{(\sigma,J)\in S_1[n]} \partial\Delta^n * \mathcal{G}^{\mathcal{O}}_{[r_n^\sigma]/} \arrow[r] \arrow[d] & \coprod_{(\sigma,J)\in S_1[n]} \Delta^n * \mathcal{G}^{\mathcal{O}}_{[r_n^\sigma]/} \arrow[d] \\
\mathcal{F}_{n,0} \arrow[r] & \mathcal{F}_{n,1}
\end{tikzcd}$$

Since the upper horizontal map is $\mathfrak{O}^{\mathrm{gen}}_{\mathcal{O}}$-anodyne, so is the lower horizontal map. Let $\mathcal{M}_{(\sigma,J)}$ be the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\cdots<X_n}$ over $\Delta^n * \mathcal{G}^{\mathcal{O}}_{[r^\sigma_n]/}$ and let $\mathcal{M}^{\partial}_{(\sigma,J)}$ be the subspace over $\partial\Delta^n * \mathcal{G}^{\mathcal{O}}_{[r^\sigma_n]/}$. Then we have a pushout diagram

$$\begin{tikzcd}
\coprod_{(\sigma,J)\in S_1[n]} \mathcal{M}^{\partial}_{(\sigma,J)} \arrow[r] \arrow[d] & \coprod_{(\sigma,J)\in S_1[n]} \mathcal{M}_{(\sigma,J)} \arrow[d] \\
\overline{\mathcal{F}}_{n,0} \arrow[r] & \overline{\mathcal{F}}_{n,1}.
\end{tikzcd}$$

By Lemma 4.6.2.5 the inclusion $\mathcal{M}^{\partial}_{(\sigma,J)} \hookrightarrow \mathcal{M}_{(\sigma,J)}$ is a trivial cofibration, hence so is $\overline{\mathcal{F}}_{n,0} \to \overline{\mathcal{F}}_{n,1}$.

$k = 2$: First let $\mathcal{G}_{n,k}$ be the subset of $\mathcal{F}_{n,2}$ containing the simplices in $\mathcal{F}_{n,1}$ together with those in $S_2[n](i)$, $S'_2[n](i)$, and $S''_2[n](i)$ for all $i \geq k$, and let $\overline{\mathcal{G}}_{n,k}$ denote the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\ldots<X_n}$ over $\mathcal{G}_{n,k}$. Then it suffices to prove that the inclusions

$$\overline{\mathcal{F}}_{n,1} = \overline{\mathcal{G}}_{n,n} \subseteq \overline{\mathcal{G}}_{n,n-1} \subseteq \cdots \subseteq \overline{\mathcal{G}}_{n,1} = \overline{\mathcal{F}}_{n,2}$$

are trivial cofibrations. Next for $i = 0,1,2$ let $\mathcal{G}^i_{n,k}$ be the subset of $\mathcal{G}_{n,k}$ containing the simplices in $\mathcal{G}_{n,k+1}$ together with those in $T_j[n](k)$, $T'_j[n](k)$, $\bar{T}'_j[n](k)$, and $\bar{T}''_j[n](k)$ for $j \leq i$, and let $\overline{\mathcal{G}}^i_{n,k}$ be the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\ldots<X_n}$ over $\mathcal{G}^i_{n,k}$. It then suffices to prove that the inclusions

$$\overline{\mathcal{G}}_{n,k+1} = \overline{\mathcal{G}}^0_{n,k} \subseteq \overline{\mathcal{G}}^1_{n,k} \subseteq \overline{\mathcal{G}}^2_{n,k} = \overline{\mathcal{G}}_{n,k}$$

are trivial cofibrations. We now consider these two inclusions in turn:

- $i = 1$: Let $\mathcal{I}_{n,k,r}$ be the subset of $\mathcal{G}^1_{n,k}$ containing the simplices of $\mathcal{G}_{n,k+1}$ together with those in $T_1[n](k,s)$, $T'_1[n](k,s)$, $\bar{T}'_1[n](k,s)$, and $\bar{T}''_1[n](k,s)$ for $s \leq r$, and let $\bar{\mathcal{I}}_{n,k,r}$ denote the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\ldots<X_n}$ over $\mathcal{I}_{n,k,r}$. It then suffices to show that the inclusions

  $$\overline{\mathcal{G}}^0_{n,k} = \bar{\mathcal{I}}_{n,k,k} \subseteq \bar{\mathcal{I}}_{n,k,k+1} \subseteq \cdots \subseteq \bar{\mathcal{I}}_{n,k,n} = \overline{\mathcal{G}}^1_{n,k}$$

  are trivial cofibrations. Finally, let $\mathcal{I}'_{n,k,r}$ be the subset of $\mathcal{I}_{n,k,r}$ containing the simplices in $\mathcal{I}_{n,k,r-1}$ together with those in $T_1[n](k,r)$ and $T'_1[n](k,r)$, and let $\overline{\mathcal{I}}'_{n,k,r}$ denote the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\ldots<X_n}$ over $\mathcal{I}'_{n,k,r}$. We then wish to show that the inclusions $\overline{\mathcal{I}}_{n,k,r} \hookrightarrow \overline{\mathcal{I}}'_{n,k,r} \hookrightarrow \overline{\mathcal{I}}_{n,k,r+1}$ are trivial cofibrations. Observe that for (σ, J) in $T_1[n](k,r)$ there exists a unique simplex (τ, K) in $T'_1[n](k,r)$ such that $(\sigma, J) = d_r(\tau, K)$. Moreover, for $j \neq r$ the face $d_j(\tau, K)$ is contained in $\mathcal{I}_{n,k,r-1}$:

  - for $i < k-1$ we have d_i in $T'_1[n-1](k-1, r-1)$,
  - we have d_{k-1} in $S'_3[n-1](r-1)$, or possibly in $T'_1[n-1](k-1, n-1)$ (or in $S'_3[n-1](r-1)$ if $k = 1$)
  - we have d_k in $S'_3[n-1](r-1)$,
  - for $k < i < r-1$ we have d_i in $T'_1[n-1](k, r-1)$,
  - we have d_{r-1} in $T_1[n](k, r-1)$ or $T'_1[n-1](k, r-1)$,
  - for $r < i < n+1$ we have d_i in $T'_1[n-1](k, r)$,

– we have d_{n+1} in $T_2[r]$.

Thus we get a pushout diagram

$$\begin{tikzcd}
\coprod_{(\tau,K)\in T_1'[n](k,r)} \Lambda^{n+1}_r \arrow[r] \arrow[d] & \coprod_{(\tau,K)\in T_1'[n](k,r)} \Delta^{n+1} \arrow[d] \\
\mathcal{I}_{n,k,r-1} \arrow[r] & \mathcal{I}'_{n,k,r}.
\end{tikzcd}$$

This means we have a pushout diagram

$$\begin{tikzcd}
\coprod_{(\tau,K)\in T_1'[n](k,r)} \mathbb{\Delta}^{\mathrm{op}}_{X_0<\cdots<X_n} \times_{\mathbb{\Delta}^{\mathrm{op}}[n]} \Lambda^{n+1}_r \arrow[r] \arrow[d] & \coprod_{(\tau,K)\in T_1'[n](k,r)} \mathbb{\Delta}^{\mathrm{op}}_{X_0<\cdots<X_n} \times_{\mathbb{\Delta}^{\mathrm{op}}[n]} \Delta^{n+1} \arrow[d] \\
\overline{\mathcal{I}}_{n,k,r-1} \arrow[r] & \overline{\mathcal{I}}'_{n,k,r}.
\end{tikzcd}$$

By [Lur11, Lemma 2.4.4.6] the upper horizontal map is a categorical equivalence, and so a trivial cofibration, hence so is the lower horizontal map. Similarly, for each (σ, J) in $\bar{T}_1'[n](k,r)$ there exists a unique simplex (τ, K) in $\bar{T}_1''[n](k,r)$ such that $(\sigma, J) = d_r(\tau, K)$ and for $j \neq r$ the face $d_j(\tau, K)$ is contained in $\mathcal{I}'_{n,k,r}$. We therefore have another pushout diagram

$$\begin{tikzcd}
\coprod_{(\tau,K)\in \bar{T}_1''[n](k,r)} \Lambda^{n+2}_r \arrow[r] \arrow[d] & \coprod_{(\tau,K)\in \bar{T}_1''[n](k,r)} \Delta^{n+2} \arrow[d] \\
\mathcal{I}'_{n,k,r} \arrow[r] & \mathcal{I}_{n,k,r}.
\end{tikzcd}$$

By the same argument again, this implies that the map $\overline{\mathcal{I}}'_{n,k,r} \to \overline{\mathcal{I}}_{n,k,r}$ is also a trivial cofibration.

- $i = 2$: Observe that for (τ, K) in $T_2'[n](k)$ the faces $d_i(\tau, K)$ for $i < n+1$ are in $\mathcal{G}^1_{n,k}$: for $i < n$ we have d_i in $\mathcal{F}(n-1)$ since these faces are narrow with final map inert, and d_n is in $T_1[n](k,n)$ or $\mathcal{F}(n-1)$. The same holds for $(\tau, K) \in T_2''[n](k)$, thus for all (σ, J) in $T_2[n](k)$ the map $\pi^{\partial}_{(\sigma,J)}$ factors through $\mathcal{G}^1_{n,k}$. This means we have a pushout diagram

$$\begin{tikzcd}
\coprod_{(\sigma,J)\in T_2[n](k)} \partial\Delta^n * \mathcal{G}^{\mathcal{O}}_{[r_n^\sigma]/} \arrow[r] \arrow[d] & \coprod_{(\sigma,J)\in T_2[n](k)} \Delta^n * \mathcal{G}^{\mathcal{O}}_{[r_n^\sigma]/} \arrow[d] \\
\mathcal{G}^1_{n,k} \arrow[r] & \mathcal{G}^2_{n,k}.
\end{tikzcd}$$

By the same argument as in the case $k = 1$, it follows that $\overline{\mathcal{G}}^1_{n,k} \to \overline{\mathcal{G}}^2_{n,k}$ is a trivial cofibration.

$k = 3$: Let $\mathcal{H}_{n,k}$ be the subset of $\mathcal{F}_{n,3}$ containing the simplices in $\mathcal{F}_{n,2}$ together with those in $S_3[n](l)$, $S_3'[n](l)$, $\bar{S}_3'[n](l)$, and $\bar{S}_3''[n](l)$ for $l \leq k$, and let $\overline{\mathcal{H}}_{n,k}$ denote the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\cdots<X_n}$ over $\mathcal{H}_{n,k}$. It then suffices to show that the inclusions

$$\overline{\mathcal{F}}_{n,2} = \overline{\mathcal{H}}_{n,0} \subseteq \overline{\mathcal{H}}_{n,1} \subseteq \cdots \subseteq \overline{\mathcal{H}}_{n,n} = \overline{\mathcal{F}}_{n,3}$$

are trivial cofibrations. Let $\mathcal{H}'_{n,k}$ be the subset of $\mathcal{H}_{n,k}$ containing the simplices in $\mathcal{H}_{n,k-1}$ together with those in $S_3[n](k)$ and $S_3'[n](k)$, and let $\overline{\mathcal{H}}'_{n,k}$ denote the subspace of $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\cdots<X_n}$ over $\mathcal{H}'_{n,k}$. We then wish to show that the inclusions $\overline{\mathcal{H}}_{n,k-1} \to \overline{\mathcal{H}}'_{n,k} \to \overline{\mathcal{H}}_{n,k}$ are trivial cofibrations. Observe that for (σ, J) in $S_3[n](k)$ there exists a unique simplex (τ, K) in $S_3'[n](k)$ such that $(\sigma, J) = d_k(\tau, K)$. Moreover, for $j \neq k$ the face $d_j(\tau, K)$ is contained in $\mathcal{H}_{n,k-1}$:

- for $j < k-1$ we have d_j in $S_3'[n-1](k-1)$,
- we have d_{k-1} in $S_3[n](k-1)$ or $S_3'[n-1](k-1)$, or in $\mathcal{N}$ if $k = 1$ (since all narrow simplices all of whose maps are active are in $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{II}}[n]$)
- for $k < j < n+1$ we have d_j in $S_3'[n-1](k)$,
- we have d_{n+1} in $T_2[n](k)$.

Thus we get a pushout diagram

$$\begin{tikzcd}
\coprod_{(\tau,K)\in S_3'[n](k)} \Lambda^{n+1}_k \arrow[r] \arrow[d] & \coprod_{(\tau,K)\in S_3'[n](k)} \Delta^{n+1} \arrow[d] \\
\mathcal{H}_{n,k-1} \arrow[r] & \mathcal{H}'_{n,k}.
\end{tikzcd}$$

Using [Lur11, Lemma 2.4.4.6] as above, this implies that $\overline{\mathcal{H}}_{n,k-1} \to \overline{\mathcal{H}}'_{n,k}$ is a trivial cofibration. Similarly, for (σ, J) in $\bar{S}_3'[n](k)$ there exists a unique simplex (τ, K) in $\bar{S}_3''[n](k)$ such that $(\sigma, J) = d_k(\tau, K)$. Moreover, for $j \neq k$ the face $d_j(\tau, K)$ is contained in $\mathcal{H}'_{n,k}$. This gives another pushout diagram

$$\begin{tikzcd}
\coprod_{(\tau,K)\in \bar{S}_3''[n](k)} \Lambda^{n+2}_k \arrow[r] \arrow[d] & \coprod_{(\tau,K)\in \bar{S}_3''[n](k)} \Delta^{n+2} \arrow[d] \\
\mathcal{H}'_{n,k} \arrow[r] & \mathcal{H}_{n,k}.
\end{tikzcd}$$

By the same argument it follows that $\overline{\mathcal{H}}'_{n,k} \to \overline{\mathcal{H}}_{n,k}$ is a trivial cofibration.

$k = 4$: Observe that for (τ, K) in $S_4'[n]$ the faces $d_i(\tau, K)$ for $i < n+1$ are in $\mathcal{F}_{n,3}$, since this contains all narrow n-simplices of $\mathbb{\Lambda}^{\mathrm{op}}[n]$. Similarly for (τ, K) in $S_4''[n]$ the faces $d_i(\tau, K)$ for $i \neq n+1$ are in $\mathcal{F}_{n,3}$. Thus for all (σ, J) in $S_4[n]$, the map $\pi^{\partial}_{(\sigma,J)}$ factors through $\mathcal{F}_{n,3}$, and so we have a pushout diagram

$$\begin{tikzcd}
\coprod_{(\sigma,J)\in S_4[n]} \partial\Delta^n * \mathcal{G}^{\mathcal{O}}_{[r^\sigma_n]/} \arrow[r] \arrow[d] & \coprod_{(\sigma,J)\in S_4[n]} \Delta^n * \mathcal{G}^{\mathcal{O}}_{[r^\sigma_n]/} \arrow[d] \\
\mathcal{F}_{n,3} \arrow[r] & \mathcal{F}_{n,4}.
\end{tikzcd}$$

By the same argument as in the case $k = 1$, it follows that $\overline{\mathcal{F}}_{n,3} \to \overline{\mathcal{F}}_{n,4}$ is a trivial cofibration. □

Corollary 4.6.2.6. Suppose $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category. For any spaces $X_0, \ldots, X_n$, and any $\mathbb{\Delta}^{\mathrm{op},\amalg}_{X_0<\cdots<X_n}$-algebra A in $\mathcal{V}^{\otimes}$, the adjunction morphism $A \to \kappa^*\kappa_!A$ is an equivalence.

Proof. By Theorem 4.6.2.4, we may regard κ as the inclusion $\mathbb{\Lambda}^{\mathrm{op}}_{X_0<\cdots<X_n} \to \mathbb{\Delta}^{\mathrm{op}}_{X_0<\cdots<X_n}$. It is clear that this morphism has the Kan extension property, so we have a description of free algebras in terms of operadic colimits. Using this it is easy to see that $A(\xi) \to \kappa_!A(\xi)$ is an equivalence for $\xi \in \mathbb{\Lambda}^{\mathrm{op}}_{X_0<\cdots<X_n}$. □

4.6.3 The Double ∞-Category of $\mathcal{V}$-∞-Categories

We will now construct a double ∞-category $\mathrm{CAT}(\mathcal{V})$ whose objects are $\mathcal{V}$-∞-categories and whose vertical and horizontal morphisms are functors and correspondences, respectively. From this we can extract an $(\infty,2)$-category $\mathrm{CORR}^{\mathcal{V}}_{\infty}$ whose mapping ∞-categories are the ∞-categories $\mathrm{Corr}^{\mathcal{V}}(\mathcal{C},\mathcal{D})$ defined above.

Definition 4.6.3.1. It is easy to see that the full subcategory of $(\mathrm{Opd}^{\mathcal{O},\mathrm{gen}}_{\infty})_{/\mathbb{\Delta}^{\mathrm{op}}[n]}$ spanned by the objects $\mathbb{\Delta}^{\mathrm{op}}_{X_0<\ldots<X_n}$ is equivalent to $\mathcal{S}^{\times(n+1)}$. Define $\mathrm{ALG}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})[n]$ by the pullback

$$\begin{tikzcd}
\mathrm{ALG}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})[n] \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathcal{O}}_{/\mathbb{\Delta}^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes}) \arrow[d] \\
\mathcal{S}^{\times(n+1)} \arrow[r] & (\mathrm{Opd}^{\mathcal{O},\mathrm{gen}}_{\infty})_{/\mathbb{\Delta}^{\mathrm{op}}[n]}.
\end{tikzcd}$$

This defines a simplicial ∞-category $\mathrm{ALG}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$.

Definition 4.6.3.2. Let $\mathrm{CORR}(\mathcal{V}^{\otimes})[n]$ be the full subcategory of $\mathrm{ALG}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})[n]$ spanned by those algebras that are composites and whose restrictions to $\mathrm{Alg}^{\mathcal{O}}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ are all complete $\mathcal{V}$-∞-categories. These are clearly closed under the functors induced by morphisms in $\mathbb{\Delta}^{\mathrm{op}}$ and so form a simplicial ∞-category $\mathrm{CORR}(\mathcal{V}^{\otimes})$.

Lemma 4.6.3.3. Suppose $\mathcal{V}^{\otimes}$ is a presentably monoidal ∞-category. The simplicial ∞-category $\mathrm{CORR}(\mathcal{V}^{\otimes})$ is a Segal object.

Proof. We must show that the Segal morphisms

$$\mathrm{CORR}(\mathcal{V}^{\otimes})[n] \to \mathrm{CORR}(\mathcal{V}^{\otimes})[1] \times_{\mathrm{CORR}(\mathcal{V}^{\otimes})[0]} \cdots \times_{\mathrm{CORR}(\mathcal{V}^{\otimes})[0]} \mathrm{CORR}(\mathcal{V}^{\otimes})[1]$$

are equivalences. Since this functor clearly preserves Cartesian arrows over $\mathcal{S}^{\times(n+1)}$ it suffices to show that it induces an equivalence on fibres. Given spaces $X_0, \ldots, X_n$, we thus have to show that we get an equivalence of ∞-categories

$$\mathrm{Alg}^{\mathrm{O}}_{\Delta^{\mathrm{op}}_{X_0<\cdots<X_n}/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes})^{\mathrm{comp}} \xrightarrow{\kappa^*} \mathrm{Alg}^{\mathrm{O}}_{\Delta^{\mathrm{op},\amalg}_{X_0<\cdots<X_n}/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes}),$$

where $\mathrm{Alg}^{\mathrm{O}}_{\Delta^{\mathrm{op}}_{X_0<\cdots<X_n}/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes})^{\mathrm{comp}}$ denotes the full subcategory of $\mathrm{Alg}^{\mathrm{O}}_{\Delta^{\mathrm{op}}_{X_0<\cdots<X_n}/\Delta^{\mathrm{op}}[n]}(\mathcal{V}^{\otimes})$ spanned by the algebras that are composites, i.e. in the image of $\kappa_!$. By Corollary 4.6.2.6, every object A on the right-hand side is the image of $\kappa_! A$, so this functor is essentially surjective. To see that it is fully faithful, suppose A and B are two $\Delta^{\mathrm{op},\amalg}_{X_0<\cdots<X_n}$-algebras in $\mathcal{V}^{\otimes}$; then we must show that

$$\mathrm{Map}(\kappa_! A, \kappa_! B) \to \mathrm{Map}(\kappa^*\kappa_! A, \kappa^*\kappa_! A)$$

is an equivalence of spaces. Under the equivalence $\mathrm{Map}(\kappa_! A, \kappa_! B) \simeq \mathrm{Map}(A, \kappa^*\kappa_! B)$ given by the adjunction this corresponds to composition with the unit $A \to \kappa^*\kappa_! A$. This is an equivalence by Corollary 4.6.2.6, which completes the proof. □

Definition 4.6.3.4. Write $\mathrm{CORR}^{\mathcal{V}}_{\infty}$ for the horizontal sub-$(\infty,2)$-category of $\mathrm{CORR}(\mathcal{V}^{\otimes})$, given by restricting the 0th ∞-category $\mathrm{CORR}(\mathcal{V}^{\otimes})[0] \simeq \mathrm{Cat}^{\mathcal{V}}_{\infty}$ to the space $\iota\mathrm{Cat}^{\mathcal{V}}_{\infty}$. This is an $(\infty,2)$-category of $\mathcal{V}$-∞-categories and correspondences. We denote its underlying ∞-category by $\mathrm{Corr}^{\mathcal{V}}_{\infty}$.

Lemma 4.6.3.5. The vertical sub-$(\infty,2)$-category of $\mathrm{CORR}(\mathcal{V}^{\otimes})$ is the $(\infty,2)$-category $\mathrm{CAT}^{\mathcal{V}}_{\infty}$ of $\mathcal{V}$-∞-categories and functors.

Proof. The vertical sub-$(\infty,2)$-category is obtained by taking the full subcategories of the ∞-categories $\mathrm{CORR}(\mathcal{V}^{\otimes})[n]$ spanned by the objects that are degeneracies of the objects in $\mathrm{CORR}(\mathcal{V}^{\otimes})[0] \simeq \mathrm{Cat}^{\mathcal{V}}_{\infty}$. But these degenerate objects are precisely the $\Delta^{\mathrm{op}}_X[n]$-algebras $\pi_n^*\mathcal{C}$ where $\mathcal{C}$ is a complete Δ^{op}_X-algebra in $\mathcal{V}^{\otimes}$. □

Chapter 5

Enriched (∞, n)-Categories

In this brief chapter we indicate how the theory of $\mathbb{O}(n)$-∞-operads leads to a non-iterative theory of enriched (∞, n)-categories. We do not, however, go very far in developing this in this thesis.

5.1 n-Categorical Algebras

In this section we use the theory of generalized ∞-operads developed in Chapter 3 to define *n-categorical algebras* in §5.1.1 and construct ∞-categories of these in §5.1.2. We show that n-categorical algebras in spaces are equivalent to Segal $\mathbb{O}(n)$-spaces (i.e. Θ_n-spaces) in §5.1.3. In §5.1.4 we introduce a notion of *completeness* for n-categorical algebras — we claim complete n-categorical algebras give the correct notion of enriched (∞, n)-categories, but do not make much progress towards proving this here.

5.1.1 The Φ-Multiple ∞-Categories $\mathcal{L}^{\Phi}_{\mathcal{X}}$

To define categorical algebras above, we used certain double ∞-categories $\mathbb{\Delta}^{\mathrm{op}}_X$, where X is a space. Here we will generalize this as follows: given a perfect operator category Φ and a clean atom $A \in \Phi$, we will define analogous Φ-multiple ∞-categories $\mathcal{L}^{\Phi}_X$, where X is a Segal $\Phi_{/A}$-space. When Φ is $\mathbb{O}(n)$ and A is the $(n-1)$-cell $C^{\mathbb{O}(n)}_{n-1}$, this gives $\mathbb{O}(n)$-multiple ∞-categories $\Theta^{\mathrm{op}}_{n,X}$ where X is a Segal $\mathbb{O}(n-1)$-space; we will use these to define *n-categorical algebras* in $\mathbb{E}_n$-monoidal ∞-categories.

Lemma 5.1.1.1. Let Φ be a perfect operator category and A a clean atom of Φ, and suppose $\mathcal{C}$ is an ∞-category with finite limits. Write j^A for the inclusion $\mathcal{L}^{\Phi_{/A}} \hookrightarrow \mathcal{L}^{\Phi}$ induced by the inclusion $\Phi_{/A} \hookrightarrow \Phi$. Then right Kan extension along j^A takes $\Phi_{/A}$-category objects in $\mathcal{C}$ to Φ-category objects in $\mathcal{C}$.

Remark 5.1.1.2. In the situation above, the functor $j^{A,*}\colon \mathrm{Cat}^{\Phi}(\mathcal{C}) \to \mathrm{Cat}^{\Phi_{/A}}(\mathcal{C})$ induced by composition with j^A clearly preserves limits, and so has a right adjoint. Since right Kan extension j^A_* along j^A preserves category objects, it follows that this right adjoint is simply given by j^A_*.

Proof. We first introduce the notation in the following diagram for the obvious inclusions of categories:

$$\begin{tikzcd}
\mathcal{G}^{\Phi} \arrow[r, "\gamma"] & \mathcal{L}^{\Phi}_{\mathrm{int}} \arrow[r, "\lambda"] & \mathcal{L}^{\Phi} \\
\mathcal{G}^{\Phi/A} \arrow[u, "g"] \arrow[r, "\bar{\gamma}"'] & \mathcal{L}^{\Phi/A}_{\mathrm{int}} \arrow[u, "l"] \arrow[r, "\bar{\lambda}"'] & \mathcal{L}^{\Phi/A} \arrow[u, "j^A"']
\end{tikzcd}$$

Let $F\colon \mathcal{L}^{\Phi_{/A}} \to \mathcal{C}$ be a $\Phi_{/A}$-category object. We must show that $j^A_* F$ is a Φ-category object, i.e. that $j^A_* F|_{\mathcal{L}^{\Phi}_{\mathrm{int}}} = \lambda^* j^A_* F$ is a right Kan extension of $j^A_* F|_{\mathcal{G}^{\Phi}} = \gamma^*\lambda^* j^A_* F$ along γ.

There is a natural transformation $\lambda^* j^A_* F \to l_*\bar{\lambda}^* F$ whose adjunct $j^A_* F \to \lambda_* l_* \bar{\lambda}^* F \simeq j^A_* \bar{\lambda}_* \bar{\lambda}^* F$ is j^A_* applied to the unit for $\bar{\lambda}^* \dashv \bar{\lambda}_*$. On an object $I \in \mathcal{L}^{\Phi}$ this is the natural map from the limit of F over $\mathcal{L}^{\Phi_{/A}}_{I/}$ to the limit over $(\mathcal{L}^{\Phi_{/A}}_{\mathrm{int}})_{I/}$. But if I is not in $\mathcal{L}^{\Phi_{/A}}$ then there are no active maps from I to an object of $\mathcal{L}^{\Phi_{/A}}$, hence if $f\colon I \to J$ is a morphism in $\mathcal{L}^{\Phi}$ with $J \in \mathcal{L}^{\Phi_{/A}}$ and $I \rightarrowtail J' \to J$ is the inert-active factorization of f, then J' is also in $\mathcal{L}^{\Phi_{/A}}$. Thus $(\mathcal{L}^{\Phi_{/A}}_{\mathrm{int}})_{I/} \to \mathcal{L}^{\Phi_{/A}}_{I/}$ is right cofinal and so $\lambda^* j^A_* F \simeq l_*\bar{\lambda}^* F$ since this is true pointwise on objects.

Similarly $\gamma^*\lambda^* j^A_* F \simeq g_*\bar{\gamma}^*\bar{\lambda}^* F$. But since F is a category object, $\bar{\lambda}^* F$ is the right Kan extension $\bar{\gamma}_*\bar{\gamma}^*\bar{\lambda}^* F$, hence we get

$$\lambda^* j^A_* F \simeq l_*\bar{\lambda}^* F \simeq l_*\bar{\gamma}_*\bar{\gamma}^*\bar{\lambda}^* F \simeq \gamma_* g_*\bar{\gamma}^*\bar{\lambda}^* F \simeq \gamma_*\gamma^*\lambda^* j^A_* F,$$

as required. □

Definition 5.1.1.3. If $\mathcal{C}$ is a $\Phi_{/A}$-category object in Cat_∞, we let $\mathcal{L}^{\Phi}_{\mathcal{C}} \to \mathcal{L}^{\Phi}$ be a coCartesian fibration associated to the Φ-category object $j^A_*\mathcal{C}$.

Example 5.1.1.4. If Φ is $\mathbb{O}$ and A is $\emptyset$ then $\mathcal{L}^{\mathbb{O}}_{\mathcal{C}}$ is $\triangle^{\mathrm{op}}_{\mathcal{C}}$ for $\mathcal{C}$ an ∞-category.

Notation 5.1.1.5. If Φ is $\mathbb{O}(n)$ and A is $C^{\mathbb{O}(n)}_{n-1}$ (so $\Phi_{/A}$ is $\mathbb{O}(n-1)$), then we write $\Theta^{\mathrm{op}}_{n,\mathcal{C}}$ for $\mathcal{L}^{\mathbb{O}(n)}_{\mathcal{C}}$, where $\mathcal{C}$ is an $\mathbb{O}(n-1)$-category object in Cat_∞.

Lemma 5.1.1.6. Suppose $\mathcal{C}$ is a $\Phi_{/A}$-category object in Cat_∞. Then the coCartesian fibration $\mathcal{L}^{\Phi}_{\mathcal{C}} \to \mathcal{L}^{\Phi}$ is a Φ-multiple ∞-category.

Proof. It follows from Lemma 5.1.1.1 that $j^A_*\mathcal{C}$ is a Φ-category object. The corresponding coCartesian fibration is therefore a Φ-multiple ∞-category. □

Definition 5.1.1.7. Let $\mathcal{V}^{\otimes}$ be an $\mathbb{O}(n)$-monoidal ∞-category and let X be a Segal $\mathbb{O}(n-1)$-space. An *n-categorical algebra* in $\mathcal{V}^{\otimes}$ with underlying Segal $\mathbb{O}(n-1)$-space X is a $\Theta^{\mathrm{op}}_{n,X}$-algebra in $\mathcal{V}^{\otimes}$.

Remark 5.1.1.8. This definition clearly does not require $\mathcal{V}^{\otimes}$ to be an $\mathbb{O}(n)$-monoidal ∞-category — we can define n-categorical algebras in any generalized $\mathbb{O}(n)$-∞-operad as $\Theta^{\mathrm{op}}_{n,X}$-algebras. We will not consider this generalization here, however.

Proposition 5.1.1.9. The functor $\Theta^{\mathrm{op}}_{n,(-)}\colon \mathrm{Seg}^{\mathbb{O}(n-1)}_\infty \to \mathrm{Opd}^{\mathbb{O}(n),\mathrm{gen}}_\infty$ preserves filtered colimits.

Proof. Suppose we have a filtered diagram of ∞-categories $p\colon \mathcal{I} \to \mathrm{Seg}_\infty^{\mathbb{O}(n-1)}$ with colimit $\mathcal{C}$. Since $\Theta_{n,\mathcal{C}}^{\mathrm{op}}$ is a generalized $\mathbb{O}(n)$-∞-operad, by Lemma 3.2.5.1 it suffices to show that $\Theta_{n,\mathcal{C}}^{\mathrm{op}}$ is the colimit of $\Theta_{n,(-)}^{\mathrm{op}}$ in Cat_∞. Now this composite functor

$$\mathrm{Seg}_\infty^{\mathbb{O}(n-1)} \xrightarrow{\Theta_{n,(-)}^{\mathrm{op}}} \mathrm{Opd}_\infty^{\mathbb{O},\mathrm{gen}} \to \mathrm{Cat}_\infty$$

factors as

$$\mathrm{Seg}_\infty^{\mathbb{O}(n-1)} \xrightarrow{j_*} \mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{CoCart}(\Theta_n^{\mathrm{op}}) \xrightarrow{q} \mathrm{Cat}_\infty,$$

where $\mathrm{CoCart}(\Theta_n^{\mathrm{op}})$ is the ∞-category of coCartesian fibrations over Θ_n^{op} and the rightmost functor q is the forgetful functor that sends a fibration $\mathcal{E} \to \Theta_n^{\mathrm{op}}$ to the ∞-category $\mathcal{E}$. By Example 2.1.12.15 the functor q preserves colimits. It thus suffices to prove that j_* preserves filtered colimits. Colimits in functor categories are computed pointwise, so to see this it suffices to show that for each $I \in \Theta_n^{\mathrm{op}}$ the composite functor $\mathrm{Seg}_\infty^{\mathbb{O}(n-1)} \to \mathrm{Cat}_\infty$ induced by composing with evaluation at I preserves filtered colimits. It is easy to see that the inclusion $\mathrm{Seg}_\infty^{\mathbb{O}(n-1)} \to \mathrm{Fun}(\Theta_{n-1}^{\mathrm{op}}, \mathcal{S})$ preserves filtered colimits, since we are localizing with respect to morphisms between compact objects, so it suffices to consider filtered colimits in $\mathrm{Fun}(\Theta_{n-1}^{\mathrm{op}}, \mathcal{S})$, which are computed pointwise. But $j_*(-)(I)$ is the limit of a finite diagram, and so commutes with filtered colimits in Cat_∞ or $\mathcal{S}$. □

5.1.2 The ∞-Category of n-Categorical Algebras

In this subsection we use the *algebra fibration*

$$\mathrm{Alg}^{\mathbb{O}(n)}(\mathcal{V}^\otimes) \to \mathrm{Opd}_\infty^{\mathbb{O}(n)}$$

to define an ∞-category of n-categorical algebras, and then show that this has various useful properties.

Definition 5.1.2.1. Suppose $\mathcal{V}^\otimes$ is an $\mathbb{E}_n$-monoidal ∞-category; to avoid clutter we will also write $\mathcal{V}^\otimes$ for the associated $\mathbb{O}(n)$-monoidal ∞-category $u^{\mathbb{O}(n),*}\mathcal{V}^\otimes$. The ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ is defined by the pullback square

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes) \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathbb{O}(n)}(\mathcal{V}^\otimes) \arrow[d] \\
\mathrm{Seg}_\infty^{\mathbb{O}(n-1)} \arrow[r, "L\Theta_{n,(-)}^{\mathrm{op}}"'] & \mathrm{Opd}_\infty^{\mathbb{O}(n)}.
\end{tikzcd}$$

where the lower horizontal map sends a Segal $\mathbb{O}(n-1)$-space X to the $\mathbb{O}(n)$-∞-operad $L\Theta_{n,X}^{\mathrm{op}}$ associated to the generalized $\mathbb{O}(n)$-∞-operad $\Theta_{n,X}^{\mathrm{op}}$. The objects of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ are thus n-categorical algebras in $\mathcal{V}^\otimes$. We will refer to $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ as the *∞-category of n-categorical algebras* in $\mathcal{V}^\otimes$.

Remark 5.1.2.2. Since $\mathcal{V}^\otimes$ is an $\mathbb{O}(n)$-monoidal ∞-category, and so in particular an $\mathbb{O}(n)$-∞-operad, we could equivalently have defined $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ using the analogue of the algebra fibration over the base $\mathrm{Opd}_{\infty}^{\mathbb{O}(n),\mathrm{gen}}$, since there is natural equivalence $\mathrm{Alg}^{\mathbb{O}(n)}_{\Theta^{\mathrm{op}}_{n,X}}(\mathcal{V}^{\otimes}) \simeq \mathrm{Alg}^{\mathbb{O}(n)}_{L\Theta^{\mathrm{op}}_{n,X}}(\mathcal{V}^{\otimes})$.

Our next goal is to prove that the ∞-category $\mathrm{Alg}^{\mathbb{O}(n)}_{\mathrm{cat}}(\mathcal{V}^{\otimes})$ is presentable if $\mathcal{V}^{\otimes}$ is presentably $\mathbb{E}_n$-monoidal; to do this we first introduce the ∞-category of *n-graphs* in $\mathcal{V}$:

Definition 5.1.2.3. Let $\mathcal{V}^{\otimes}$ be an $\mathbb{E}_n$-monoidal ∞-category. The ∞-category $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ of *$\mathcal{V}$-n-graphs* is defined by the pullback

$$\begin{tikzcd}
\mathrm{Graph}_{\infty}^{n,\mathcal{V}} \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathbb{O}(n)}_{\mathrm{triv}}(\mathcal{V}^{\otimes}) \arrow[d] \\
\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)} \arrow[r, "L\Theta^{\mathrm{op}}_{n,(-)}"'] & \mathrm{Opd}_{\infty}^{\mathbb{O}(n)}.
\end{tikzcd}$$

Thus the fibre of $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ at $X \in \mathcal{S}$ is equivalent to $\mathrm{Fun}((\Theta^{\mathrm{op}}_{n,X})_{C_n}, \mathcal{V})$.

Remark 5.1.2.4. If X is a Segal $\mathbb{O}(n-1)$-space, we can describe $(\Theta^{\mathrm{op}}_{n,X})_{C_n}$ as the limit of the diagram of spaces

$$\begin{tikzcd}
X(C_{n-1}) \arrow[r] \arrow[dr] & X(C_{n-2}) \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & X(C_0) \\
X(C_{n-1}) \arrow[r] \arrow[ur] & X(C_{n-2}) \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & X(C_0).
\end{tikzcd}$$

Lemma 5.1.2.5. Suppose $\mathcal{V}$ is an accessible ∞-category. Then the ∞-category $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ is accessible.

Proof. Let $\mathcal{F} \to \mathcal{S}$ be the Cartesian fibration associated to the functor $\mathcal{S} \to \mathrm{Cat}_{\infty}$ sending X to $\mathrm{Fun}(X, \mathcal{V})$. Then there is a pullback square

$$\begin{tikzcd}
\mathrm{Graph}_{\infty}^{n,\mathcal{V}} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)} \arrow[r, "\phi"'] & \mathcal{S},
\end{tikzcd}$$

where the lower horizontal map is the functor ϕ that sends X to $(\Theta^{\mathrm{op}}_{n,X})_{C_n}$.

The ∞-category $\mathcal{F}$ is accessible, and the projection $\mathcal{F} \to \mathcal{S}$ is an accessible functor, by Theorem 2.1.11.1. Moreover, since filtered colimits in $\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$ are computed pointwise, and finite limits in $\mathcal{S}$ commute with filtered colimits, the functor ϕ preserves filtered colimits and so is accesible. The pullback $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ is therefore accesible and the projection $\mathrm{Graph}_{\infty}^{n,\mathcal{V}} \to \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$ is an accessible functor, by [Lur09a, Proposition 5.4.6.6]. □

Proposition 5.1.2.6. Suppose $\mathcal{V}^{\otimes}$ is an $\mathbb{E}_n$-monoidal ∞-category compatible with small colimits. Then $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ has all small colimits. Moreover, if $\mathcal{V}$ is presentable then so is $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$.

Proof. By Lemma 3.2.8.5, the fibration $\pi\colon \mathrm{Alg}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes}) \to \mathrm{Opd}_{\infty}^{\mathbb{O}(n)}$ is both Cartesian and coCartesian, hence the same is true of its pullback $p\colon \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes}) \to \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$. Moreover, its fibres $\mathrm{Alg}_{\Theta_{n,X}^{\mathrm{op}}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ have all colimits by Corollary 3.2.7.6 and the functors $f_!$ induced by morphisms f in $\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$ preserve colimits, being left adjoints. Thus p satisfies the conditions of Lemma 2.1.5.10, which implies that $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ has small colimits.

Since the functor $\tau^*\colon \mathrm{Alg}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}_{\mathrm{triv}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ preserves filtered colimits by Corollary 3.2.8.10, it is clear that so does its pullback $U\colon \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes}) \to \mathrm{Graph}_{\infty}^{n,\mathcal{V}}$. Moreover, the pullback of the left adjoint $\tau_!$ of τ^* gives a functor $F\colon \mathrm{Graph}_{\infty}^{n,\mathcal{V}} \to \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ left adjoint to U; this preserves compact objects by Lemma 2.1.7.11.

Every object of $\mathrm{Alg}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is a (sifted) colimit of objects in the image of

$$\tau_!\colon \mathrm{Alg}_{\mathrm{triv}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes}) \to \mathrm{Alg}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes}),$$

hence every object of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is a (sifted) colimit of objects in the image of F. The ∞-category $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ is accessible by Lemma 5.1.2.5; suppose it is generated under colimits by κ-compact objects. Since F preserves colimits it follows that every object of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is the colimit of objects that are the images of κ-compact objects of $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ under F. As the functor F preserves κ-compact objects, this means there is a small subcategory of κ-compact objects of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ — namely the images of κ-compact objects of $\mathrm{Graph}_{\infty}^{n,\mathcal{V}}$ — such that every object of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is a colimit of objects in this ∞-category. In other words, the ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is κ-accessible. $\square$

Now we show that $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is functorial in $\mathcal{V}^{\otimes}$:

Definition 5.1.2.7. As in §3.2.8, let $\mathrm{Alg}_{\mathrm{co}}^{\mathbb{O}(n)} \to \mathrm{Opd}_{\infty}^{\mathbb{O}(n)} \times (\widehat{\mathrm{Opd}}_{\infty}^{\mathbb{O}(n)})^{\mathrm{op}}$ be a Cartesian fibration classifying the functor $\mathrm{Alg}_{(-)}^{\mathbb{O}(n)}(-)$. Let $\mathrm{Alg}_{\mathrm{cat,co}}^{\mathbb{O}(n)}$ be the pullback

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{cat,co}}^{\mathbb{O}(n)} \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathrm{co}}^{\mathbb{O}(n)} \arrow[d] \\
\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)} \times (\widehat{\mathrm{Mon}}_{\infty}^{\mathbb{O}(n),\mathrm{lax}})^{\mathrm{op}} \arrow[r] & \mathrm{Opd}_{\infty}^{\mathbb{O}(n)} \times (\widehat{\mathrm{Opd}}_{\infty}^{\mathbb{O}(n)})^{\mathrm{op}}.
\end{tikzcd}$$

Lemma 5.1.2.8. $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$ is functorial in $\mathcal{V}^{\otimes}$ with respect to lax monoidal functors.

Proof. The composite $\mathrm{Alg}_{\mathrm{cat,co}}^{\mathbb{O}(n)} \to (\widehat{\mathrm{Mon}}_{\infty}^{\mathbb{O}(n),\mathrm{lax}})^{\mathrm{op}}$ is a Cartesian fibration classifying a functor $\mathcal{V}^{\otimes} \mapsto \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$. $\square$

Proposition 5.1.2.9. $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(-)$ is lax monoidal with respect to the Cartesian product of $\mathbb{O}(n)$-monoidal ∞-categories.

Proof. The functor $\Theta^{\mathrm{op}}_{n,(-)} \colon \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)} \to \mathrm{Opd}_{\infty}^{\mathbb{O}(n),\mathrm{gen}}$ preserves products, and if we define $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(-)$ using the version of $\mathrm{Alg}^{\mathbb{O}(n)}(-)$ fibred over $\mathrm{Opd}_{\infty}^{\mathbb{O}(n),\mathrm{gen}}$ we see by the same proof as that of Proposition 3.2.8.13 that this functor is lax monoidal with respect to the Cartesian product of generalized $\mathbb{O}(n)$-∞-operads. Thus the pullback $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(-)$ is also lax monoidal. □

5.1.3 n-Categorical Algebras in Spaces

In this subsection we prove that the ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{S}^{\times})$ of n-categorical algebras in spaces is equivalent to the ∞-category $\mathrm{Seg}_{\infty}^{\mathbb{O}(n)}$ of Segal $\mathbb{O}(n)$-spaces.

If $\mathcal{V}$ is a Cartesian monoidal ∞-category, we can construct a Cartesian fibration

$$\mathrm{Mnd}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}) \to \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$$

with fibre at X the ∞-category $\mathrm{Mnd}_{\Theta^{\mathrm{op}}_{n,X}}^{\mathbb{O}(n)}(\mathcal{V})$ of $\Theta^{\mathrm{op}}_{n,X}$-monoids in $\mathcal{V}$, in the same way as we defined $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\times})$ above. This has a natural equivalence over $\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$ with the ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\times})$.

We can also define a Cartesian fibration $\mathrm{Mon}_{\infty}^{\mathbb{O}(n),\mathrm{cat}} \to \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$ whose fibre at X is the ∞-category $\mathrm{Mon}_{\infty}^{\mathbb{O}(n),\Theta^{\mathrm{op}}_{n,X}}$ of $\Theta^{\mathrm{op}}_{n,X}$-monoidal ∞-categories. Using the equivalence between functors to $\mathcal{S}$ and left fibrations, we can identify $\mathrm{Mnd}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{S})$ with the full subcategory $\mathrm{LMon}_{\infty}^{\mathbb{O}(n),\mathrm{cat}}$ of $\mathrm{Mon}_{\infty}^{\mathbb{O}(n),\mathrm{cat}}$ spanned by those $\Theta^{\mathrm{op}}_{n,X}$-monoidal ∞-categories that are left fibrations.

Similarly, we can identify the ∞-category $\mathrm{Seg}_{\infty}^{\mathbb{O}(n)}$ of Segal $\mathbb{O}(n)$-spaces with the full subcategory $\mathrm{LMult}_{\infty}^{\mathbb{O}(n)}$ of $\mathrm{Mult}_{\infty}^{\mathbb{O}(n)}$ spanned by the $\mathbb{O}(n)$-multiple ∞-categories that are left fibrations.

There is an obvious functor $p \colon \mathrm{LMon}_{\infty}^{\mathbb{O}(n),\mathrm{cat}} \to \mathrm{LMult}_{\infty}^{\mathbb{O}(n)}$ given by composing a $\Theta^{\mathrm{op}}_{n,X}$-monoidal ∞-category $\mathcal{C} \to \Theta^{\mathrm{op}}_{n,X}$ that is a left fibration with the map $\Theta^{\mathrm{op}}_{n,X} \to \Theta^{\mathrm{op}}_{n}$, which is also a left fibration and an $\mathbb{O}(n)$-multiple ∞-category.

Proposition 5.1.3.1. This functor $p \colon \mathrm{LMon}_{\infty}^{\mathbb{O}(n),\mathrm{cat}} \to \mathrm{LMult}_{\infty}^{\mathbb{O}(n)}$ is an equivalence.

Proof. Let j denote the usual inclusion $\Theta^{\mathrm{op}}_{n-1} \hookrightarrow \Theta^{\mathrm{op}}_{n}$. Then there is an adjunction

$$j^* \colon \mathrm{Seg}_{\infty}^{\mathbb{O}(n)} \rightleftarrows \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)} \colon j_*,$$

and $\Theta^{\mathrm{op}}_{n,X}$ is the object of $\mathrm{LMult}_{\infty}^{\mathbb{O}(n)}$ corresponding to j_*X. Moreover, j^* is a Cartesian fibration by Lemma 2.1.6.4; if $A \in \mathrm{Seg}_{\infty}^{\mathbb{O}(n)}$, a Cartesian arrow with target A over $X \to j^*A$ is given by taking the pullback of $A \to j_*j^*A$ along $j_*X \to j_*j^*A$.

To prove that p is an equivalence, we must show that it is fully faithful and essentially surjective. We thus have to prove that the map

$$\mathrm{Map}_{\mathrm{LMon}_{\infty}^{\mathbb{O}(n),\mathrm{cat}}}(A, B) \to \mathrm{Map}_{\mathrm{LMult}_{\infty}^{\mathbb{O}(n)}}(p(A), p(B))$$

is an equivalence. Since it is clear that the functor p preserves Cartesian morphisms over $\mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$, it suffices to show that the induced maps on fibres over $f \colon j^*p(A) \to j^*p(B)$

are equivalences. But this is clear: on both sides the fibre at f can be identified with the space of those maps over $\Theta^{\mathrm{op}}_{n,j^*A}$ from A to the pullback of B along $\Theta^{\mathrm{op}}_{n,f}$ that preserve inert morphisms.

It remains to prove that p is essentially surjective. Suppose $\alpha\colon \mathcal{A} \to \Theta_n^{\mathrm{op}}$ is an object of $\mathrm{LMult}_\infty^{\mathbb{O}(n)}$. The adjunction $j^* \dashv j_*$ induces a map $h\colon \mathcal{A} \to \Theta^{\mathrm{op}}_{n,j^*A}$; this is equivalent to a left fibration by Proposition 2.1.4.4 and so α is in the essential image of p. $\square$

Corollary 5.1.3.2. The composite functor $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{S}^\times) \to \mathrm{Seg}_\infty^{\mathbb{O}(n)}$ is an equivalence.

5.1.4 Complete n-Categorical Algebras

In this subsection we define *complete* n-categorical algebras — the full subcategory $\mathrm{Cat}^{\mathcal{V}}_{\mathbb{O}(n)}$ of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ spanned by the complete n-categorical algebras should be the "correct" ∞-category of $\mathcal{V}$-(∞, n)-categories. However, we will unfortunately not be able to show that $\mathrm{Cat}^{\mathcal{V}}_{\mathbb{O}(n)}$ is a localization of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$, let alone the localization with respect to an appropriate notion of "fully faithful and essentially surjective" morphisms. I hope to return to this question, as well as the related problem of comparing $\mathrm{Cat}^{\mathcal{V}}_{\mathbb{O}(n)}$ to the ∞-category $\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)}$ obtained by iterated enrichment, after more of the machinery of $\mathbb{O}(n)$-∞-operads has been developed.

Definition 5.1.4.1. Suppose $\mathcal{V}^\otimes \to \Theta_n^{\mathrm{op}}$ is an $\mathbb{O}(n)$-monoidal ∞-category and X is a Segal $\mathbb{O}(n-1)$-space. The *trivial n-categorical algebra* $E_X^{\mathcal{V}}$ with underlying Segal $\mathbb{O}(n-1)$-space X is defined as the composite

$$\Theta^{\mathrm{op}}_{n,X} \to \Theta_n^{\mathrm{op}} \xrightarrow{I} \mathcal{V}^\otimes,$$

where I is the unit of $\mathcal{V}^\otimes$. This gives a functor $E^{\mathcal{V}}_{(-)}\colon \mathrm{Seg}_\infty^{\mathbb{O}(n-1)} \to \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$.

Remark 5.1.4.2. The n-categorical algebra $E_X^{\mathcal{S}}$ can be described as the (∞, n)-category constructed from the $(\infty, n-1)$-category X by adjoining a unique n-morphism between any two parallel $(n-1)$-morphisms in X. In particular, all parallel $(n-1)$-morphisms in X are equivalent in $E_X^{\mathcal{S}}$.

The identity map $\Theta_n^{\mathrm{op}} \to \Theta_n^{\mathrm{op}}$ is the unique $\mathbb{O}(n)$-monoidal structure on the point $*$. This is the unit for the Cartesian product of $\mathbb{O}(n)$-monoidal ∞-categories, and so for every $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}^\otimes$ the ∞-category $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\Theta_n^{\mathrm{op}})$, since $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(-)$ is lax monoidal by Proposition 4.1.3.9. Clearly the only $*$-∞-categories are of the form E_X^* for Segal $\mathbb{O}(n-1)$-spaces X; we can identify the $\mathcal{V}$-∞-category $E_X^{\mathcal{V}}$ with the tensor $E_X^* \otimes I_{\mathcal{V}}$:

Lemma 5.1.4.3. For any $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}^\otimes$ and Segal $\mathbb{O}(n-1)$-space X, we have $E_X^{\mathcal{V}} \simeq E_X^* \otimes I_{\mathcal{V}}$. Moreover, if $\mathcal{V}^\otimes$ is presentably monoidal (so $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{S}^\times)$), then $E_X^{\mathcal{V}} \simeq E_X^{\mathcal{S}} \otimes I_{\mathcal{V}}$.

Proof. Considering the construction of the external product in $\mathrm{Alg}^{\mathbb{O}(n)}$, we see that $E_X^* \otimes I_{\mathcal{V}}$ is given by

$$E_X^* \times_{\Theta_n^{\mathrm{op}}} I_{\mathcal{V}}\colon \Theta^{\mathrm{op}}_{n,X} \times_{\Theta_n^{\mathrm{op}}} \Theta_n^{\mathrm{op}} \to \Theta_n^{\mathrm{op}} \times_{\Theta_n^{\mathrm{op}}} \mathcal{V}^\otimes \simeq \mathcal{V}^\otimes.$$

We can factor this as

$$\Theta_{n,X}^{\mathrm{op}} \times_{\Theta_n^{\mathrm{op}}} \Theta_n^{\mathrm{op}} \xrightarrow{E_X^* \times_{\Theta_n^{\mathrm{op}}} \mathrm{id}} \Theta_n^{\mathrm{op}} \times_{\Theta_n^{\mathrm{op}}} \Theta_n^{\mathrm{op}} \xrightarrow{\mathrm{id} \times_{\Theta_n^{\mathrm{op}}} I_{\mathcal{V}}} \Theta_n^{\mathrm{op}} \times_{\Theta_n^{\mathrm{op}}} \mathcal{V}^{\otimes},$$

which is clearly the same as $E_X^{\mathcal{V}}$.

In the presentable case, we have

$$E_X^{\mathcal{S}} \otimes I_{\mathcal{V}} \simeq (E_X^* \otimes I_{\mathcal{S}}) \otimes I_{\mathcal{V}} \simeq E_X^* \otimes (I_{\mathcal{S}} \otimes I_{\mathcal{V}}) \simeq E_X^* \otimes I_{\mathcal{V}} \simeq E_X^{\mathcal{V}},$$

since it is easy to see that the tensorings with $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\Theta_n^{\mathrm{op}})$ and $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{S}^{\times})$ are compatible. □

Recall that for $A \in \Theta_n^{\mathrm{op}}$ we have a Segal $\mathbb{O}(n-1)$-space A^* given by

$$A^*(B) := \mathrm{Hom}_{\Theta_n^{\mathrm{op}}}(A, B),$$

giving a functor $(-)^* \colon \Theta_n \to \mathrm{Seg}_{\infty}^{\mathbb{O}(n-1)}$. We write $E^A := E_{A^*}$, thus $E^{(-)}$ is a functor $\Theta_n \to \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})$.

Remark 5.1.4.4. When $\mathcal{V}$ is $\mathcal{S}$, it is easy to see that E^A as defined here corresponds to E^A as defined in §2.2.3 under the equivalence of §5.1.3.

Definition 5.1.4.5. Suppose $\mathcal{V}^{\otimes}$ is an $\mathbb{O}(n)$-monoidal ∞-category and $\mathcal{C}$ is an n-categorical algebra in $\mathcal{V}^{\otimes}$. An *n-equivalence* in $\mathcal{C}$ is a morphism $E^{C_n} \to \mathcal{C}$.

Definition 5.1.4.6. Given an n-categorical algebra $\mathcal{C}$ in $\mathcal{V}$, we write $\iota_{\bullet}\mathcal{C}$ for the functor $\Theta_n^{\mathrm{op}} \to \mathcal{S}$ given by $\mathrm{Map}(E^{(-)}, \mathcal{C})$.

Lemma 5.1.4.7. Let $\mathcal{C}$ be an n-categorical algebra in $\mathcal{V}$ with underlying Segal $\mathbb{O}(n-1)$-space X, and let A be an object of $\Theta_{n-1}^{\mathrm{op}} \subseteq \Theta_n^{\mathrm{op}}$. Then the map

$$\iota_A\mathcal{C} := \mathrm{Map}(E^A, \mathcal{C}) \to \mathrm{Map}(A^*, X) \simeq X(A)$$

is an equivalence.

Proof. It suffices to check that the homotopy fibres of this map are contractible. By [Lur09a, Proposition 2.4.4.2] the homotopy fibre at $p \colon A^* \to X$ is

$$\mathrm{Map}_{\mathrm{Alg}^{\mathbb{O}(n)}_{\Theta^{\mathrm{op}}_{n,A^*}}}(E^A, p^*\mathcal{C}).$$

Since A is in $\Theta_{n-1}^{\mathrm{op}}$, there are no parallel $(n-1)$-morphism in the (∞, n)-category E^A, which means that E^A is equivalent to the initial $\Theta_{n,A^*}^{\mathrm{op}}$-algebra. Thus the fibre at p is indeed contractible. □

The restriction of $\iota_{\bullet}\mathcal{C}$ to $\Theta_{n-1}^{\mathrm{op}}$ is thus equivalent to the underlying Segal $\mathbb{O}(n-1)$-space of $\mathcal{C}$.

Definition 5.1.4.8. Let $\mathcal{C}$ be an n-categorical algebra in an $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}$. The *classifying Segal $\mathbb{O}(n-1)$-space of n-equivalences* $\iota\mathcal{C}$ of $\mathcal{C}$ is the left Kan extension $p_!\iota_{\bullet}\mathcal{C}$ of the Segal $\mathbb{O}(n)$-space $\iota_{\bullet}\mathcal{C}$ along $p \colon \Theta_n^{\mathrm{op}} \to \Theta_{n-1}^{\mathrm{op}}$.

Definition 5.1.4.9. Let $\mathcal{C}$ be a $\Theta_{n,X}^{\mathrm{op}}$-algebra in an $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}$. We say $\mathcal{C}$ is *n-complete* if the natural map $X \simeq \iota_\bullet \mathcal{C}|_{\Theta_{n-1}^{\mathrm{op}}} \to \iota\mathcal{C}$ is an equivalence.

Conjecture 5.1.4.10. An n-categorical algebra is n-complete if and only if it is local with respect to the map $E^{C_n} \to E^{C_{n-1}}$.

Remark 5.1.4.11. We would like to deduce this from the case where $\mathcal{V}$ is $\mathcal{S}$, i.e. Proposition 2.2.3.16. In §4.2.1 we were able to carry out such a reduction because we knew that if $\mathcal{V}^\otimes$ is presentably monoidal then $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}}(\mathcal{V}^\otimes)$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}}(\mathcal{S}^\times)$ in a colimit-preserving way — to see this we needed to know that composition with a strong monoidal functor gives a colimit-preserving functor on algebras, and that the functor $L\mathbb{\Delta}_{(-)}^{\mathrm{op}}$ preserves products. However, we do not yet know how to prove the analogues of these two statements in the setting of $\mathbb{O}(n)$-∞-operads, and so we are currently unable to prove Conjecture 5.1.4.10.

Definition 5.1.4.12. If $\mathcal{C}$ is an n-categorical algebra in an $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}^\otimes$, we say that $\mathcal{C}$ is *complete* if $\mathcal{C}$ is n-complete and the Θ_{n-1}-space $\iota\mathcal{C}$ is complete. We write $\mathrm{Cat}_{\mathbb{O}(n)}^{\mathcal{V}}$ for the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ spanned by the complete n-categorical algebras.

Remark 5.1.4.13. If $\mathcal{V}$ is the ∞-category $\mathcal{S}$ of spaces, then the complete n-categorical algebras correspond to the complete Θ_n-spaces under the equivalence of §5.1.3.

Conjecture 5.1.4.14. Suppose $\mathcal{V}^\otimes$ is a presentably $\mathbb{E}_n$-monoidal ∞-category, so there is a (strong) monoidal functor $t\colon \mathcal{S} \to \mathcal{V}$, which induces a functor

$$t_*\colon \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{S}^\times) \to \mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes).$$

Let ϵ_k denote the morphism of n-categorical algebras in $\mathcal{S}$ corresponding to the morphism ϵ_k of Θ_n-spaces of Definition 2.2.3.15 under the equivalence of §5.1.3. Then $\mathrm{Cat}_{\mathbb{O}(n)}^{\mathcal{V}}$ is the localization of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ with respect to $t_*\epsilon_k$ for $k = 1, \ldots, n$. In particular, $\mathrm{Cat}_{\mathbb{O}(n)}^{\mathcal{V}}$ is an accessible localization of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ and so is a presentable ∞-category.

Definition 5.1.4.15. A morphism $\phi\colon \mathcal{C} \to \mathcal{D}$ of n-categorical algebras in an $\mathbb{E}_n$-monoidal ∞-category $\mathcal{V}^\otimes$ is *fully faithful and essentially surjective* if ϕ is Cartesian with respect to the projection $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes) \to \mathrm{Seg}_\infty^{\mathbb{O}(n-1)}$ and the morphism $\iota\phi$ of Segal $\mathbb{O}(n-1)$-spaces is fully faithful and essentially surjective in the sense of Definition 2.2.3.17.

Conjecture 5.1.4.16. Suppose $\mathcal{V}^\otimes$ is a presentably $\mathbb{E}_n$-monoidal ∞-category. The fully faithful and essentially surjective morphisms in $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ constitute precisely the saturated class of morphisms generated by $t_*\epsilon_k$, $k = 1, \ldots, n$. In particular $\mathrm{Cat}_{\mathbb{O}(n)}^{\mathcal{V}}$ is the localization of $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ with respect to the fully faithful and essentially surjective morphisms.

5.2 n-Correspondences

In this section we will briefly discuss the analogue of correspondences for n-categorical algebras.

5.2.1 The Φ-Multiple ∞-Categories $\mathcal{L}^{\Phi}[I, \{X_\alpha\}]$

Here we define the Φ-multiple ∞-categories we will use below, in the case $\Phi = \mathbb{O}(n)$, to define correspondences between n-categorical algebras:

Definition 5.2.1.1. Suppose Φ is a self-categorical perfect operator category. Let $h_I \colon \mathcal{L}^{\Phi} \to \mathrm{Set}$ be the representable functor $\mathrm{Hom}_{\mathcal{L}^{\Phi}}(I, -)$, and let $\mathcal{L}^{\Phi}[I] \to \mathcal{L}^{\Phi}$ be a coGrotendieck fibration associated to $\hat{h}_I$; then $\mathcal{L}^{\Phi}[I]$ is a Φ-multiple ∞-category.

Example 5.2.1.2. If Φ is $\mathbb{O}$ then $\mathcal{L}^{\mathbb{O}}[n]$ is $\mathbb{\Delta}^{\mathrm{op}}[n]$.

Definition 5.2.1.3. Suppose Φ is a self-categorical perfect operator category that has an initial object $\emptyset$, and let A be a clean atom of Φ. Given a morphism $\alpha \colon I \nrightarrow \emptyset$ in $\mathcal{L}^{\Phi}$ there is a functor $\mathcal{L}^{\Phi} \to \mathcal{L}^{\Phi}[I]$ that sends J to the composite $I \nrightarrow \emptyset \nrightarrow J$. Restricting to $\mathcal{L}^{\Phi_{/A}} \subseteq \mathcal{L}^{\Phi}$ we get a functor $\gamma \colon \coprod_\alpha \mathcal{L}^{\Phi_{/A}} \to \mathcal{L}^{\Phi}[I]$. Given Segal $\Phi_{/A}$-spaces $\mathcal{X}_\alpha$ for $\alpha \colon I \nrightarrow \emptyset$ consider their disjoint union $F_{(X_\alpha)} \colon \coprod_\alpha \mathcal{L}^{\Phi_{/A}} \to \mathcal{S}$. The right Kan extension $j_* F_{(X_\alpha)}$ is clearly a Segal Φ-space. We write $\mathcal{L}^{\Phi}[I, \{\mathcal{X}_\alpha\}] \to \mathcal{L}^{\Phi}[I]$ for the left fibration associated to $j_* F_{(X_\alpha)}$; this is a Φ-multiple ∞-category.

Example 5.2.1.4. If $\Phi = \mathbb{O}$ then $\mathcal{L}^{\mathbb{O}}[\mathbf{n}, (X_0, \ldots, X_n)]$ is $\mathbb{\Delta}^{\mathrm{op}}_{X_0 < \cdots < X_n}$.

Definition 5.2.1.5. If $\Phi = \mathbb{O}(n)$, we write $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]$ for $\mathcal{L}^{\mathbb{O}(n)}[I, (X_\alpha)]$.

5.2.2 Correspondences

We now use the generalized ∞-operads introduced above to define correspondences between n-categorical algebras:

Definition 5.2.2.1. Let $\mathcal{V}^{\otimes}$ be an $\mathbb{O}(n)$-monoidal ∞-category. A *k-correspondence* ($1 \leq k \leq n$) between two n-categorical algebras $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{V}^{\otimes}$ is a $\Theta_n^{\mathrm{op}}[C_k, (X, Y)]$-algebra $\mathcal{M}$ in $\mathcal{V}^{\otimes}$, where X and Y are the underlying Segal $\mathbb{O}(n-1)$-spaces of $\mathcal{C}$ and $\mathcal{D}$, respectively, such that $\mathcal{M}$ restricts to $\mathcal{C}$ and $\mathcal{D}$ when pulled back along the two maps $C_k \nrightarrow C_0$.

Definition 5.2.2.2. For $I \in \Theta_n^{\mathrm{op}}$, let $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]^{\amalg}$ denote the colimit of generalized $\mathbb{O}(n)$-∞-operads

$$\operatorname*{colim}_{p \colon I \nrightarrow A \in \mathcal{G}^{\mathbb{O}(n)}_{I/}} \Theta_n^{\mathrm{op}}[A, (X_i^p)]$$

where X_i^p denotes X_α when α is p composed with a map $A \nrightarrow \emptyset$. Let $\kappa \colon \Theta_n^{\mathrm{op}}[I, (X_\alpha)]^{\amalg} \to \Theta_n^{\mathrm{op}}[I, (X_\alpha)]$ denote the obvious inclusion. We say a $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]$-algebra $\mathcal{M}$ in an $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}^{\otimes}$ is a *composite* if it is the left operadic Kan extension of its restriction to $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]^{\amalg}$, i.e. the adjunction morphism $\kappa_! \kappa^* \mathcal{M} \to \mathcal{M}$ is an equivalence.

Definition 5.2.2.3. It is easy to see that that the full subcategory of $(\mathrm{Opd}_\infty^{\mathbb{O}(n),\mathrm{gen}})_{/\Theta_n^{\mathrm{op}}[I]}$ spanned by the objects $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]$ is equivalent to $(\mathrm{Seg}_\infty^{\mathbb{O}(n-1)})^{\times k}$, where k is the number of morphisms $I \nrightarrow \emptyset$. Define $\mathrm{ALG}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})[I]$ by the pullback

$$\begin{tikzcd}
\mathrm{ALG}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^{\otimes})[I] \arrow[r] \arrow[d] & \mathrm{Alg}^{\mathbb{O}(n)}_{/\Theta_n^{\mathrm{op}}[I]}(\mathcal{V}^{\otimes}) \arrow[d] \\
(\mathrm{Seg}_\infty^{\mathbb{O}(n-1)})^{\times k} \arrow[r] & (\mathrm{Opd}_\infty^{\mathbb{O}(n),\mathrm{gen}})_{/\Theta_n^{\mathrm{op}}[I]}.
\end{tikzcd}$$

This defines a functor $\Theta_n^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

Definition 5.2.2.4. Suppose $\mathcal{V}^\otimes$ is an $\mathbb{O}(n)$-monoidal ∞-category. For $I \in \Theta_n^{\mathrm{op}}$, we write $\mathrm{CORR}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)[I]$ for the full subcategory of $\mathrm{ALG}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)[I]$ spanned by those algebras that are composites and whose restrictions to $\mathrm{Alg}_{\mathrm{cat}}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ are complete n-categorical algebras.

Conjecture 5.2.2.5. For any $I \in \Theta_n^{\mathrm{op}}$ and any $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]^{\amalg}$-algebra $\mathcal{M}$ in an $\mathbb{O}(n)$-monoidal ∞-category $\mathcal{V}^\otimes$, the adjunction morphism $\mathcal{M} \to \kappa^*\kappa_!\mathcal{M}$ is an equivalence.

Remark 5.2.2.6. This would follow from an $\mathbb{O}(n)$-analogue of Theorem 4.6.2.4, which can probably be proved by essentially the same proof as that result, but we will not attempt to carry out such an argument here.

Assuming this we can show the following, by the same argument as in the proof of Lemma 4.6.3.3:

Lemma 5.2.2.7. The functor $\mathrm{CORR}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)[\bullet] \colon \Theta_n^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is an $\mathbb{O}(n)$-category object.

Remark 5.2.2.8. By looking at the subcategory of $\Theta_n^{\mathrm{op}}[I, (X_\alpha)]$-algebras of the form $\pi_I^*\mathcal{C}$ where π_I is the projection $\Theta_n^{\mathrm{op}}[I] \to \Theta_n^{\mathrm{op}}$, we can extract from the $\mathbb{O}(n)$-multiple ∞-category $\mathrm{CORR}^{\mathbb{O}(n)}(\mathcal{V}^\otimes)$ an $(\infty, n+1)$-category of $\mathcal{V}$-∞-categories, functors, natural transformations, etc.

Bibliography

[Bac10] Hugo V. Bacard, *Segal enriched categories I* (2010), available at `arXiv:1009.3673`.

[BD95] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105.

[BK12] C. Barwick and D. M. Kan, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 42–68.

[BK] ______, *Partial Model Categories and their Simplicial Nerves*, available at `arXiv:1102.2512`.

[BSP11] Clark Barwick and Christopher Schommer-Pries, *On the unicity of the homotopy theory of higher categories* (2011), available at `arXiv:1112.0040`.

[Bar13] Clark Barwick, *From operator categories to topological operads* (2013), available at `arXiv:1302.5756`.

[Ber07] Clemens Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), no. 1, 230–270.

[BM12] Clemens Berger and Ieke Moerdijk, *On the homotopy theory of enriched categories* (2012), available at `arXiv:1201.2134`.

[Ber07] Julia E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058.

[CS10] G. S. H. Cruttwell and Michael A. Shulman, *A unified framework for generalized multicategories*, Theory Appl. Categ. **24** (2010), No. 21, 580–655.

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004.

[DK80a] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284.

[DK80b] ______, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35.

[DK84] ______, *Homotopy theory and simplicial groupoids*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no. 4, 379–385.

[GH] David Gepner and Rune Haugseng, *Enriched ∞-categories*. In preparation; draft version available from `math.mit.edu/~haugseng`.

[GM12] Bertrand Guillou and J. P. May, *Permutative G-categories in equivariant infinite loop space theory* (2012), available at `arXiv:1207.3459`.

[GM11a] ______, *Models of G-spectra as presheaves of spectra* (2011), available at `arXiv:1110.3571`.

[GM11b] ______, *Enriched model categories and diagram categories* (2011), available at `arXiv:1110.3567`.

[HS98] André Hirschowitz and Carlos Simpson, *Descente pour les n-champs* (1998), available at `arXiv:math/9807049`.

[JT07] André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326.

[Lei02] Tom Leinster, *Generalized enrichment of categories*, J. Pure Appl. Algebra **168** (2002), no. 2-3, 391–406. Category theory 1999 (Coimbra).

[Lei04] ______, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.

[Lur09a] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available at `http://math.harvard.edu/~lurie/papers/highertopoi.pdf`.

[Lur09b] ———, *(∞,2)-Categories and the Goodwillie Calculus I* (2009), available at `http://math.harvard.edu/~lurie/papers/GoodwillieI.pdf`.

[Lur11] ———, *Higher Algebra*, 2011. Available at `http://math.harvard.edu/~lurie/papers/higheralgebra.pdf`.

[MP89] Michael Makkai and Robert Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, Providence, RI, 1989.

[Mor11] Jack Morava, *The cosmic Galois group as Koszul dual to Waldhausen's A(pt)* (2011), available at `arXiv:1108.4627`.

[Mur11] Fernando Muro, *Homotopy theory of nonsymmetric operads*, Algebr. Geom. Topol. **11** (2011), no. 3, 1541–1599.

[Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic).

[Rez10] ———, *A Cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010), no. 1, 521–571.

[Roi94] Agustí Roig, *Model category structures in bifibred categories*, J. Pure Appl. Algebra **95** (1994), no. 2, 203–223.

[SS00] Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511.

[Sim98] Carlos Simpson, *On the Breen-Baez-Dolan stabilization hypothesis for Tamsamani's weak n-categories* (1998), available at `arXiv:math/9810058`.

[Sim12] ———, *Homotopy theory of higher categories*, New Mathematical Monographs, vol. 19, Cambridge University Press, Cambridge, 2012.

[Sta12] Alexandru E. Stanculescu, *Constructing model categories with prescribed fibrant objects* (2012), available at `arXiv:1208.6005`.

[Web04] Mark Weber, *Generic morphisms, parametric representations and weakly Cartesian monads*, Theory Appl. Categ. **13** (2004), No. 14, 191–234.